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# ON PROPER BOUNDARY POINTS OF THE SPECTRUM AND COMPLEMENTED EIGENSPACES

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## 1. Introduction.

Let $X$ be a Banach space and $B(X)$ the Banach algebra of all bounded linear operators on $X$. The following definition was given by Schechter in [10] for operators on a Hilbert space. We assume that $T \in B(X)$. The spectrum, resolvent set and kernel of $T$ are denoted by $\mathrm{Sp}(T)$, $\mathrm{Res}(T)$ and $N(T)$.

DEFINITION 1. A point $\lambda \in \partial \mathrm{Sp}(T)$ is a *proper* boundary point of $\mathrm{Sp}(T)$ if there exists a bounded sequence $\{\lambda_n\} \subset \mathrm{Res}(T)$ such that

$$\|(\lambda_n - \lambda)(\lambda_n - T)^{-1}\| \to 1 .$$

We denote by $\mathrm{Pr}(T)$ the set of all proper boundary points of $\mathrm{Sp}(T)$.

Let $0 \in \mathrm{Pr}(T)$. If $X$ is reflexive, $N(T)$ is complemented. In fact $X = N(T) \oplus (TX)^-$ (Lemma 2 and [7 Corollary VII.7.5]). We will show that $N(T^*)$ is always complemented in $X^*$. We also extend this result to a countable commuting family of operators and give an application to normal operators on a Hilbert space.

In a non-reflexive space $(TX)^-$ is not in general a complement of $N(T)$ (when a complement exists). An example of this is given by the derivation of a hermitian operator with infinite spectrum on a Hilbert space $H$ (see [1]). It is a hermitian operator on $B(H)$ with complemented kernel (Lemma 3 and Theorem 3).

We recall that an element $a$ of a unital Banach algebra $A$ is called *hermitian* if its numerical range

$$V(A,a) = \{f(a) : f \in A^* \text{ and } \|f\| = f(1) = 1\} \subset \mathbf{R}$$

or, equivalently, $\|e^{ita}\| = 1$ for all real $t$ ([3, p. 46]). If $a = h + ik$ where $h$ and $k$ are commuting hermitian elements, $a$ is *normal* (see [5] and [3]).

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If $T$ is a hermitian operator, $\mathrm{Sp}\,(T)=\mathrm{Pr}\,(T)$ by the following lemma. The proof is essentially the same as in [10, p. 43].

LEMMA 1. $\partial V(B(X), T)\cap \mathrm{Sp}\,(T)\subset \mathrm{Pr}\,(T)$.

PROOF. Let $\lambda$ be in the boundary of the numerical range $V=V(B(X), T)$ and in $\mathrm{Sp}\,(T)$. Since $V$ is convex, there is an element $\alpha \notin V$ such that $|\alpha-\lambda|=d(\alpha, V)$. Note that $\alpha \in \mathrm{Res}\,(T)$ since $\mathrm{Sp}\,(T)\subset V$ ([5, p. 53]). By [11, p. 418]

$$\|(\alpha-T)^{-1}\| \leqq d(\alpha, V)^{-1} .$$

Hence $\|(\alpha-\lambda)(\alpha-T)^{-1}\|\leqq 1$. On the other hand $\lambda \in \mathrm{Sp}\,(T)$ implies by the Spectral mapping theorem that $1 \in \mathrm{Sp}\,((\alpha-\lambda)(\alpha-T)^{-1})$. It follows that $\|(\alpha-\lambda)(\alpha-T)^{-1}\|=1$ and so $\lambda \in \mathrm{Pr}\,(T)$.

LEMMA 2. *The sequence $\{\lambda_n\}$ in Definition 1 can be chosen so that $\lambda_n \to \lambda$.*

PROOF. Let $\{\lambda_n\}$ be a bounded sequence $\subset \mathrm{Res}\,(T)$ such that $\|(\lambda_n-\lambda)(\lambda_n-T)^{-1}\| \to 1$. There exists a convergent subsequence $\{\lambda_{n_k}\}$. Let $\alpha=\lim \lambda_{n_k}$.

If $\alpha \in \mathrm{Sp}\,(T)$, $\alpha=\lambda$, since the sequence $\{(\lambda_{n_k}-\lambda)(\lambda_{n_k}-T)^{-1}\}$ is bounded in $B(X)$ (see [7, Corollary VII.3.3]). Hence $\lambda_{n_k} \to \lambda$.

Suppose $\alpha \in \mathrm{Res}\,(T)$. Then

$$\|(\alpha-\lambda)(\alpha-T)^{-1}\| = 1 . \tag{1}$$

Let $\beta=\lambda+t(\alpha-\lambda)$, $t \in (0,1]$. By (1) $\beta \in \mathrm{Res}\,(T)$. Since (1) implies $\|(\alpha-T)x\| \geqq |\alpha-\lambda|\|x\|$ $(x \in X)$, we have

$$\begin{aligned}\|(\beta-T)x\| = \|[\alpha-T+(1-t)(\lambda-\alpha)]x\| &\geqq \|(\alpha-T)x\|-(1-t)|\lambda-\alpha|\,\|x\| \\ &\geqq t|\lambda-\alpha|\,\|x\| = \|(\beta-\lambda)x\|\end{aligned}$$

for $x \in X$. Therefore $\|(\beta-\lambda)(\beta-T)^{-1}\|\leqq 1$. Since $\lambda \in \mathrm{Sp}\,(T)$ this norm $=1$. To complete the proof we choose a sequence of elements $\beta$ converging to $\lambda$.

## 2. The main theorems.

We shall make use of a Banach (generalised) limit on the space $l^\infty$ of all bounded complex sequences (see [2] for example).

NOTATIONS. If $F\subset B(X)$, $[F]^*=\{S^* : S \in F\}$ and

$$\mathrm{com}\,F = \{V\in B(X) : VS=SV \text{ for all } S \in F\} ,$$

the commutant of $F$.

THEOREM 1. *If $\lambda \in \mathrm{Pr}\,(T)$, there is a projection $P$ onto $N(T^*-\lambda)$ such that $\|P\| \leqq 1$ and $P \in \mathrm{com}\,[\mathrm{com}\,\{T\}]^*$.*

PROOF. We may assume that $\lambda=0$ since the general case is then obtained by considering $T-\lambda$ instead of $T$. By Lemma 2 there exists a sequence $\{\lambda_n\}$ such that $\lambda_n \to 0$ and $\|W_n\| \to 1$ where $W_n=\lambda_n(\lambda_n-T)^{-1}$.

Let Lim be a Banach limit on $l^\infty$. Since for fixed elements $x \in X, f \in X^*$ the sequence $\{(W_n^* f)(x)\}$ is bounded, we can define an operator $P$ on $X^*$ by

$$(Pf)(x) = \mathrm{Lim}\,(W_n^* f)(x) \qquad (x \in X,\ f \in X^*) \tag{2}$$

and then $\|P\| \leqq 1$.

Since $TW_n=W_nT=\lambda_n(W_n-I) \to 0$, we have

$$(T^*Pf)(x) = (Pf)(Tx) = \mathrm{Lim}\, f(W_nTx) = 0$$

and so

$$T^*P = 0\,. \tag{3}$$

It follows that

$$P = (I-W_n^*)P+W_n^*P = \lambda_n^{-1}W_n^*T^*P+W_n^*P = W_n^*P$$

which by (2) gives $P^2=P$. That $PX^*=N(T^*)$ follows easily from (2) and (3). Consequently $P$ is a projection onto $N(T^*)$.

Let $V \in \mathrm{com}\,\{T\}$. Then

$$\begin{aligned}(PV^*f)(x) &= \mathrm{Lim}\,(W_n^*V^*f)(x) \qquad \text{(by (2))}\\ &= \mathrm{Lim}\,(V^*W_n^*f)(x) = \mathrm{Lim}\,(W_n^*f)(Vx)\\ &= (Pf)(Vx) = (V^*Pf)(x) \qquad \text{(by (2))}\end{aligned}$$

for $x \in X$ and $f \in X^*$. Hence $PV^*=V^*P$.

REMARK. In the case of Theorem 1, $X^*=N(T^*-\lambda)\oplus Y$ where $Y$ is a closed subspace of $X^*$ and $((T^*-\lambda)X^*)^- \subset Y$.

We need the following properties of a normal operator ([6, Lemma 3]). Let $T$ be normal with $T=H+iK$ where $H$ and $K$ are commuting hermitian operators. Then

$$N(T) = N(H) \cap N(K) \tag{N1}$$

$$\mathrm{com}\,\{T\} = \mathrm{com}\,\{H\} \cap \mathrm{com}\,\{K\}\,. \tag{N2}$$

THEOREM 2. *Let $F=\{T_i : i \in \mathbf{N}\}$ be a family of pairwise commuting operators* *on $X$ such that for each $i$ $\lambda_i \in \mathrm{Sp}(T_i)$ and either $\lambda_i \in \mathrm{Pr}(T_i)$ or $T_i$ is normal. Then there exists a projection $P$ onto the space*

$$M = \bigcap_{i=1}^{\infty} N(T_i^* - \lambda_i)$$

*such that $\|P\| \leqq 1$ and $P \in \mathrm{com}[\mathrm{com}\, F]^*$.*

PROOF. a) We prove first the theorem in the case when $0 \in \mathrm{Pr}(T_i)$ for each $i \in \mathbf{N}$. By Theorem 1 there exist projections $P_i$ onto $N(T_i^*)$ with the properties: $\|P_i\| \leqq 1$ and

$$P_i \in \mathrm{com}[\mathrm{com}\{T_i\}]^* \quad (i \in \mathbf{N}) . \tag{4}$$

Let Lim be a fixed Banach limit on $l^\infty$. Since the sequence $\{(P_n P_{n-1} \ldots P_1 f)(x)\}$ is bounded we can define an operator $Q$ on $X^*$ by

$$(Qf)(x) = \mathrm{Lim}\,(P_n P_{n-1} \ldots P_1 f)(x) \quad (x \in X,\ f \in X^*) \cdot$$

and then $Q$ is bounded with $\|Q\| \leqq 1$.

We will show that $Q$ is a projection onto $M$. For $i = 1, 2, \ldots$

$$(T_i^* Q f)(x) = (Qf)(T_i x) = \mathrm{Lim}\,(T_i^* P_n P_{n-1} \ldots P_1 f)(x) = 0$$

since $T_i^* P_k = P_k T_i^*$ $(k \in \mathbf{N})$ and $T_i^* P_i = 0$. Hence

$$T_i^* Q = 0 \quad (i \in \mathbf{N}) . \tag{5}$$

This implies $QX^* \subset N(T_i^*) = P_i X^*$ and so $Q = P_i Q$ $(i \in \mathbf{N})$. It follows from the definition of $Q$ that $Q^2 = Q$. We clearly have $QX^* \subset M$. If on the other hand $f \in M = \bigcap N(T_i^*)$, $f = P_i f$ for each $i \in \mathbf{N}$ and we obtain $f = Qf$. Hence $QX^* = M$.

The property $Q \in \mathrm{com}[\mathrm{com}\, F]^*$ follows easily from (4).

b) The general case. We may assume that $\lambda_i = 0$ $(i \in \mathbf{N})$. By (N1) and (N2) the space $M$ can be expressed in the form $\bigcap_{i=1}^{\infty} N(S_i^*)$ where $0 \in \mathrm{Pr}(S_i)$ for all $i$ and

$$\mathrm{com}\{S_i : i \in \mathbf{N}\} = \mathrm{com}\{T_i : i \in \mathbf{N}\} .$$

The result follows when a) is applied to the family $\{S_i : i \in \mathbf{N}\}$.

Obviously $\|P\| = 1$ except for the case when $M = \{0\}$ and $P = 0$.

## 3. An application.

If $a$ and $a'$ are elements of a unital Banach algebra $A$, we denote the operator $x \mapsto ax - xa'$ on $A$ by $\delta$ or $\delta(a, a')$.

LEMMA 3. *If $a$ and $a'$ are hermitian (normal) elements of $A$, $\delta(a,a')$ is a hermitian (normal) operator on $A$.*

The proof is straight forward. Note that if $a=h+ik$ and $a'=h'+ik'$, then

$$\delta(a,a') = \delta(h,h')+i\delta(k,k') .$$

Let $H$ be a Hilbert space and let

$$F_1 = \{N_i : i \in \mathbf{N}\} \quad \text{and} \quad F_2 = \{N_i' : i \in \mathbf{N}\}$$

be two families of normal operators on $H$ such that in each family the elements commute pairwise. Then the operators $\delta(N_i, N_i')$, $i=1,2,\ldots$, also commute.

The space $B(H)$ can be isometrically identified with the dual space of the trace class $\tau(H)$ equipped with the trace norm [9, p. 47]. Then an operator $T \in B(H)$ is identified with the linear form $t \mapsto \operatorname{trace}(tT)$ $(t \in \tau(H))$. For $T, T' \in B(H)$ the restriction $\delta(T',T) \mid \tau(H)$ is a bounded operator on $\tau(H)$ and

$$\delta(T,T') = (-\delta(T',T) \mid \tau(H))^* .$$

We omit the proofs. It follows from [4, p. 2] that a restriction of a hermitian operator is hermitian. Hence if $T$ and $T'$ are hermitian we deduce from Lemma 3 that $\delta(T',T) \mid \tau(H)$ is hermitian on $\tau(H)$.

We conclude that $\delta_i = \delta(N_i', N_i) \mid \tau(H)$ is normal and its adjoint is $-\delta(N_i, N_i')$, $i=1,2,\ldots$. Applying Theorem 2 to the family $F=\{\delta_i : i \in \mathbf{N}\}$ we obtain

THEOREM 3. *There exists a projection $p$ onto the space*

$$\{S \in B(H) : N_i S = S N_i',\ i \in \mathbf{N}\}$$

*such that* $\|p\| \leqq 1$ *and* $p \in \operatorname{com}[\operatorname{com} F]^*$.

We refer to [13] for another construction of a projection onto the commutant of a normal operator. There always exist projections of norm one onto von Neumann algebras (see [8]). These projections have a property similar to that of $p$ in the following corollary [12].

COROLLARY. *$p$ has the property: Given $U \in \operatorname{com} F_1$, $V \in \operatorname{com} F_2$*

$$p(USV) = Up(S)V \quad (S \in B(H)) .$$

PROOF. Let $\mu=\mu(U,V)$ be the operator $S \mapsto USV$ $(S \in B(H))$. It can be shown that $\mu$ is the adjoint of the operator $\mu(V,U) \mid \tau(H)$ and $\mu(V,U) \mid \tau(H)$ commutes with $F$. Since $p \in \operatorname{com}[\operatorname{com} F]^*$ we conclude that $p\mu=\mu p$.

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