

ON M-IDEALS AND THE ALFSEN–EFFROS STRUCTURE TOPOLOGY

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Abstract.

The M-ideals of a G-space are characterized, and it is shown that a Banach space V is a G-space if and only if the Alfsen–Effros structure topology on the extreme points of the dual ball is Hausdorff.

Introduction.

Let V be a real Banach space and $\hat{c}_e V_1^*$ the extreme points of the unit ball of V^* . Alfsen and Effros used in [3] the w^* -closed L-summands of V^* to define a structure topology on $\hat{c}_e V_1^*$. In this topology a $p \in \hat{c}_e V_1^*$ can never be separated from its negative, hence is it sometimes more convenient to identify p and $-p$ and use the quotient space $(\hat{c}_e V_1^*)_\sigma$. The Alfsen–Effros structure topology has two important special cases:

1) If K is a compact convex set in a locally convex Hausdorff space, then there is a facial topology on $\hat{c}_e K$ (see [1] or [2]). Let $V = A(K)$ (the set of all continuous affine functions on K), then it is known that $(\hat{c}_e V_1^*)_\sigma$ is homeomorphic to $\partial_e K$.

2) If V is a Lindenstrauss space (i.e. V^* isometric to an $L_1(\mu)$ -space), then the structure topology on $\hat{c}_e V_1^*$ introduced by Effros in [5] coincides with the Alfsen–Effros structure topology (see [3, p. 168]).

A subspace N of V is an L-summand if there is a subspace N' of V such that $N \cap N' = \{0\}$, $N + N' = V$, and for each $p \in N$, $q \in N'$

$$\|p + q\| = \|p\| + \|q\|$$

From the symmetry of the definition we see that N' is an L-summand. N' is unique and hence we call it the complementary L-summand of N .

A closed subspace J of V is an M-ideal if its annihilator J^\perp is an L-summand of V^* . In Corollary 5 we give a condition on the M-ideals of V that is sufficient to ensure that V is a preidual $L_1(\mu)$ -space. We then use an analog of this condition or rather its equivalent formulation on the w^* -closed L-summands

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of V^* , to define a separation axiom for topological spaces. We call it the splitting property and it follows that (Corollary 7) V^* is isometric to an $L_1(\mu)$ -space if $(\partial_e V_1^*)_\sigma$ has this property. The splitting property is stronger than T_1 and weaker than Hausdorff.

In section 2 we characterize the M-ideals of a G-space (Theorem 9). We show that a closed subspace J of V ($=\{f \in C(X) : f(x_\alpha) = \lambda_\alpha f(y_\alpha)\}$) is an M-ideal if and only if

$$J = \{f \in V : f(x) = 0 \text{ for all } x \in F\} \quad \text{for some closed set } F \subseteq X.$$

Theorem 10 is our main result. We there generalize results ([2, Theorem 6.2] and [5, Theorem 6.3]) on the facial topology of $\partial_e K$ and the structure topology on $(\partial_e V_1^*)_\sigma$ (V a predual $L_1(\mu)$ -space). We show that a Banach space V is a G-space if and only if $(\partial_e V_1^*)_\sigma$ is Hausdorff. We also show that in a G-space the intersection of any family of M-ideals is an M-ideal, and we raise the problem whether the G-spaces can be characterized in this way. Finally in Theorem 12 we show that $(\partial_e V_1^*)_\sigma$ is perfectly normal if V is a separable G-space.

In section 3 we give some examples of Banach spaces where $(\partial_e V_1^*)_\sigma$ has the splitting property, and examples of families of M-ideals such that their intersection is not an M-ideal.

Part of this paper is from the authors cand. real. thesis prepared in the period 1974-76 at the University of Oslo under direction of professor Erik Alfsen. Most of it is based on Åsvald Lima's paper *Intersection properties of balls and subspaces in Banach spaces* [11], and many of the results must be regarded as corollaries of his results. The author wants to thank Erik Alfsen, Åsvald Lima and in particular Gunnar Olsen for encouragements, discussions and valuable suggestions.

1. L-summands and $L_1(\mu)$ -spaces.

In [11, Theorem 5.8] has Lima given several characterizations of Lindenstrauss spaces. To make the geometrical content clearer we shall here reformulate one of them. First we need some lemmas about L-summands.

LEMMA 1. *Let L , M and N be L-summands in a Banach space V . Then:*

- (i) *If $N \cap L = \{0\}$ then $L \subseteq N'$*
- (ii) *If $N + L = V$ then $N' \subseteq L$*
- (iii) *If $N \cap L = \{0\}$ and $N + L = V$ then $L = N'$*
- (iv) *If $N \cap L = \{0\}$ and $N \cap M = \{0\}$ then $N \cap (L + M) = \{0\}$*
- (v) *If $N \subseteq L$ then there exists an L-summand M such that $N + M = L$ and $N \cap M = \{0\}$*

PROOF. The definition of an L-summand and simple verification.

LEMMA 2. Let N be an L-summand with $\dim N \geq 2$. Then N can be written as a direct sum of two L-summands both different from N if and only if there exists an L-summand L such that $L \subseteq N$ and $\{0\} \neq L \neq N$.

PROOF. If $N = L + M$ and $L, M \neq N$, then $L \subseteq N$ and $\{0\} \neq L \neq N$. Conversely if $L \subseteq N$ and $\{0\} \neq L \neq N$, then from Lemma 1(v) there is an M such that $N = L + M$, $L \cap M = \{0\}$ and $M \neq N$.

THEOREM 3. Let V be a real Banach space. Then the following statements are equivalent:

- (i) V^* is isometric to an $L_1(\mu)$ -space.
- (ii) $[0, 1] \partial_e V_1^* = \bigcap \{ (L \cup L') \cap V_1^* : L \text{ is an L-summand in } V^* \}$ and $\text{span}(p)$ is an L-summand for all $p \in \partial_e V_1^*$.
- (iii) If N is any L-summand in V^* of dimension ≥ 2 , then N can be written as a direct sum of two L-summands both different from N .

PROOF. The equivalence between (i) and (ii) is proved in [11, Theorem 5.8].

(ii) \Rightarrow (iii) Suppose (ii) is true, and suppose there is an L-summand N in V^* of dimension ≥ 2 that cannot be written as a direct sum of two smaller L-summands. Let L be any L-summand in V^* . We cannot have $N \cap L \neq \{0\}$ and $N \cap L \neq N$ according to Lemma 2, hence $N \subseteq L$ or $N \cap L = \{0\}$. Then from Lemma 1(i) $N \subseteq L$ or $N \subseteq L'$, that is $N \subseteq L \cup L'$. Thus

$$N \cap V_1^* \subseteq \bigcap \{ (L \cup L') \cap V_1^* : L \text{ is an L-summand in } V^* \},$$

and from (ii)

$$N \cap V_1^* \subseteq [0, 1] \partial_e V_1^* .$$

Let $p \in N$ and $\|p\| = 1$, then $p \in \partial_e V_1^*$. From (ii) $\text{span}(p)$ is an L-summand, and $\{0\} \neq \text{span}(p) \neq N$. But this is impossible by Lemma 2, and we have got a contradiction.

(iii) \Rightarrow (ii) Suppose (iii) is true. Since we always have

$$[0, 1] \partial_e V_1^* \subseteq \bigcap \{ (L \cup L') \cap V_1^* : L \text{ is an L-summand in } V^* \} \quad ([11, p. 34])$$

it is enough to prove the converse inclusion. Suppose then $p \notin [0, 1] \partial_e V_1^*$. We may assume $\|p\| \leq 1$. Let $N(p)$ be the intersection of all L-summands containing p . Then $N(p)$ is an L-summand ([3, Prop. I. 1.13]). $\dim N(p) \geq 1$ since $p \neq 0$. If $\dim N(p) = 1$, then $N(p) = \text{span}(p)$ and

$$N(p) \cap \partial_e V_1^* = \partial_e(N(p) \cap V_1^*) = \{ \pm \|p\|^{-1} p \},$$

hence $\|p\|^{-1}p \in \partial_e V_1^*$. But that is impossible since $p \notin [0, 1]\partial_e V_1^*$. Hence $\dim N(p) \geq 2$ and there are L-summands N and $M, N \cap M = \{0\}, N + M = N(p)$ and $N, M \neq N(p)$. Now $p \notin N$ and $p \notin M$ since $N(p)$ is the smallest L-summand containing p . Define $L = N + N(p)$, then L is an L-summand and $p \notin L$ since $p \in N(p)$ and $p \notin N$. It is also easy to prove (by using Lemma 1) that $L' = M$ and hence $p \notin L'$. Together $p \notin L \cup L'$ and we have

$$p \notin \bigcap \{ (L \cup L') \cap V_1^* : L \text{ is an L-summand in } V^* \}.$$

It remains to prove that $\text{span}(p)$ is an L-summand for all $p \in \partial_e V_1^*$. Let $N(p)$ as before be the smallest L-summand containing p . Suppose $\dim N(p) \geq 2$. Then there are L-summands $N, M, N + M = N(p)$ and $N, M \neq N(p)$. We always have

$$p \in N(p) \cap \partial_e V_1^* = (N + M) \cap \partial_e V_1^* = (N \cap \partial_e V_1^*) \cup (M \cap \partial_e V_1^*).$$

Hence $p \in N$ or $p \in M$. But that contradicts the choice of $N(p)$. Thus $\dim N(p) = 1, N(p) = \text{span}(p)$ and $\text{span}(p)$ is an L-summand. The proof is complete.

We are now turning to the w^* -closed L-summands in V^* .

DEFINITION. We say that we can split a w^* -closed L-summand N in V^* if there exist w^* -closed L-summands L and $M, L, M \neq N$ such that $N = L + M$.

REMARK. A simplex K is said to be prime (see [1, p. 164]) if for any two closed faces F_1, F_2 such that $K = \text{conv}(F_1 \cup F_2)$, necessarily $F_1 = K$ or $F_2 = K$. This definition can be extended. Let F, F_1 and F_2 be closed splitfaces of a compact convex set K . We say that F is prime if $F = \text{conv}(F_1 \cup F_2)$ implies $F_1 = F$ or $F_2 = F$. Let J be the M-ideal in $V = A(K)$ defined by (see [3, p. 100])

$$J = \{a \in A(K) : a(x) = 0 \text{ for all } x \in F\},$$

and let $N = J^\circ$. Then we can split the w^* -closed L-summand N if and only if F is not prime.

THEOREM 4. Let V be a real Banach space, then V^* is isometric to an $L_1(\mu)$ -space if we can split every w^* -closed L-summand in V^* of dimension ≥ 2 .

PROOF. Let N be any L-summand in V^* with $\dim N \geq 2$ and let N^* be the intersection of all w^* -closed L-summands that contains N (there does exist one since V^* is an L-summand). N^* is a w^* -closed L-summand, and $\dim N^* \geq 2$ since $N \subseteq N^*$. Hence there exist w^* -closed L-summands L and $M, L, M \neq N^*$ and $L + M = N^*$. Since N^* is the smallest w^* -closed L-summand containing N , we have $N \subseteq L$ and $N \subseteq M$. Now $N \cap L \neq \{0\}$ or $N \cap M \neq \{0\}$ because if $N \cap L = \{0\}$ and $N \cap M = \{0\}$ then from Lemma 1(iv)

$$\{0\} = N \cap (L + M) = N \cap N^* = N.$$

Without loss of generality we may assume $N \cap L \neq \{0\}$, hence $\{0\} \neq N \cap L \neq N$. By using Lemma 2 we verify statement (iii) of Theorem 3, and V^* is isometric to an $L_1(\mu)$ -space.

If J is an M -ideal in a Banach space V then the annihilator J° is a w^* -closed L -summand, and if N is a w^* -closed L -summand in V^* then there exists an M -ideal J in V such that $J^\circ = N$. Hence we can find a property for the M -ideals in V that is an analog to the split property for w^* -closed L -summands in V^* .

DEFINITION. We say that an M -ideal $J \subseteq V$ is reducible if there exist M -ideals J_1 and J_2 , $J \neq J_1, J_2$ such that $J = J_1 \cap J_2$. An M -ideal is irreducible if it is not reducible. (This definition is due to Alfsen.)

COROLLARY 5. A Banach space V is isometric to a predual $L_1(\mu)$ -space if every irreducible M -ideal $J \neq V$ is a hyperplane (i.e. $\text{codim } J = 1$).

PROOF. Use Theorem 4, the comments above and the fact that $J_1^\circ + J_2^\circ = (J_1 \cap J_2)^\circ$ for all M -ideals J_1 and J_2 in V ([11], Lemma 6.18).

DEFINITION. We say that a topological space has the splitting property if for every closed set F that contains more than one point, there exist closed sets F_1, F_2 and $F_1, F_2 \neq F$ such that $F_1 \cup F_2 = F$.

If Y has the splitting property then Y is T_1 , because $\overline{\{y\}}$ can be written as a union of two smaller closed sets if it contains more than one point, and that is impossible since $\overline{\{y\}}$ is the smallest closed set containing y . It is not difficult to prove that a T_1 -space where all convergent nets have at most finitely many limitpoints will enjoy this property, and hence every Hausdorff space has the splitting property. But such a space is generally not Hausdorff.

LEMMA 6. Let V be a real Banach space. Then $(\partial_e V_1^*)_\sigma$ has the splitting property if and only if we can split every w^* -closed L -summand in V^* of dimension ≥ 2 .

PROOF. A set in $\partial_e V_1^*$ is (structure) closed if and only if it is of the form $N \cap \partial_e V_1^*$ where N is a w^* -closed L -summand. Hence the splitting property on $(\partial_e V_1^*)_\sigma$ is equivalent to the property that to all w^* -closed L -summands N of dimension ≥ 2 there exist w^* -closed L -summands N_1 and N_2 such that

$$(1.1) \quad N \cap \partial_e V_1^* = (N_1 \cap \partial_e V_1^*) \cup (N_2 \cap \partial_e V_1^*)$$

and

$$(1.2) \quad N_i \cap \partial_e V_1^* \neq N \cap \partial_e V_1^* \quad i=1,2.$$

Suppose (1.1) and (1.2) are true. $N_1 + N_2$ is a w^* -closed L -summand (see [11, Lemma 6.18]) and from [3, II Prop. 1.15]

$$\begin{aligned} \partial_e(N \cap V_1^*) &= N \cap \partial_e V_1^* = (N_1 \cap \partial_e V_1^*) \cup (N_2 \cap \partial_e V_1^*) \\ &= (N_1 + N_2) \cap \partial_e V_1^* = \partial_e((N_1 + N_2) \cap V_1^*) \end{aligned}$$

and hence $N \cap V_1^* = (N_1 + N_2) \cap V_1^*$ (The Krein–Milman theorem). Now $N = N_1 + N_2$ since N and $N_1 + N_2$ are subspaces, and from (1.2) $N_1, N_2 \neq N$. If conversely $N = N_1 + N_2$ and $N_1, N_2 \neq N$, then it is trivial to prove (1.1) and (1.2).

COROLLARY 7. *Let V be a real Banach space. V^* is isometric to an $L_1(\mu)$ -space if the structure space $(\partial_e V_1^*)_\sigma$ has the splitting property.*

PROOF. Use Lemma 6 and Theorem 4.

2. G-spaces.

A real Banach space V is said to be a G-space if there exists a compact Hausdorff space X and a set

$$S = \{(x_\alpha, y_\alpha, \lambda_\alpha)\} \subseteq X \times X \times [-1, 1]$$

such that V is isometric to the space

$$A = \{f \in C(X) : f(x_\alpha) = \lambda_\alpha f(y_\alpha) \text{ for all } (x_\alpha, y_\alpha, \lambda_\alpha) \in S\}.$$

A G-space is isometric to a predual $L_1(\mu)$ -space. This was first proved by Lindenstrauss [13, Theorem 6.9]. Lima has given a new proof in [11, Theorem 7.10].

In [3] a subspace N of V^* is defined to be hereditary if $q \in N$ and $\|p\| + \|q - p\| = \|q\|$ implies $p \in N$.

LEMMA 8. *Let X be compact Hausdorff and $V \subseteq C(X)$ a Banach space and let $J \subseteq V$ be closed subspace such that J° is hereditary. Then there exists a closed set $F \subseteq X$ such that*

$$J = \{f \in V : f(x) = 0 \text{ for all } x \in F\}.$$

PROOF. Let $\varepsilon_x, x \in X$ be the point measure and define $F = \{x \in X : \varepsilon_x \in J^\circ\}$, F is closed since J° is w^* -closed. Now $f \in J$ is equivalent to $p(f) = 0$ for all $p \in J^\circ$. Let $f \in J$ and x any point in F , then $\varepsilon_x \in J^\circ$ and hence $0 = \varepsilon_x(f) = f(x)$, thus $f(x) = 0$ for all $x \in F$. Assume conversely that $f(x) = 0$ for all $x \in F$. Let $p \in \partial_e(J^\circ \cap V_1^*)$, then $p \in J^\circ \cap \partial_e V_1^*$ since J° is hereditary ([3, Prop. II 1.15])

and hence there exists $x \in F$ and $\lambda, |\lambda|=1$, such that $p = \lambda \varepsilon_x$, and so $p(f) = \lambda \varepsilon_x(f) = \lambda f(x) = 0$. From the Krein–Milman theorem every $q \in J^\circ$ is the w^* -limit of a net $\{p_\alpha\}$ where each p_α is linear combination of points from $\partial_c(J^\circ \cap V_1^*)$, thus $q(f) = 0$ and hence $f \in J$.

If F is a subset of X and $V \subseteq C(X)$, we define

$$J_F = \{f \in V : f(x) = 0 \text{ for all } x \in F\} .$$

Let \bar{F} be the closure of F , then $J_F = J_{\bar{F}}$ since all $f \in V$ are continuous on X .

THEOREM 9. *Let $V \subseteq C(X)$ be a G-space (i.e.*

$$V = \{f \in C(X) : f(x_\alpha) = \lambda_\alpha f(y_\alpha)\} .$$

A closed subspace J of V is an M-ideal if and only if $J = J_F$ for some closed set $F \subseteq X$.

PROOF. “only if” follows from Lemma 8 since J° is an L-summand and hence hereditary. Let F be any closed subset of X , and let $J = J_F$. It now suffices to prove that J is a semi M-ideal since all semi M-ideals in a predual $L_1(\mu)$ -space are M-ideals. (A consequence of [11, Theorem 5.5]). Choose any functions $f \in J, g \in V$ with $\|f\| \leq 1, \|g\| \leq 1$. If we now are able to prove

$$(2.1) \quad J \cap B(g+f, 1) \cap B(g-f, 1) \neq \emptyset$$

then we can use Theorem 6.15 of [11] (with $\varepsilon = 0$) to conclude that J is a semi M-ideal. Define h_1, h_2 and h by $h_1(x) = g(x) + f(x), h_2(x) = g(x) - f(x)$ and

$$h(x) = \max(h_1(x), h_2(x), 0) + \min(h_1(x), h_2(x), 0) - g(x), \quad x \in X .$$

Now $h + g \in V$ ([13, Lemma 6.7]) and hence $h \in V$. Let $x \in F$, then $f(x) = 0$ and

$$h(x) = \max(g(x), 0) + \min(g(x), 0) - g(x) = 0 ,$$

and so $h \in J$. Let $x \in X$, then

$$h_1(x) - h(x) = \begin{cases} g(x) & \text{if } h_1(x) \leq h_2(x) \leq 0 \text{ or } 0 \leq h_2(x) \leq h_1(x) \\ f(x) & \text{if } h_1(x) \leq 0 \leq h_2(x) \text{ or } h_2(x) \leq 0 \leq h_1(x) \\ h_1(x) + f(x) & \text{if } h_2(x) \leq h_1(x) \leq 0 \text{ or } 0 \leq h_1(x) \leq h_2(x) \end{cases}$$

In the third case we have $|h_1(x) + f(x)| \leq \max(|f(x)|, |g(x)|)$ and hence $|h_1(x) - h(x)| \leq 1$ for all x since $\|f\| \leq 1$ and $\|g\| \leq 1$. Thus

$$\|h_1 - h\| \leq 1 \quad \text{or} \quad \|g + f - h\| \leq 1$$

and $h \in B(g+f, 1)$. Similarly we prove $h \in B(g-f, 1)$ and so we have (2.1).

REMARK. Let $Y = \{x \in X : \varepsilon_x \in \widehat{\partial}_e V_1^*\}$, a set $F \subseteq Y$ is said to contain all its extreme points if $\bigcap \{f^{-1}(0) \cap Y : f \in J_F\} = F$. Such a set has to be relatively closed in Y . Now it follows from Theorem 9 that a closed $J \subseteq V$ is an M-summand if and only if $J = J_F$ for some relatively open-closed $F \subseteq Y$ and both F and $\mathcal{C}F$ contain all their extreme points. If $0 \notin \widehat{\partial}_e V_1^*$ (w^* -closure) and X is connected, then V does not contain any nontrivial M-summands. But if $0 \in \widehat{\partial}_e V_1^*$ or X is not connected, we can have, but we need not have, any nontrivial M-summands.

THEOREM 10. Let V be a real Banach space. The following statements are related in this way: (i) \Leftrightarrow (ii) and (i) \Rightarrow (iii)

- (i) V is a G-space
- (ii) $(\widehat{\partial}_e V_1^*)_\sigma$ is Hausdorff
- (iii) The intersection of any family of M-ideals is an M-ideal and $\ker(p)$ is an M-ideal for all $p \in \widehat{\partial}_e V_1^*$.

PROOF. (ii) \Rightarrow (i) If $(\widehat{\partial}_e V_1^*)_\sigma$ is Hausdorff, then it has the splitting property and hence by Corollary 7, V^* is isometric to an $L_1(\mu)$ -space. Since the structure topology on $(\widehat{\partial}_e V_1^*)_\sigma$ coincides with the biface topology we can use [5] Theorem 6.3 to conclude that V is a G-space.

(i) \Rightarrow (ii) If two Banach spaces are isometric, then the structurespaces are homeomorphic, hence it is sufficient to prove it for a G-space $V \subseteq C(X)$ (i.e. $V = \{f \in C(X) : f(x_2) = \lambda_2 f(y_2)\}$). Let $p_1, p_2 \in \widehat{\partial}_e V_1^*$ be linearly independent points, then there exist $x_1, x_2 \in X$ such that $p_i = \lambda_i \varepsilon_{x_i}$, $|\lambda_i| = 1$, $i = 1, 2$. Without loss of generality we may assume $\lambda_1 = \lambda_2 = 1$. Choose w^* -continuous linear functionals, i.e. $f, g \in V$ such that $f(p_1) = g(p_1) = f(p_2) = -g(p_2) = 1$. Define

$$F_1 = \{x \in X : f(x) \geq 0, g(x) \geq 0 \text{ or } f(x) \leq 0, g(x) \leq 0\}$$

and

$$F_2 = \{x \in X : f(x) \geq 0, g(x) \leq 0 \text{ or } f(x) \leq 0, g(x) \geq 0\},$$

and $N_i = J_{F_i}$, $i = 1, 2$. N_1 and N_2 are by Theorem 9 w^* -closed L-summands. Now

$$(N_1 \cap \widehat{\partial}_e V_1^*) \cup (N_2 \cap \widehat{\partial}_e V_1^*) = \widehat{\partial}_e V_1^*$$

since $F_1 \cup F_2 = X$, and $p_i \in N_i \cap \widehat{\partial}_e V_1^*$, $i = 1, 2$ since $x_1 \in F_1$ and $x_2 \in F_2$. Define

$$h(x) = \max(f(x), g(x), 0) + \min(f(x), g(x), 0) - f(x) - g(x) \quad \text{for all } x \in X.$$

Then $h \in J_{F_2}$ and $h(x_1) = -1$, hence $p_1 \notin N_2 \cap \widehat{\partial}_e V_1^*$. In a similar way we find $p_2 \notin N_1 \cap \widehat{\partial}_e V_1^*$. Thus $(\widehat{\partial}_e V_1^*)_\sigma$ is Hausdorff.

(i) \Rightarrow (iii) It suffices to prove it for G-spaces

$$V = \{f \in C(X) : f(x_2) = \lambda_2 f(y_2)\}.$$

Let $\{J_\gamma\}$ be any family of M-ideals in V , then there exists by Theorem 9 a family $\{F_\gamma\}$ of closed sets in X such that $J_\gamma = J_{F_\gamma}$ for each γ . Now

$$\bigcap_\gamma J_\gamma = \bigcap_\gamma J_{F_\gamma} = J_{\cup F_\gamma} = J_{\overline{\cup F_\gamma}}$$

and hence $\bigcap_\gamma J_\gamma$ is an M-ideal by Theorem 9.

Let $p \in \partial_e V_1^*$, then $p = \lambda \varepsilon_x$, $x \in X$ and

$$\ker(p) = \{f \in V : f(x) = 0\} = J_{\{x\}}.$$

Hence $\ker(p)$ is an M-ideal by Theorem 9.

REMARK 1. (i) \Rightarrow (ii) was proved by Effros [5] in the separable case, and later generally by Fakhoury [7] and Taylor [16]. The main idea in our proof is from Taylor.

REMARK 2. We do not know whether (iii) \Rightarrow (i) is true or not. This is a more general form of a problem raised by Effros [6, p. 115] and solved in the separable case by Gleit [9]. He proved that if V is a separable simplex space then V is an M-space if and only if the intersection of any family of M-ideals is an M-ideal. Statement (iii) can also be formulated in terms of L-summands in V^* , that is $\overline{\sum N_\gamma}$ (w^* -closure) is an L-summand for any family $\{N_\gamma\}$ of w^* -closed L-summands (this is similar to Størmer's axiom for compact convex sets, see [1, p. 146]) and $\text{span}(p)$ is an L-summand for all $p \in \partial_e V_1^*$. Lima has proved such a result for compact convex sets, [12, Theorem 20].

REMARK 3. Roy proves in [15, Lemma 4] that for a G-space V the family $\{U_f : f \in V\}$ where

$$U_f = \{p \in \partial_e V_1^* : f(p) \neq 0\},$$

is a basis for the structure topology on $\partial_e V_1^*$. Now it is not difficult to prove that a Banach space V satisfies statement (iii) if and only if the family $\{U_f : f \in V\}$ is a basis for the structure topology on $\partial_e V_1^*$.

The following corollary was first proved by Alfsen and Andersen (see [1, Theorem II, 7.8] or [2, Theorem 6.2]).

COROLLARY 11. *Let K be a compact convex set in a locally convex Hausdorff space. Then the facial topology of $\partial_e K$ is Hausdorff if and only if K is a Bauer simplex.*

PROOF. Let $V = A(K)$, then $\partial_e K$ with facial topology is homeomorphic to

$(\partial_e V_1^*)_\sigma$ with structure topology. Now $A(K)$ is a G -space if and only if K is a Bauer simplex, and the corollary follows from Theorem 10.

Effros proved as mentioned above, that $(\partial_e V_1^*)_\sigma$ is Hausdorff if V is a separable G -space. Roy has pointed out ([15, p. 145]) that a slight change in his proof gives that $(\partial_e V_1^*)_\sigma$ is in fact a normal space. We will show that $(\partial_e V_1^*)_\sigma$ is perfectly normal, and our proof is almost a copy of a part of the proof Gleit made for [8, Prop. 1.6].

THEOREM 12. *$(\partial_e V_1^*)_\sigma$ is perfectly normal if V is a separable G -space.*

PROOF. From the above remarks it suffices to show that each closed set is a G_δ . Let N be any w^* -closed L -summand in V^* . The w^* -topology on V_1^* is metrizable, and Gleit constructs the following metric that generates the w^* -topology.

$$d(p, q) = \sum 2^{-n} |a_n(p) - a_n(q)| \quad p, q \in V_1^*$$

where $\{a_n\}$ is dense in the unit ball of $A(V_1^*)$. Then he defines

$$f(p) = d(p, N \cap V_1^*) \quad p \in V_1^*$$

and shows that this f is a continuous and convex function and $f(0) = 0$. Define

$$C_n = \left\{ p \in \partial_e V_1^* : f(p) \geq \frac{1}{n} \right\} \quad n = 1, 2, \dots$$

then each C_n is structurally compact in $\partial_e V_1^*$ (see [5, Prop. 4.8]). Now $d(-p, q) = d(p, -q)$ and hence $f(p) = f(-p)$ and thus each C_n is symmetric. Since $(\partial_e V_1^*)_\sigma$ is Hausdorff each C_n is structurally closed, and

$$U_n = \left\{ p \in \partial_e V_1^* : f(p) < \frac{1}{n} \right\}$$

structurally open. Now $N \cap \partial_e V_1^* = \bigcap_n U_n$ and hence $N \cap \partial_e V_1^*$ is a G_δ .

3. The splitting property.

Let X be compact Hausdorff, $x_0 \in X$ and $\mu, \|\mu\| \leq 1$ a regular Borel measure on X with $\mu(\{x_0\}) = 0$. Define $V \subseteq C(X)$ by

$$V = \{ f \in C(X) : f(x_0) = \mu(f) \} .$$

Then V is a Banach space, and it is possible by using [11, Theorem 6.17] and our Lemma 8 to prove the following Proposition:

PROPOSITION 13. Let V be as above and $J \subseteq V$ a closed subspace. Then J is an M-ideal if and only if $J = J_F$ for some closed $F \subseteq X$ where $x_0 \notin F$, or $x_0 \in F$ and $\text{supp } \mu \setminus F$ contains at most one point.

COROLLARY 14. $(\partial_e V_1^*)_\sigma$ is homeomorphic to the space $Y = X \setminus \{x_0\}$ where all the sets $F \cap Y$, F closed in X , and $x_0 \notin F$ or $(\{x_0\} \cup \text{supp } \mu) \subseteq F$ form the closed sets of the topology on Y .

PROOF. If $\text{supp } \mu \setminus F$ contains just one point x , then $F' = F \cup \{x\}$ is closed and $J_F = J_{F'}$.

COROLLARY 15. Assume $\text{supp } \mu$ contains more than one point. Then $(\partial_e V_1^*)_\sigma$ has the splitting property if and only if $x_0 \notin \text{supp } \mu$, and $(\partial_e V_1^*)_\sigma$ is never Hausdorff.

PROOF. If $x_0 \notin \text{supp } \mu$ then it is simple verification to show that Y (defined in Corollary 14) has the splitting property. If $x_0 \in \text{supp } \mu$ then it is impossible to split the closed set $\text{supp } \mu$. (We all the time assume $\mu(\{x_0\}) = 0$). Y is never Hausdorff since it is impossible to separate the points of $\text{supp } \mu$.

From Corollary 15 and Corollary 7 we have that

$$V = \{f \in C(X) : f(x_0) = \mu(f)\}$$

is isometric to a predual $L_1(\mu)$ -space if $x_0 \notin \text{supp } \mu$. This is also true when $x_0 \in \text{supp } \mu$. A more general result was proved by Gleit [10] and Bednar and Lacey [4]. They proved if for each $i = 1, 2, \dots, n$, μ_i is a regular Borel measure on X , $\|\mu_i\| \leq 1$, $x_i \in X$ and $|\mu_i|(\{x_1, x_2, \dots, x_n\}) = 0$, then

$$V = \{f \in C(X) : f(x_i) = \mu_i(f), i = 1, \dots, n\}$$

is isometric to a predual $L_1(\mu)$ -space and a simplex space if all μ_i are positive. Lima has pointed out that the condition $|\mu_i|(\{x_1, \dots, x_n\}) = 0, i = 1, \dots, n$, is not necessary for the conclusion that V is isometric to a predual $L_1(\mu)$ -space.

PROPOSITION 13 can now in a natural way be extended to finitely many measures μ_i .

EXAMPLE 1. Let m be the Lebesgue measure on $[0, 1]$ and

$$V = \{f \in C([0, 1]) : f(\frac{1}{2}) = m(f)\},$$

$(\partial_e V_1^*)_\sigma$ does not have the splitting property since $\frac{1}{2} \in [0, 1] = \text{supp } m$. Perdrizet [14] used this space as an example of a simplex space with a family of M-ideals such that the intersection is not an M-ideal. Let $F_n = \{\frac{1}{2} + \frac{1}{n}\}, n = 2, 3, \dots$,

then by Proposition 13, J_{F_n} is an M-ideal for each n , but $\bigcap J_{F_n} = J_F$ where $F = (\bigcup F_n) \cup \{\frac{1}{2}\}$, is not an M-ideal since $\frac{1}{2} \in F$ and $[0, 1] \setminus F = \text{supp } m \setminus F$ contains more than one point.

EXAMPLE 2. Let $\mu = \frac{1}{2}\varepsilon_0 + \frac{1}{2}\varepsilon_1$, and

$$V = \{f \in C([0, 1]) : f(\frac{1}{2}) = \mu(f)\},$$

$(\partial_e V_1^*)_\sigma$ has the splitting property since $\frac{1}{2} \notin \{0, 1\} = \text{supp } \mu$. Let F_n , $n=3, 4, \dots$ be as above, then by Proposition 13, J_{F_n} is an M-ideal for each n , but $\bigcap J_{F_n}$ is not an M-ideal by the same reason as above.

EXAMPLE 3. Alfsen gives in [1, Proposition II 7.17] an example of a prime simplex. If μ is positive, $\|\mu\| = 1$ and $\mu(\{x_0\}) = 0$ then

$$V = \{f \in C(X) : f(x_0) = \mu(f)\}$$

is a simplex space with unit, and hence

$$K = \{p \in V^* : p \text{ positive and } \|p\| = 1\}$$

is a simplex. Assume $\text{supp } \mu = X$. From Corollary 15 and an earlier remark we have that K is prime. Alfsen used in his example $X = N \cup \{\infty\}$ (the one point compactification of the natural numbers), $\mu = \sum 2^{-n}\varepsilon_n$ and $x_0 = \infty$. Now $\partial_e K$ with the facial topology is homeomorphic to $(\partial_e V_1^*)_\sigma$ and hence by Corollary 14 homeomorphic to N with closed sets N, \emptyset and the finite ones (because a closed infinite subset of X must contain ∞).

REFERENCES

1. E. Alfsen, *Compact convex sets and boundary integrals* (Ergebnisse Math. Grenzgebiete 57), Springer-Verlag, Berlin - Heidelberg - New York, 1971.
2. E. Alfsen and T. B. Andersen, *Split faces of compact convex sets*, Proc. London Math. Soc. (3) 21 (1970), 415-442.
3. E. Alfsen and E. Effros, *Structure in real Banach spaces*, part I & II, Ann. of Math. 96 (1972), 98-173.
4. J. B. Bednar and H. E. Lacey, *Concerning Banach spaces whose duals are abstract L-spaces*, Pacific J. Math. 41 (1972), 13-24.
5. E. Effros, *On a class of real Banach spaces*, Israel J. Math. 9 (1971), 430-458.
6. E. Effros, *Structure in simplexes*, Acta Math. 117 (1967), 103-121.
7. H. Fakhoury, *Préduaux de L-espace, Notion de Centre*, J. Functional Analysis 9 (1972), 189-207.
8. A. Gleit, *Topologies on the extreme points of compact convex sets*, part I, Math. Scand. 31 (1972), 209-219.
9. A. Gleit, *A characterization of M-spaces in the class of separable simplex spaces*, Trans. Amer. Math. Soc. 169 (1972), 25-33.
10. A. Gleit, *On the existence of simplex spaces*, Israel J. Math. 9 (1971), 199-209.

11. Á. Lima, *Intersection properties of balls and subspaces in Banach spaces*, Trans. Amer. Math. Soc. 227 (1977), 1–62.
12. Á. Lima, *On continuous convex functions and split faces*, Proc. London Math. Soc. 25 (1972), 27–40.
13. J. Lindenstrauss, *Extensions of compact operators*, Memoirs Amer. Math. Soc. 48 (1964).
14. F. Perdrizet, *Espaces de Banach ordonné et idéaux*, J. de Math. 49 (1970), 61–98.
15. N. Roy, *A characterization of square Banach spaces*, Israel. J. Math. 17 (1974), 142–148.
16. P. Taylor, *A characterization of G-spaces*, Israel. J. Math. 10 (1971), 131–134.

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