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ON THE BURNSIDE RING AND STABLE COHOMOTOPY OF A FINITE GROUP

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0. Introduction.

In this paper we study the connection between permutation representations of finite groups G and stable cohomotopy of the classifying space BG, analogous to the connection between the character ring of G and the K-theory of BG.

Suppose S is a finite G-set. Each ordering of S gives a homomorphism ϱ: $G \to \Sigma_{|S|}$, where Σ_n denotes the permutation group in n letters and $|S|$ is the cardinality of S. Different orderings give conjugate maps, as do isomorphic G-sets. Hence the homotopy class of $B\varrho$: $BG \to B\Sigma_{|S|}$ only depends on the isomorphism class of S.

The disjoint union $\coprod_{n \geqq 0} B\Sigma_n$ is a monoid and its group completion $\Omega B(\coprod_{n \geqq 0} B\Sigma_n)$ is homotopy equivalent (as an H-space) to the space $QS^0 = \lim_{n \to \infty} \Omega^n S^n$ of stable self maps of spheres. Let $i: \coprod_{n \geqq 0} B\Sigma_n \to QS^0$ be the resulting H-map and form the composition

$$\alpha_G(S)\colon BG \to B\Sigma_{|S|} \to \coprod_{n \geqq 0} B\Sigma_n \xrightarrow{i} QS^0 .$$

Disjoint union and Cartesian product turn the equivalence classes of G-sets into a semiring, whose associated ring is the Burnside ring $A(G)$ ([17], [7]), and the correspondence $S \to \alpha_G(S)$ defines an additive map

$$\alpha_G\colon A(G) \to [BG, QS^0] .$$

$[BG, QS^0]$ is by definition the stable cohomotopy $\pi_S^0(BG)$.

The space QS^0 admits besides the loop addition the smash product, homotopic to the product given by composition of maps. If $\coprod_{n \geqq 0} B\Sigma_n$ is equipped with the monoid structure induced from the homomorphisms $\Sigma_n \times \Sigma_m \to \Sigma_{nm}$, then i respects both structures and α_G is a ring homomorphism.

The space QS^0 splits into a disjoint union of homotopy equivalent spaces $Q_n S^0$, $n \in \mathbf{Z}$, where $Q_n S^0$ denotes the subspace of degree n maps. Thus we have

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an augmentation $[BG, QS^0] \to \mathbf{Z}$. Taking cardinality of G-sets defines an augmentation of $A(G)$, and α_G is clearly augmentation preserving. The $(n+1)$-fold products become trivial on the n-skeleton B_nG, so α_G factors as

$$\alpha_G\colon A(G)/I^{n+1}(G) \to [B_nG, QS^0]\ .$$

Passing to the limit we get a map from the $I(G)$-adic completion

$$\hat{\alpha}_G\colon \hat{A}(G) \to \lim_{\leftarrow} [B_nG, QS^0] = [BG, QS^0]$$

(the last isomorphism follows from the finiteness of $[B_nG, Q_0S^0]$).

The map $\hat{\alpha}_G$ is analogous to the isomorphism from the completed representation ring $\hat{R}(G)$ to $K(BG)$ [1], and some time ago G. Segal made the

CONJECTURE. $\hat{\alpha}_G\colon \hat{A}(G) \to [BG, QS^0]$ is an isomorphism.

The full conjecture seems very hard and is probably out of reach at the moment. In this paper we study the injectivity of $\hat{\alpha}_G$.

First we reduce the problem to p-groups by showing

THEOREM A. *If $\hat{\alpha}_{G_P}$ is injective for the Sylow subgroups G_P of G, then $\hat{\alpha}_G$ is injective.*

This is proved by showing that $\hat{A}(G)$ embeds into $\oplus_p \hat{A}(G_p)$ via the restriction maps, and compatibly with $BG_p \to BG$.

For cyclic groups the natural map $\hat{A}(G) \to \hat{R}(G)$ is injective, and using Atiyah's result, $\hat{R}(G) \cong K^*(BG)$, we deduce

THEOREM B. *$\hat{\alpha}_G$ is injective for cyclic groups G.*

One cannot hope to detect the maps $\alpha_G(x)\colon BG \to Q_0S^0$ for $x \in \operatorname{Ker}(\hat{A}(G) \to \hat{R}(G))$ by K-theory, since the space Q_0S^0 splits as $J \times \operatorname{cok} J$ with $\operatorname{cok} J$ a K-theory point, and at least for groups G of odd order $[BG, J]$ embeds into $[BG, BU \times \mathbf{Z}] = \hat{R}(G)$. By studying induced maps in homology we prove

THEOREM C. *$\hat{\alpha}_G$ is injective for elementary abelian groups $G = (\mathbf{Z}/p)^n$.*

For Theorem C we need an induction machine, which tells that if α_H is injective for all genuine subgroups H of a p-group G and furthermore α_G is injective on a specific summand $\mathbf{Z}x \subset A(G)$, then $\hat{\alpha}_G$ is injective. It is the maps $\alpha_G(nx)$ that induce nontrivial maps in $\mathbf{Z}/p$-homology. We show even more: one gets a host of homologically distinct elements $B((\mathbf{Z}/p)^n) \to \operatorname{cok} J_p$ for $n \geqq 2$.

THEOREM D. *If* $G=(Z/p)^n$, *and* $\hat{A}_0(G)=\operatorname{Ker}(\hat{A}(G)\to\hat{R}(G))$, *then* $\hat{\alpha}_G$ *maps* $\hat{A}_0(G)$ ($2\hat{A}_0(G)$ *if* $p=2$) *injectively into* $[BG,\operatorname{cok} J_p]$.

We note that Theorems A and B combine to show that $\hat{\alpha}_G$ is injective for the groups with cyclic Sylow subgroups. They are all metacyclic. Theorem C enlarges the class of groups for which $\hat{\alpha}_G$ is injective to include e.g. A_5, the alternating group on 5 letters.

The smallest groups we cannot settle with our method are $Z/4\times Z/2$, the dihedral D8 and the quaternionic Q8 of order 8. For these groups the maps $\alpha_G(nx)$ induce zero both in homology and K-theory. One wonders if connective K-theory or unitary bordism theory could settle these cases.

The representation ring $R(G)$ and K-theory $K(X)$ admit the structure of a λ-ring. Atiyah, Tall and Segal [3], [4] have explored the algebraic nature of such rings showing that one gets exponential isomorphisms ϱ_k: $\hat{I}(G)\xrightarrow{\cong}1+\hat{I}(G)$ for any p-group G, and $KSO(X)\hat{}\xrightarrow{\cong}(1+KSO(X))\hat{}$ for any finite complex X, where $\hat{}$ denotes the p-adic completion.

Now $A(G)$ has also λ-operations, yielding λ-operations on stable cohomotopy $\pi_S^0(X)=[X,QS^0]$, see [19]. Unfortunately the λ-ring $A(G)$ is not "special", and this breaks down the algebraic program above. However, it is interesting to identify the maps ϱ_k: $Q_0S^0\to SG_p$. We give a character argument to show

THEOREM E. ϱ_k: $Q_0S^0\to SG_p$ *is the composition* $Q_0S^0\xrightarrow{e}J_p\xrightarrow{\alpha_p}SG_p$.

The paper is divided into 4 sections. The first contains generalities on the Burnside ring: its characters, functorial properties and topology. The main theorem is 1.15 which shows that $\hat{A}(G)$ is detected by p-groups.

In section 2 we study the λ-ring structure of $A(G)$ and note that the natural map $A(G)\to R(G)$ is a λ-homomorphism. We describe the characters of the associated operations λ^n, ψ^n and ϱ_k, we show they induce operations in the (zero degree) stable cohomotopy π_S^0 (2.12) and prove Theorem E (2.20). The characters of λ^n and ψ^n were obtained independently by C. Siebeneicher [20].

The third section is devoted to the study of $\hat{\alpha}_G$ and contains the proofs of theorems A and B (3.2 and 3.3). We set up the induction machinery needed to prove theorems C and D in section 4 (4.15, 4.22 and 4.23).

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1. The Burnside ring.

Let G be a finite group. A G-action on a finite set S is a homomorphism from

G to Σ_S, the permutation group of S. A G-set is a finite set with a G-action. Two G-sets S and T are isomorphic if there exists a G-equivariant bijection $f\colon S \to T$, or in other words if the diagram

$$\begin{array}{ccc} & & \Sigma_S \\ & \nearrow & \\ G & & \downarrow \Sigma_f \\ & \searrow & \\ & & \Sigma_T \end{array}$$

commutes. The equivalence classes of G-sets form a commutative semiring $A^+(G)$ under disjoint union and cartesian product. The associated ring $A(G)$ is called the Burnside ring of G.

The additive structure of $A(G)$ is easily described. Breaking G-sets into G-orbits one sees that $A(G)$ is a free abelian group with basis consisting of cosets G/H, one for each conjugacy class of subgroups H of G. We fix a set $C(G)$ of representatives.

For the multiplicative structure we introduce characters, following W. Burnside (who called them marks [5 p. 236]). Let H be a subgroup of G and S be a G-set. Then we set

$$\chi_H(S) = |S^H|, \tag{1.1}$$

the number of elements in S fixed by H. The character χ_H extends to give a ring homomorphism

$$\chi_H\colon A(G) \to \mathbf{Z}.$$

On the additive generators of $A(G)$ we have

$$\chi_{H_1}(G/H_2) = |\{gH_2 \mid H_1 g H_2 = gH_2\}| = |\{gH_2 \mid g^{-1}H_1 g \subset H_2\}|$$

or

$$\chi_{H_1}(G/H_2) = |\{gH_2 \mid H_1^g \subset H_2\}|. \tag{1.2}$$

This shows that χ_H depends only on the conjugacy class of H.

The homomorphisms χ_H define together a homomorphism

$$\chi\colon A(G) \to \bigoplus_{C(G)} \mathbf{Z}.$$

The first statement of the following theorem is due to W. Burnside [5] and the second one to A. Dress (unpublished, cf. [7]).

THEOREM 1.3. *The homomorphism* $\chi\colon A(G) \to \oplus_{C(G)} \mathbf{Z}$ *is an embedding with* *finite cokernel. Its image consists of families* $(d_H)_{H\in C(G)}$ *satisfying the congruences*

$$d_H \equiv - \sum_{\substack{H\lhd K\leqq G\\ K/H \text{ cyclic} \neq 1}} \varphi(|K/H|)d_K \qquad (\operatorname{mod} |N_G(H)/H|)$$

where we set $d_K=d_{K'}$ *for* $K\sim K'\in C(G)$ *and* φ *is the Euler function.*

PROOF. To prove the injectivity, it is enough to show that non-isomorphic G-sets S and T cannot have the same characters. Write $S=\sum m_H G/H$, $T=\sum n_H G/H$ with H running over $C(G)$ and let $H_0\in C(G)$ be maximal with respect to $m_{H_0}\neq n_{H_0}$. Note from (1.2) that $\chi_{H_1}(G/H_2)$ is non-zero if and only if H_1 is conjugate to a subgroup of H_2, denoted by $H_1\lesssim H_2$. Then

$$\chi_{H_0}(S) = m_{H_0}\chi_{H_0}(G/H_0)+\sum_{H_0\lesssim H,\, H\neq H_0} m_H\chi_{H_0}(G/H)$$

and

$$\chi_{H_0}(T) = n_{H_0}\chi_{H_0}(G/H_0)+\sum_{H_0\lesssim H,\, H\neq H_0} n_H\chi_{H_0}(G/H)$$

are different as the sum terms coincide but $m_{H_0}\neq n_{H_0}$.

Let S be a G-set. We want to show that the numbers $\chi_H(S)$ satisfy the congruences for each subgroup H. If $H<K$ then the K-fixed points of S are contained in the $N_G(H)/H$-set S^H so that we are reduced to the case $H=e$..By a theorem of W. Burnside [5 p. 191] $|G|^{-1}\sum_{g\in G}|S^g|$ is an integer, namely the number of G-orbits in S. This implies the congruence

$$|S| = \chi_e(S) \equiv -\sum_{g\neq 1}\chi_{\langle g\rangle}(S) = -\sum_{K\leqq G \text{ cyclic}\neq 1}\varphi(|K|)\chi_K(S) \ (\operatorname{mod} |G|)$$

where $\varphi(|K|)$ is the number of generators in K.

Conversely, we construct $x\in A(G)$ with given characters (d_H) by adding increasing orbits. Start with d_G points with trivial G-action. Choose a total order on $C(G)$ extending $\lesssim$. Assume we have $y\in A(G)$ such that $\chi_H(y)=d_H$ for H greater than H_0. Since the characters of y and the numbers (d_H) both fulfil the congruences, we have

$$\chi_{H_0}(y) \equiv d_{H_0} \ (\operatorname{mod} |N_G(H_0)/H_0|)\,,$$

say $d_{H_0}=\chi_{H_0}(y)+n|N_G(H_0)/H_0|$. We add n copies of G/H_0 to y. This does not change the earlier adjusted characters, but

$$\chi_{H_0}(y+nG/H_0) = \chi_{H_0}(y)+n|N_G(H_0)/H_0| = d_{H_0}$$

by (1.2). This completes the induction.

Finally χ has finite cokernel since $\oplus |G| \mathbf{Z} \subset \operatorname{Im} \chi$ by the congruences. The theorem is proved.

REMARK. In [17] G. Segal defined a ring ω_G^0 in terms of equivariant stable homotopy. It coincides with $A(G)$ as both are characterized by theorem 1.3. For a proof and generalization to compact Lie groups, see [16, Theorem 3].

If $f\colon H \to G$ is a homomorphism of finite groups, then the pull-back f^*S of a G-set S has the same underlying set with H-action

$$H \xrightarrow{f} G \to \Sigma_S .$$

The induced maps $f^*\colon A(G) \to A(H)$ make A into a contravariant functor. The characters of f^*x are

$$\chi_U(f^*x) = \chi_{f(U)}(x), \qquad U \leqq H,\ x \in A(G)\,. \tag{1.4}$$

In the special case of a subgroup $i\colon H \to G$ we call i^* the restriction homomorphism, and denote it by Res_H^G.

There is also a covariant induction homomorphism Ind_H^G or f_* for inclusions of subgroups $f\colon H \to G$. On the coset basis it is given by

$$\operatorname{Ind}_H^G(H/H_1) = G/H_1, \qquad H_1 \leqq H \leqq G$$

It is easily checked from (1.2) that the characters of f_*y are

$$\chi_U(f_*y) = \sum_{U^g \leqq H} \chi_{U^g}(y), \qquad U \leqq G,\ y \in A(H) \tag{1.5}$$

where g runs through representatives of G/H.

The homomorphisms Res and Ind are related in the same fashion as the restriction and induction maps in representation theory or cohomology of finite groups. If H is a subgroup of G, then the Frobenius reciprocity

$$\operatorname{Ind}_H^G(y \cdot \operatorname{Res}_H^G(x)) = \operatorname{Ind}_H^G(y) \cdot x, \qquad y \in A(H),\ x \in A(G) \tag{1.6}$$

holds. Further, if K is another subgroup of G, let $Kg_1H, \ldots, Kg_rH \subset G$ be the double cosets of $G \bmod (K, H)$. Then we have

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G(x) = \sum_{i=1}^{r} \operatorname{Ind}_{K \cap g_iHg_i^{-1}}^K(c_{g_i}^* \operatorname{Res}_{g_i^{-1}Kg_i \cap H}^H(x)), \qquad x \in A(H) \tag{1.7}$$

where c_{g_i} is conjugation by g_i. The proofs of (1.6) and (1.7) are analogous to the corresponding formulas of representation theory.

Each G-set S can be considered as a linear representation of G over a field k by extending the G-action on the canonical basis of k^S by linearity. As the trace of a permutation matrix is equal to the number of 1's along the diagonal, the linear character of k^S can be read off from the characters χ_H of (1.1) for H cyclic:

$$\chi_{k^S}(g) = \chi_{\langle g\rangle}(S) .$$

If k has characteristic 0, then the elements in $R_k(G)$ are detected by their linear characters. We conclude from theorem 1.3

LEMMA 1.8. *Let k be a field of characteristic 0. Then the kernel of the natural map $A(G) \to R_k(G)$ coincides with the kernel of the restriction map*

$$\text{Res}\colon A(G) \to \bigoplus_{C \leqq G} A(C)$$

to the cyclic subgroups of G.

In Segal's conjecture the Burnside ring $A(G)$ is compared with the stable cohomotopy ring $\pi_S^0(BG)$. As the latter is complete (see section 3), we study here the algebraic process of completing $A(G)$.

The character χ_e just counts the number of points in a G-set and defines an augmentation $\varepsilon\colon A(G) \to \mathbf{Z}$. This is a split surjection so $A(G) = \mathbf{Z} \oplus I(G)$ where $I(G)$ is the augmentation ideal $\varepsilon^{-1}(0)$.

We give the ring $A(G)$ the usual $I(G)$-adic topology, letting the powers $I(G)^n$ be a neighbourhood basis of 0. The completion of $A(G)$ is defined as the inverse limit

$$\hat{A}(G) = \lim_{\substack{\leftarrow \\ n}} A(G)/I(G)^n .$$

We shall study the kernel of $A(G) \to \hat{A}(G)$. First we recall a result from commutative algebra (see e.g. [22 p. 262, Corollary to Theorem 8]). Let A be a Noetherian ring with no nilpotents and $\mathfrak{m} \subset A$ a prime ideal. Then the kernel of the natural map from A to the $\mathfrak{m}$-adic completion $\hat{A} = \lim\limits_{\leftarrow} A/\mathfrak{m}^n$ is

$$\bigcap_{n=0}^{\infty} \mathfrak{m}^n = \bigcap_{\mathfrak{p}_j + \mathfrak{m} \neq A} \mathfrak{p}_j$$

where $\mathfrak{p}_j$ runs over such minimal prime ideals of A that $\mathfrak{p}_j + \mathfrak{m} \neq A$.

The ring $A(G)$ is Noetherian as a finitely generated abelian group. A. Dress determined in [7] the prime ideal structure of $A(G)$. There are two types of prime ideals in $A(G)$: the minimal ones

$$\mathfrak{p}_{U,0} = \{x \in A(G) \mid \chi_U(x) = 0\}$$

for $U \leqq G$, and the maximal ones

$$\mathfrak{p}_{U,p} = \{x \in A(G) \mid \chi_U(x) \equiv 0 \pmod{p}\}$$

for $U \leqq G$ and p a prime. Furthermore,

$$
(1.9)\qquad \begin{aligned} \mathfrak{p}_{U,0} &= \mathfrak{p}_{V,0} && \text{if and only if } U \sim V, \\ \mathfrak{p}_{U,p} &= \mathfrak{p}_{V,q} && \text{if and only if } p=q \text{ and } U^p \sim V^p \end{aligned}
$$

where U^p is the smallest normal subgroup of U with U/U^p a p-group, and $\mathfrak{p}_{U,0} \subset \mathfrak{p}_{U,p}$ together with (1.9) accounts for all inclusions between prime ideals in $A(G)$.

PROPOSITION 1.10. *The kernel of $A(G) \to \hat{A}(G)$ coincides with the kernel of the restriction map*

$$
\mathrm{Res}\colon A(G) \to \bigoplus_{G_p \leqq G} A(G_p)
$$

to the Sylow subgroups G_p of G.

PROOF. It follows from the above that the kernel is

$$
\bigcap_{n=0}^{\infty} I(G)^n = \bigcap_{\mathfrak{p}_{U,0}+I(G) \neq A(G)} \mathfrak{p}_{U,0} \,.
$$

Now $I(G) = \mathfrak{p}_{e,0}$ and if the ideal $\mathfrak{p}_{U,0} + \mathfrak{p}_{e,0}$ is proper then it is contained in a maximal ideal $\mathfrak{p}_{V,p}$. By (1.9) this implies that $\mathfrak{p}_{U,p} = \mathfrak{p}_{e,p} = \mathfrak{p}_{V,p}$, hence $U^p = e = V^p$ and U is a p-group. Conversely, if U is a p-group then $\mathfrak{p}_{U,0} + \mathfrak{p}_{e,0} \subset \mathfrak{p}_{U,p}$. Thus

$$
\mathrm{Ker}\,(A(G) \to \hat{A}(G)) = \bigcap_{U \leqq G \ p\text{-group}} \mathrm{Ker}\,\chi_U \,,
$$

and the claim follows from theorem 1.3.

COROLLARY 1.11. *If G is a p-group then $A(G) \to \hat{A}(G)$ is a monomorphism.*

In the case of a p-group G the $I(G)$-adic completion is the familiar p-adic one:

$$
\hat{A}(G) = \mathbf{Z} \oplus (\hat{\mathbf{Z}}_p \otimes_{\mathbf{Z}} I(G))
$$

where $\hat{\mathbf{Z}}_p = \varprojlim_n \mathbf{Z}/p^n$ denotes the p-adic integers:

PROPOSITION 1.12. *If G is a p-group then the $I(G)$-adic topology of $A(G)$ is the same as its p-adic topology.*

PROOF. We have to prove that for each m there are integers n_1, n_2 such that

$$
(1)\qquad p^{n_1} I(G) \subset I(G)^m
$$

$$
(2)\qquad I(G)^{n_2} \subset p^m I(G) \,.
$$

The first relation follows from Atiyah's

LEMMA 1.13. *For any group G,* $|G|I(G)^n \subset I(G)^{n+1}$.

(This is a consequence of the reciprocity formula 1.6, see [1p. 269, Proposition 6.13]).

To get the inclusion (2), we note that for any $H \leqq G$ and $U \leqq G$

$$\chi_U(G/H - |G/H|) \equiv 0 \pmod{p}$$

since the complement of $(G/H)^U$ consists of non-trivial U-orbits, hence $\chi_U(I(G)) \subset p\mathbf{Z}$. As $\chi = \oplus_{U \leqq G} \chi_U$ is a ring homomorphism we have $\chi(I(G)^n) \subset \oplus_{e \neq U \leqq G} p^n \mathbf{Z}$ and it is enough to prove

$$\bigoplus_{e \neq U \leqq G} |G|\mathbf{Z} \subset \chi(I(G)) .$$

This follows immediately from the congruences 1.3. The proof of 1.12 is complete.

The completion $\hat{A}(G_p)$ is now described for p-groups G_p. Next we shall show that if G is an arbitrary finite group, then $\hat{A}(G)$ embeds into the sum $\oplus \hat{A}(G_p)$, taken over the Sylow subgroups G_p of G. This is done by completing the map of proposition 1.10.

Let A be a Noetherian ring and $\mathfrak{m} \subset A$ an ideal. The $\mathfrak{m}$-adic completion of a finitely generated A-module M is defined to be $\hat{M} = \lim_{\leftarrow n} M/\mathfrak{m}^n M$. It is a basic fact that Noetherian completion is an exact functor [1 p. 258, Proposition 3.16].

If $H \leqq G$, then $A(H)$ is an $A(G)$-module via the restriction homomorphism $\varrho = \mathrm{Res}_H^G \colon A(G) \to A(H)$. In the following proof we distinguish the prime ideals $\mathfrak{p}_{U,p}$ of $A(H)$ and $A(G)$ by upper indices, so that $\mathfrak{p}^H_{U,p} \subset A(H)$ and $\mathfrak{p}^G_{U,p} \subset A(G)$.

PROPOSITION 1.14. *Let H be a subgroup of G. Then the* $I(H)$*-adic topology of* $A(H)$ *is the same as its* $I(G)$*-adic topology.*

PROOF. It is enough to show that the radicals of the ideals $\varrho(I(G))$ and $I(H)$ coincide [22 p. 256]. This means that each prime ideal $\mathfrak{p} \subset A(H)$ either contains the both ideals or none. Since $\varrho(I(G)) \subset I(H)$, one way is trivial. Let $\mathfrak{p}$ be a prime ideal of $A(H)$ with $\varrho(I(G)) \subset \mathfrak{p}$. We claim that $I(H) \subset \mathfrak{p}$. We know that $\mathfrak{p}$ is of the form $\mathfrak{p}^H_{U,0}$ or $\mathfrak{p}^H_{U,p}$ with some subgroup $U \leqq H$ and some prime p. If $\mathfrak{p} = \mathfrak{p}^H_{U,0}$, then

$$\mathfrak{p}^G_{e,0} = I(G) \subset \varrho^{-1}(\mathfrak{p}) = \mathfrak{p}^G_{U,0}$$

which implies $U=e$ and $\mathfrak{p}=\mathfrak{p}^H_{e,0}=I(H)$ by (1.9). Similarly, if $\mathfrak{p}=\mathfrak{p}^H_{U,p}$, then

$$\mathfrak{p}^G_{e,0} = I(G) \subset \varrho^{-1}(\mathfrak{p}) = \mathfrak{p}^G_{U,p}$$

and U must be a p-group by (1.9), whence $\mathfrak{p}=\mathfrak{p}^H_{U,p}=\mathfrak{p}^H_{e,p}$. In both cases $I(H) \subset \mathfrak{p}$, are claimed.

THEOREM 1.15. *Let G be a finite group and $\{G_p\}$ its Sylow subgroups. Then the completion of the restriction maps* $\mathrm{Res}^G_{G_p}$ *gives an injective homomorphism*

$$0 \to \hat{A}(G) \to \bigoplus_p \hat{A}(G_p) .$$

PROOF. By 1.10 $A(G)/\bigcap_{n=0}^{\infty} I^n(G)$ maps injectively into $\oplus_p A(G_p)$. Both modules have $I(G)$-adic topology by 1.14. The claim follows since Noetherian completion is an exact functor.

We close the chapter with some examples. The first two are abelian p-groups. The last one illustrates the restrictions to Sylow subgroups and completion.

EXAMPLE 1.16. The cyclic group $\mathbf{Z}/p^n$. It has a unique subgroup of order p^{n-m}, $0 \leqq m \leqq n$. Let η_m be the quotient ($\eta_0=1$). We have additively

$$A(\mathbf{Z}/p^n) = \mathbf{Z}\oplus\mathbf{Z}\eta_1\oplus\ldots\oplus\mathbf{Z}\eta_n .$$

From the characters

$$\chi_{\mathbf{Z}/p^i}(\eta_m) = \begin{cases} p^m, & i \leqq n-m \\ 0, & i > n-m \end{cases}$$

one gets the multiplication $\eta_l \cdot \eta_m = p^l \eta_m$ for $l \leqq m$.

EXAMPLE 1.17. The elementary abelian group $(\mathbf{Z}/p)^n$. It can be interpreted geometrically as a vector space over the finite field F_p, with subgroups corresponding to linear subspaces. The number of m-dimensional planes is

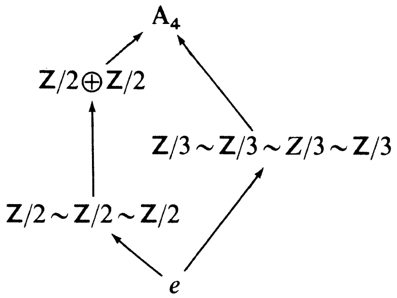
$$G(m,n) = \frac{(p^n-1)(p^n-p)\ldots(p^n-p^{m-1})}{(p^m-1)(p^m-p)\ldots(p^m-p^{m-1})}$$
$$= \frac{(p^n-1)(p^{n-1}-1)\ldots(p^{n-m+1}-1)}{(p^m-1)(p^{m-1}-1)\ldots(p-1)}$$

(G stands for Grassmann). $A((\mathbf{Z}/p)^n)$ is additively generated by the m-

dimensional quotient planes η^i_m, $0 \leqq m \leqq n$, $1 \leqq i \leqq G(n-m, n) = G(m, n)$, and

$$F^n_p/V_1 \times F^n_p/V_2 = |F^n_p/V_1 + V_2| F^n_p/V_1 \cap V_2 .$$

EXAMPLE 1.18. The alternating group A_4. The diagram of subgroups is



and the character table is given in table 1.19 where the small letters 1, a, b, c, and d denote the cosets A_4/H in the given order.

Table 1.19.

A_4/H \ χ_H	A_4	$Z/2 \oplus Z/2$	$Z/3$	$Z/2$	e
1	1	1	1	1	1
a	0	3	0	3	3
b	0	0	1	0	4
c	0	0	0	2	6
d	0	0	0	0	12

The Burnside rings of the Sylow subgroups of A_4 are described in the preceding examples:

$$A(Z/2 \oplus Z/2) = Z \oplus Z\eta^1_1 \oplus Z\eta^2_1 \oplus Z\eta^3_1 \oplus Z\eta_2$$

$$A(Z/3) = Z \oplus Z\xi .$$

The restriction map $A(G) \to A(G_2) \oplus A(G_3)$ is read from the character table using (1.4) $\chi_H(i^*x) = \chi_H(x)$. The result is

$$\begin{array}{ll} 1 \to (1,1) & \\ a \to (3, \xi) & a_1 \to (0, \underline{\xi}) \\ b \to (\eta_2, 1+\xi), \quad \text{or} & b_1 \to (\underline{\eta}_2, 0) \\ c \to (\eta^1_1 + \eta^2_1 + \eta^3_1, 2\xi) & c_1 \to (\underline{\eta}^1_1 + \underline{\eta}^2_1 + \underline{\eta}^3_1, 0) \\ d \to (3\eta_2, 4\xi) & d_1 \to (0,0) \end{array}$$

in the basis $a_1=a-3$, $b_1=b-a-1$, $c_1=c-2a$, $d_1=d-3b-a+3$ for $I(A_4)$. Here $\underline{x}$ denotes $x-\varepsilon(x)\in I(G_p)$. This shows that the image of $A(G)$ in $A(G_2)\oplus A(G_3)$ consists precisely of the *stable* elements. These are the pairs (x_2, x_3) with

(1.20) (1) $\varepsilon(x_2)=\varepsilon(x_2)$
(2) $\chi_{H_1}(x_p)=\chi_{H_2}(x_p)$, if $H_1<G_p$ and $H_2<G_p$ are conjugate in G.

The condition (2) rules out $\underline{\eta}_1^1$, $\underline{\eta}_1^2$ and $\underline{\eta}_1^3$ since they have different characters on the three $\mathbf{Z}/2<\mathbf{Z}/2\oplus\mathbf{Z}/2$ which are conjugate in A_4.

It might be interesting to know whether (1.20) characterizes the image of $A(G)$ in $\oplus_p A(G_p)$ in general. If the Segal conjecture $\hat{A}(G)=\pi_S^0(BG)$ is true, then this holds at least on the completion level by general properties of cohomology theories on BG (see [11, 1.7]).

Finally, the multiplication table for $I(A_4)$

	a_1	b_1	c_1	d_1
a_1	$-3a_1$	d_1	0	$-3d_1$
b_1		$-4b_1-d_1$	$-4c_1$	$-d_1$
c_1			$6b_1-10c_1+2d_1$	0
d_1				$3d_1$

shows that $\hat{A}(G)=\mathbf{Z}\oplus\hat{\mathbf{Z}}_3a_1\oplus\hat{\mathbf{Z}}_2b_1\oplus\hat{\mathbf{Z}}_2c_1$ and $I(G)^\infty=\mathbf{Z}d_1$.

2. λ-Operations on the Burnside ring.

Let G be a finite group. If k is a field and V is a representation of G over k then the exterior powers $\lambda^n V$ are also G-representations. We want to construct operations in $A(G)$ which under the natural map $A(G)\to R_k(G)$ correspond to the exterior powers. As it is not clear how to make sense of the relation $x\wedge y=-y\wedge x$ in a G-set, we consider first the symmetric powers s^nV where no signs are needed.

Let $S=\{s_1,\ldots,s_l\}$ be a G-set. The vector space $s^n(k^S)$ has a basis consisting of monomials of degree n in $s_i\in s^1(k^S)$, considered as elements of the symmetric algebra $s(k^S)$. We define the nth symmetric power of S as

$$s^n(S) = S^n/\Sigma_n$$

with the diagonal G-action. It is clear that

(2.1) $$s^1(S) = S$$

(2.2) $$s^n(S\cup T) = \sum_{i=0}^{n} s^i(S)s^{n-i}(T).$$

We assign to S the formal power series

(2.3) $$s_t(S) = 1 + \sum_{n \geqq 1} s^n(S)t^n \in A(G)[[t]] .$$

It is invertible as the leading coefficient is 1, and (2.2) shows that $s_t(S \cup T) = s_t(S) \cdot s_t(T)$. The homomorphism s_t is uniquely extended to $A(G)$ by $s_t(S-T) = s_t(S) \cdot s_t(T)^{-1}$.

In the representation ring $R_k(G)$ the symmetric powers are connected to the exterior powers by means of the identity

$$\lambda_t(V)s_{-t}(V) = 1 .$$

Thus we are lead to

DEFINITION 2.4. The nth exterior power of $x \in A(G)$, denoted by $\lambda^n(x)$, is the coefficient of t^n in the series $\lambda_t(x) = s_{-t}(x)^{-1}$.

The formulae (2.1)–(2.3) translate to give

(2.5)
$$\text{(i)} \quad \lambda^0(x) = 1$$
$$\text{(ii)} \quad \lambda^1(x) = x$$
$$\text{(iii)} \quad \lambda^n(x+y) = \sum_{i=0}^{n} \lambda^i(x)\lambda^{n-i}(y) .$$

This is summarised in saying that the operations λ^n, $n \geqq 1$, give $A(G)$ the structure of a λ-ring [4]. By construction they are natural with respect to induced maps, and $A(G) \to R_k(G)$ is a λ-homomorphism.

We shall calculate the character of $\lambda^n(x)$, $\chi_U(\lambda^n(x))$. By (2.5) and naturality it is enough to consider a single G-orbit $x = G/H$. A point $(s_1, \ldots, s_n) \in (G/H)^n/\Sigma_n$ is fixed under G only if it can be split up to G-orbits G/H. This implies that n is a multiple of $|G/H|$, and

(2.6) $$\chi_G(s^n(G/H)) = \begin{cases} 0, & n \not\equiv 0 \ (|G/H|) \\ 1, & n \equiv 0 \ (|G/H|) . \end{cases}$$

Thus $\chi_G(s_t(G/H)) = (1 - t^{|G/H|})^{-1}$. For a subgroup U of G we can break up S into U-orbits $\cup S_i$ so

(2.7) $$\chi_U(\lambda_t(S)) = \prod_{S_i \subset S \ U\text{-orbits}} (1 - (-t)^{|S_i|}) .$$

In particular the degree of $\chi_U(\lambda_t(S))$ is equal to $|S|$, hence $\lambda^n(S) = 0$ if $n > |S|$. Also $\varepsilon(\lambda_t(s)) = \chi_e(\lambda_t(s)) = (1+t)^{|S|}$. Thus $A(G)$ is a finite-dimensional augmented λ-ring.

We define the Adams operations $\psi^n\colon A(G) \to A(G)$, $n \geqq 1$, by

$$\psi_{-t}(x) = -t\frac{\lambda_t'(x)}{\lambda_t(x)}, \quad \text{where } \psi_t(x) = \sum_{n\geqq 1} \psi^n(x)t^n .$$

Then (2.5) implies that ψ^n is additive, $\psi^n(x+y)=\psi^n(x)+\psi^n(y)$. As to the characters, the logarithmic differentiation of (2.7) yields

$$\chi_U(\psi_t(X)) = \sum_{S_i\subset S\ U\text{-orbits}} |S_i|t^{|S_i|}(1-t^{|S_i|})^{-1} .$$

This proves

PROPOSITION 2.8. $\chi_U(\psi^n(S))=\sum_{|S_i|\,|\,n}|S_i|$, *where* $S=\bigcup S_i$ *the decomposition into U-orbits.*

COROLLARY 2.9. *The Adams operations are periodic of period dividing the order of G.*

(Indeed, the length of each U-orbit U/H is a divisor of $|G|$).

If G is a p-group and $(n,p)=1$, then the only orbits occuring in 2.8 are the U fixed points. This proves

COROLLARY 2.10. *If G is a p-group, then* $\psi^n=\mathrm{id}$ *for n relatively prime to p.*

The operations λ^n have geometrical significance: they induce natural transformations of π_S^0, the zeroth stable cohomotopy functor.

First recall the Barratt–Quillen theorem. The group completion map i: $\coprod_{n\geqq 1} B\Sigma_n \to QS^0$ gives a natural transformation of monoid-valued functors

$$\left[X, \coprod_{n\geqq 1} B\Sigma_n\right] \to [X, QS^0] .$$

Here $A(X)=[X,\coprod_{n\geq 1} B\Sigma_n]$ is the set of isomorphism classes of finite coverings of X organized to a semiring under disjoint union and fibrewise cartesian product of the total spaces, and $\pi_S^0(X)=[X,QS^0]$ is the stable cohomotopy of X (in degree 0), an abelian group with respect to loop sum. The group completion theorem states [18, Proposition 4.1]

THEOREM 2.11. *The transformation* $A \to \pi_S^0$ *is universal among transformations* $\theta\colon A \to F$, *where F is a representable abelian-group-valued homotopy functor on compact spaces, and* θ *is a transformation of monoid-valued functors.*

We deduce from this the existence of λ-operations on π_S^0 along the lines of Segal [19].

THEOREM 2.12. *There are natural transformation* $\lambda^n\colon \pi_S^0 \to \pi_S^0$ *for* $n \geqq 0$, *such that*

(i) $\lambda^0(x) = 1$
(ii) $\lambda^1(x) = x$
(iii) $\lambda^n(x+y) = \sum_{i=0}^n \lambda^i(x)\lambda^{n-i}(y)$.

PROOF. We define a transformation $\lambda^n\colon A(X) \to \pi_S^0(X)$. Assume X is connected. An m-fold covering $Y \downarrow X$ can be written as $P \times_{\Sigma_m} [m] \downarrow X$, where P is the principal Σ_m-bundle associated to Y consisting of mappings of $[m] = (1, 2, \ldots, m)$ onto the fibres of Y, and $[m]$ has the usual Σ_m-action. Let $\lambda^n([m]) = S - T \in A(\Sigma_n)$. We associate to $Y \downarrow X$ the difference

$$\lambda^n(Y) = P \underset{\Sigma_m}{\times} S - P \underset{\Sigma_m}{\times} T \in \pi_S^0(X)$$

where we have used $A \to \pi_S^0$ from 2.11.

Let us form the mapping

$$\lambda_t = \sum_{n \geqq 0} \lambda^n t^n\colon A(X) \to 1 + \pi_S^0(X)[[t]]^+ = \prod_{n \geqq 1} \pi_S^0(X).$$

It is a monoid homomorphism, when we use multiplication of power series on the right. As $1 + \pi_S^0(X)[[t]]^+$ is a representable abelian-group-valued functor, λ_t extends by theorem 2.11 to a group homomorphism

$$\lambda_t\colon \pi_S^0(X) \to 1 + \pi_S^0(X)[[t]]^+ .$$

This completes the proof of theorem 2.12.

In the articles [3] and [4] Atiyah, Tall and Segal showed that special p-adic λ-rings possess certain canonical exponential isomorphisms between the additive group $\hat{I}(G)$ and the multiplicative group $1 + \hat{I}(G)$. Unfortunately the Burnside ring $A(G)$ is special only if G is cyclic: The Adams operations ψ^n are ring homomorphisms in special λ-rings, but Siebeneicher showed that this is not true in $A(G)$ for any non-cyclic G [20, p. 232]. On the other hand, $A(G)$ embeds as a sub-λ-ring of $R(G)$ if G is cyclic.

However, it is interesting to study the exponential map ϱ_k. We first do the algebra and then identify the resulting geometric map $Q_0S^0 \to SG[1/k]$ with the composition

$$Q_0S^0 \xrightarrow{e} \operatorname{Im} J_p \xrightarrow{\alpha_p} SG\left[\frac{1}{k}\right].$$

Philosophically this is a negative result: the λ-operations on $A(G)$ do not give any information on the fibre of e, the space usually denoted $\operatorname{cok} J_p$.

Let G be a finite group. We shall encounter series of the form $\lambda_\alpha(x)=1+\sum \alpha^n\lambda^n(x)$. To show their convergence in $\hat{A}(G)$ we introduce a new topology on $A(G)$. Define the Grothendieck operations by

(2.13) $$\gamma^n(x) = \lambda^n(x+n-1)\,.$$

If $\gamma_t(x)=1+\sum_{n\geqq 1}\gamma^n(x)t^n$, then

(2.14) $$\gamma_t(x) = \lambda_{t/(1-t)}(x), \qquad \gamma_t(x+y) = \gamma_t(x)\gamma_t(y)\,.$$

The γ-operations are convenient on the augmentation ideal $I(G)$ as the generators $G/H-\varepsilon(G/H)$ have finite γ-dimension but infinite λ-dimension. In fact, (2.7) and (2.14) imply

$$\chi_U\big(\gamma_t(S-\varepsilon(S))\big) = \prod_{S_i\subset S\ U\text{-orbit}} [(1-t)^{|S_i|}-(-t)^{|S_i|}]$$

for any G-set S.

Define the γ-filtration by

(2.15) I_n is the group generated additively by $\gamma^{n_1}(x_1)\dots\gamma^{n_r}(x_r)$ with $x_i\in I(G)$, $\sum n_i\geqq n$.

Then $I_m\cdot I_n\subset I_{m+n}$, $I_0=A(G)$ and $I_1=I(G)$. Thus the filtration $(I_n)_{n\geqq 0}$ defines a topology on $A(G)$, the γ-topology.

PROPOSITION 2.16. *If G is a p-group, then the p-adic, $I(G)$-adic and γ-topologies on $A(G)$ are equivalent.*

PROOF. We proved in 1.12 that the first two topologies coincide. Atiyah [1, Corollary 12.3] shows that the γ-topology is equivalent to the $I(G)$-adic if $I(G)$ has a finite number of generators, each of finite γ-dimension.

Let G be a p-group $x\in I(G)$ and $\alpha\in\hat{\mathbf{Z}}_p$. Then the series $\gamma_\alpha(x)=1+\sum_{n\geqq 1}\alpha^n\gamma^n(x)$ converges in the γ-topology, hence also in $\hat{A}(G)$. More generally, if $\alpha\in A$ where A is a finitely generated $\hat{\mathbf{Z}}_p$-algebra, then $\gamma_\alpha(x)$ exists in $1+\hat{I}(G)\otimes_{\hat{\mathbf{Z}}_p}A$. We fix a prime k different from p and apply this to $A=\hat{\mathbf{Z}}_p[\xi]$, where ξ is a primitive kth root of 1.

DEFINITION 2.17. $\varrho_k(x)=\prod_{u^k=1,\,u\neq 1}\lambda_{-u}(x)$, $x\in I(G)$.

A priori $\varrho_k(x)$ belongs to $1+\hat{I}(G)\otimes\hat{\mathbf{Z}}_p[\xi]$. But it is invariant under the action of the Galois group of $\hat{\mathbf{Q}}_p(\xi)/\hat{\mathbf{Q}}_p$, so actually $\varrho_k(x)\in 1+\hat{I}(G)$.

We compute the character of $\varrho_k(x)$. A substitution in (2.7) yields

$$\chi_U(\varrho_k(S-\varepsilon(S))) = \prod_{\substack{u^k=1\\u\neq 1}} \left(\prod_{S_i\subset S} (1-u^{|S_i|})\right)(1-u)^{-|S|}\,.$$

As the sizes of the U-orbits S_i are 1 or multiples of p and $(k,p)=1$, $1-u^{|S_i|}$ runs through the same values as $1-u$, when S_i is fixed. Noting that $\prod_{\substack{u^k=1\\u\neq 1}} (1-u)=k$ we get

$$\chi_U(\varrho_k(S-\varepsilon(S))) = k^{o_U(S)-\varepsilon(S)} \tag{2.18}$$

where $o_U(S)$ is the number of U-orbits in S.

Next we show that ϱ_k can be obtained by a direct operation on G-sets.

PROPOSITION 2.19. *For a G-set S let $\theta_k(S)$ be the underlying set of the vector space F_k^S with the linear G-action extending the permutation of the basis. Then θ_k satisfies*

(i) $$\theta_k(S+T) = \theta_k(S)\theta_k(T)$$

(ii) $$\varepsilon\theta_k(S) = k^{\varepsilon(S)}$$

(iii) θ_k *is natural*

(iv) $$\theta_k(S) = \prod_{\substack{u^k=1\\u\neq 1}} \lambda_{-u}(S)$$

on $A^+(G)$.

PROOF. Properties (i)–(iii) are obvious. To prove (iv) it is enough to check $\chi_U(\theta_k(S))$ for $U=G$ by naturality and for a transitive G-set S by (i). A point $\sum_{x\in S} a_x x$, $a_x \in F_k$, is fixed under G only if it is of the form $a\sum_{x\in S} x$, thus

$$\chi_G(\theta_k(S)) = |F_k| = k$$

but $\chi_G(\prod \lambda_{-u}(S))=k^{o_G S}=k$.

REMARK. There is no problem about the convergence of $\lambda_{-u}(S)$ in (iv), since $\lambda_t(S)$ is a polynomial.

We return now to the stable cohomotopy interpretation. Let p and k be different primes. The operation $\theta_k\colon A^+(\Sigma_n)\to A^+(\Sigma_n)$ induces a natural transformation $\theta_k\colon A\to A$ as in theorem 2.12: if $Y\downarrow X$ is an n-fold covering, write it as $Y=P\times_{\Sigma_n}[n]$ with some principal Σ_n-bundle P and set $\theta_k(Y)=P\times_{\Sigma_n}\theta_k[n]$. By (2.19) (i) and (ii) the composite

$$A \xrightarrow{\theta_k} A \to \pi_S^0$$

is exponential and maps n-fold coverings of X to the component $[X, Q_{k^n}S^0]$.

In order to apply theorem 2.11 the elements $\theta_k(Y) \in \pi_S^0(X)$ have to be invertible in the composition product, in particular the maps in $Q_{k^n}S^0$ must have an inverse of degree k^{-n}. This can be accomplished by forming the localization $\lambda\colon QS^0 \to QS_p^0$ of the space QS^0 at p [21, sections 2 and 4]. Denote the 1-component of QS_p^0 by SG_p. Then the transformation

$$\varrho_k\colon A(X) \to [X, SG_p]$$

which takes an n-fold covering $Y \downarrow X$ to

$$X \xrightarrow{\theta_k(Y)} Q_{k^n}S^0 \xrightarrow{\lambda} Q_{k^n}S_p^0 \xrightarrow{\cdot k^{-n}} SG$$

extends to a homomorphism

$$(*) \qquad \varrho_k\colon \pi_S^0(X) \to [X, SG_p]$$

by theorem 2.11.

The restriction of (*) to $\tilde{\pi}_S^0$ corresponds to an H-map $\varrho_k\colon Q_0S^0 \to SG_p$, defined up to homotopy. The space SG_p splits as a product $J_p \times \operatorname{cok} J_p$ (this will be discussed in section 4), and we point out here

THEOREM 2.20. *The map $\varrho_k\colon Q_0S^0 \to SG_p$ factors $Q_0S^0 \xrightarrow{e} J_p \xrightarrow{\alpha_p} SG_p$.*

PROOF. Compare proposition 2.19 to [14, p. 236].

The natural homomorphism $A(G) \to R(G)$ is a λ-ring homomorphism. For the elementary abelian groups its kernel is large. We evaluate the Adams operations on $A((\mathbf{Z}/p)^n)$ in the concluding example.

EXAMPLE 2.21. Elementary abelian groups $(\mathbf{Z}/p)^n$.

Let $G = (\mathbf{Z}/p)^n$. Each generator G/H of $A(G)$ is the image of the regular representation under $\pi^*\colon A(G/H) \to A(G)$. By naturality it is thus enough to find $\psi^k(\eta^n)$, where we denote $\eta_n = G/e$ (see 1.17). From 2.8 we get

$$\chi_H(\psi^k(\eta_n)) = \begin{cases} 0, & k \not\equiv 0 \pmod{|H|} \\ p^n, & k \equiv 0 \pmod{|H|}, \end{cases} \qquad H \leqq (\mathbf{Z}/p)^n$$

which depends only on the size of H. This suggests that we begin with the sum of all cosets of cardinality p^m,

$$\eta_m^{\text{tot}} = \sum_i \eta_m^i = \sum_{|H| = p^{n-m}} G/H$$

which has the characters $\chi_H(\eta_m^{\text{tot}}) = 0$ if $|H| > p^{n-m}$, $\chi_{\mathbf{Z}/p^{n-m}}(\eta_m^{\text{tot}}) = p^m$, and then correct χ_H for smaller H by adding linear combinations of η_{m+k}^{tot}, $k > 0$. An inductive calculation shows that the element.

$$a_m = \sum_{k=0}^{n-m} (-1)^k p^{k(k-1)/2} G(k, m+k) \frac{\eta_{m+k}^{\text{tot}}}{p^{m+k}} \in A(G)\left[\frac{1}{p}\right], \tag{2.22}$$

where $G(k,m+k)$ is defined in example 1.17, has characters

$$\chi_{(\mathbf{Z}/p)^k}(a_m) = \begin{cases} 1, & k=n-m \\ 0, & k \neq n-m . \end{cases}$$

By the first formula we then get

$$\psi^{p^m h}(\eta_n) = \sum_{i=n-m}^{n} p^n a_i = p^m \sum_{k=0}^{m} (-1)^k p^{\binom{k}{2}} G(k, n-m+k-1) \eta_{n-m+k}^{\text{tot}}$$

if $(p,h)=1$, $0 \leqq m < n$ and

$$\psi^{p^n h}(\eta_n) = p^n .$$

3. The map α_G.

In this section we study the injectivity of the map $\hat{\alpha}_G$: $\hat{A}(G) \to \pi_S^0(BG)$ from the completion of the Burnside ring of a finite group G to the stable cohomotopy of its classifying space BG.

Recall the definition of α_G. Each G-set S with G-action ϱ: $G \to \Sigma_{|S|}$ gives rise to a map $\alpha_G(S)$: $BG \to QS^0$ by

$$BG \xrightarrow{B\varrho_s} B\Sigma_{|S|} \hookrightarrow \coprod_{n \geqq 0} B\Sigma_n \xrightarrow{i} QS^0 ,$$

where i is the group completion map. The homotopy class of $\alpha_G(S)$ depends only on the class of S in $A(G)$, and the correspondence $S \mapsto \alpha_G(S)$ extends to a ring homomorphism

$$\alpha_G: A(G) \to [BG, QS^0]$$

(by definition, $\pi_S^0(BG) = [BG, QS^0]$).

Alternatively, we can define $\alpha_G(S)$ as the image of the covering $EG \underset{G}{\times} S \downarrow BG$ in $\pi_S^0(BG)$ (cf. 2.11). This is quite analogous to the homomorphism

$$\alpha: R(G) \to K^*(BG)$$

studied by Atiyah in [1]: if ϱ: $G \to \mathrm{Gl}(n, \mathbf{C})$ is a complex representation of G, then $\alpha(\varrho)$ is the class of the vector bundle $EG \underset{G}{\times} (\mathbf{C}^n, \varrho)$ in $K^0(BG)$. It is no surprise that α_G and α are connected via the natural map $A(G) \to R(G)$:

PROPOSITION 3.1. *Let G be a finite group. Then the diagram*

$$\begin{array}{ccc} A(G) & \xrightarrow{\alpha_G} & \pi_S^0(BG) \\ \downarrow & & \downarrow e_* \\ R(G) & \xrightarrow{\alpha} & K^*(BG) \end{array}$$

commutes, where $e\colon QS^0 \to BU \times \mathbf{Z}$ *is induced from a unit of the unitary spectrum.*

PROOF. Let S be a G-set of cardinality n, and let $\varrho\colon G \to \Sigma_n$ be its G-action. If $P\colon \Sigma_n \to U_n$ is the permutation representation, then $\alpha_G(S)$ and $\alpha(\mathbf{C}^S)$ are represented by the upper and lower horizintal arrows in

$$\begin{array}{ccccccc} BG & \xrightarrow{B\varrho} & B\Sigma_n & \xrightarrow{i_n} & Q_nS^0 & \hookrightarrow & QS^0 \\ \| & & \downarrow{\scriptstyle BP} & & \downarrow{\scriptstyle e_n} & & \downarrow{\scriptstyle e} \\ BG & \xrightarrow{BP\circ B\varrho} & BU_n & \longrightarrow & BU \times (n) & \hookrightarrow & BU \times \mathbf{Z} \end{array}$$

But the right hand squares commute [12, Corollary 5.31].

REMARK. We shall give in section 4 a closer description of the map e (see 4.18).

In section 1 we considered the $I(G)$-adic topology on $A(G)$. If X is a CW-complex with n-skeleton X^n, we filter $\pi_S^0(X)$ by

(F) $$F^n\pi_S^0(X) = \operatorname{Ker}\,(\pi_S^0(X) \to \pi_S^0(X^{n-1}))\,.$$

Then $F^n \cdot F^m \subset F^{n+m}$ by diagonal approximation. J. W. Milnor's original construction of BG gave a CW-complex with finite skeletons B_nG. As the stable homotopy groups $\tilde{\pi}_S^0(S^n)$ are finite, so are the groups $\tilde{\pi}_S^0(B_nG)$. Hence $\pi_S^0(BG) = \varprojlim_n \pi_S^0(B_nG)$, and $\pi_S^0(BG)$ is complete in the filtration topology.

It follows from the definition that $\alpha_G(I(G)) \subset [BG, Q_0S^0] = F^1\pi_S^0(BG)$, and since α_G is a ring homomorphism, $\alpha_G(I(G)^n) \subset F^n\pi_S^0(BG)$. Thus α_G is continuous and induces a homomorphism

$$\hat{\alpha}_G\colon \hat{A}(G) \to \pi_S^0(BG)$$

between the completions.

All the maps of 3.1 are continuous homomorphisms, when $R(G)$ is equipped with augmentation ideal topology and $K^*(BG)$ with a filtration topology similar to (F). Passing to completions we have

$$\begin{array}{ccc} \hat{A}(G) & \xrightarrow{\hat{\alpha}_G} & \pi_S^0(BG) \\ \downarrow & & \downarrow{\scriptstyle e_*} \\ \hat{R}(G) & \xrightarrow{\hat{\alpha}} & K^*(BG) \end{array}$$

The main result of Atiyah [1] states that $\hat{\alpha}$ is an isomorphism. Therefore we can conclude the injectivity of $\hat{\alpha}_G$ if $\hat{A}(G)$ embeds into $\hat{R}(G)$. If G is cyclic then $A(G) \to R(G)$ is injective by Lemma 1.8. If moreover the order of G is a prime power p^n, then the augmentation ideals of both rings have the p-adic topology by Proposition 1.12 and [4, p. 277]. But then $\hat{A}(G) \to \hat{R}(G)$ is injective, since the p-adic completion is an exact functor. We have proved

THEOREM 3.2. *Let G be a cyclic group of prime power order. Then $\hat{\alpha}_G$ is injective.*

We can express theorem 3.2 by saying that the maps $\alpha_G(x)$ for cyclic G are detected by K-theory. Indeed, the proof of 3.1 shows that $\alpha(x)$: $BG \to BU \times \mathbf{Z}$ factors as

$$BG \xrightarrow{\alpha_G(x)} QS^0 \xrightarrow{e} BU \times \mathbf{Z}\,.$$

Since the map induced by $\alpha(x)$ in K-theory is non-trivial, if $x \neq 0$, so must be the one induced by $\alpha_G(x)$, too.

Next we invoke theorem 1.15 to show that the injectivity of $\hat{\alpha}_G$ can be deduced from that of $\hat{\alpha}_{G_p}$, for all Sylow subgroups G_p of G. Consider the commutative diagram

$$\begin{array}{ccc} \hat{A}(G) & \xrightarrow{\hat{\alpha}_G} & \pi_S^0(BG) \\ {\scriptstyle \mathrm{Res}}\downarrow & & \downarrow \\ \bigoplus_p \hat{A}(G_p) & \xrightarrow{\oplus \hat{\alpha}_{G_p}} & \bigoplus_p \pi_S^0(BG_p) \end{array}$$

By Theorem 1.15, Res is injective. If the maps $\hat{\alpha}_{G_p}$ are injective for all $G_p \leqq G$, then $\hat{\alpha}_G$ must be injective. Hence

THEOREM 3.3. *Let G be a finite group and $\{G_p\}$ its Sylow subgroups. If $\hat{\alpha}_{G_p}$ is injective for all $G_p \leqq G$, then $\hat{\alpha}_G$ is injective.*

Theorem 3.3 reduces the study of $\hat{\alpha}_G$ to p-groups G. First, we note

LEMMA 3.4. *Let G be a p-group. Then $\hat{\alpha}_G$ is injective if and only if α_G is injective.*

(Indeed, as $A(G)$ embeds into $\hat{A}(G)$ by Corollary 1.11, one way is trivial and the converse follows since the p-adic completion is an exact functor and $I(G)$ and $\tilde{\pi}_S^0(BG)$ both have p-adic topology, $I(G)$ by Proposition 1.12 and $\tilde{\pi}_S^0(BG)$ being profinite with BG p-local [21].)

The smallest non-trivial p-group is the cyclic $\mathbf{Z}/p$, where we can apply Theorem 3.2. Suppose inductively that α_H is injective for all genuine subgroups

H of G. By naturality of α an element in $A(G)$ which has a non-zero restriction to some $H<G$, cannot lie in the kernel of α_G. Applying Theorem 1.3 we get

LEMMA 3.5. *Let G be a p-group. Suppose α_H is injective for all genuine subgroups $H<G$. Then α_G is injective on* Ker χ_G.

To handle the rest, we have

PROPOSITION 3.6. *Let G be a p-group. There exists an element $x \in A(G)$ with $\chi_G(x)=p$ and $\chi_H(x)=0$ for $H<G$. It is induced from an epimorphism $G \to (\mathbf{Z}/p)^d$.*

PROOF. The existence of x follows from the congruences of 1.3. However, we prefer to construct it directly.

Let $\Phi(G)$ be the Frattini subgroup of G, that is, the intersection of all maximal subgroups of G. We recall some elementary facts about $\Phi(G)$ [8, III § 3]:

1) $\Phi(G)=G^p[G,G]$,
2) $\Phi(G) \lhd G$ and $G/\Phi(G)$ is a maximal elementary abelian quotient of G, say $(\mathbf{Z}/p)^d$, and
3) the elements of $\Phi(G)$ are redundant in any set of generators for G.

One can also characterize the quotient $G/\Phi(G)$ as $H_1(G; \mathbf{Z}/p)$. In $A(\mathbf{Z}/p)^d$ we write down the element

$$y = p-\eta_1^{\mathrm{tot}}+\eta_2^{\mathrm{tot}}-\ldots+(-1)^d p^{\binom{d-1}{2}}\eta_d$$

(See example 2.21, $y=pa_0$ in (2.22)). If $\pi\colon G \to G/\Phi(G)$ denotes the projection, then $\pi^*(y)$ has the required properties. Clearly $\chi_G(\pi^*(y))=(\chi_{(\mathbf{Z}/p)^d}(y))=p$. If $H<G$, then $\pi(H)<(\mathbf{Z}/p)^d$ and $\chi_H(\pi^*(y))=\chi_{\pi(H)}(y)=0$. Indeed, if $\pi(H)=(\mathbf{Z}/p)^d$, then $H\Phi(G)=G$, which implies $H=G$ by 3) above. This completes the proof of proposition 3.6.

LEMMA 3.7. $\mathbf{Z}x=\bigcap_{H<G}\operatorname{Ker}\operatorname{Res}_H^G$ *and it is a λ-ideal of $A(G)$.*

PROOF. The second claim follows from the first, since the maps Res_H^G are λ-homomorphisms. By definition $\operatorname{Res}_H^G(x)=0$ for each $H<G$. For the other containment suppose $\operatorname{Res}_H^G(y)=0$ for each $H<G$; we must show $\chi_G(y)\equiv 0 \pmod p$. Let $H<G$ be a subgroup of index p. As $H<N(H)$, H must be normal in G. The congruences of 1.3 become

$$0 = \chi_H(y) \equiv -\varphi(p)\chi_G(y) = -(p-1)\chi_G(y) \pmod p$$

thus $\chi_G(y) \equiv 0 \pmod p$.

We are ready to state the final result.

THEOREM 3.8. *Let G be a p-group. Suppose that*

1) *α_H is injective for all $H<G$*
2) *α_G is injective for the λ-ideal $\mathbf{Z}x$ described in Proposition 3.6.*

Then $\hat{\alpha}_G$ is injective.

PROOF. We first show that α_G is injective on $\mathbf{Z}x \oplus \operatorname{Ker} \chi_G$. If $m \in \mathbf{Z}$, $\chi_G(y)=0$ and $\alpha_G(mx+y)=0$, then also

$$0 = \alpha_G(mx+y)\alpha_G(y) = \alpha_G(y^2)$$

since $xy=0$ (all characters are 0). By Lemma 3.5 $y^2=0$, so $y=0$. Thus $\alpha_G(mx) =0$ and $m=0$ by assumption 2).

Now any element in $A(G)$ can be written as $n+z$ with $0 \leqq n < p$ and $z \in \mathbf{Z}x \oplus \operatorname{Ker} \chi_G$ since the latter ideal consists of z such that $\chi_G(z) \equiv 0 \pmod p$. Suppose $\alpha_G(n+z)=0$, then

$$0 = \alpha_G(z)\alpha_G(n+z) = \alpha_G(nz+z^2)$$

where $nz+z^2 \in \mathbf{Z}x \oplus \operatorname{Ker} \chi_G$. From the above $nz=-z^2$, and taking characters we get

$$\chi_H(z) = 0 \text{ or } -n \qquad \text{for all } H \leqq G\,.$$

As $\chi_G(z) \equiv 0 \pmod p$ we must have $\chi_G(z)=0$. But then $\chi_H(z)=0$ for all $H<G$: if $H<G$ is a maximal subgroup with $\chi_H(z)=-n$ then by 1.3

$$-n = \chi_H(z) \equiv -\sum \varphi(K/H)\chi_K(z) = 0 \pmod{|N(H)/H|}$$

which is impossible, since $|N(H)/H|$ is a positive power of p. Thus $z=0$ and $\alpha_G(n)=0$ implies $n=\deg \alpha_G(n)=0$.

This completes the proof of Theorem 3.8.

4. Homological study of α_G.

In this section we shall study the maps induced by $\alpha_G(x)\colon BG \to QS^0$ in homology for elementary abelian groups G. As a corollary we get that $\alpha_{(\mathbf{Z}/p)^n}$ is injective. We obtain also information relative to the splitting $Q_0S_p^0 \cong J_p \times \operatorname{cok} J_p$. We suppress the index G and write α for α_G.

Consider $\alpha(S)$ for a $(\mathbf{Z}/p)^n$-set S. The map α is additive, so we can restrict to transitive sets: $S=(\mathbf{Z}/p)^n/H$ where $H \leqq (\mathbf{Z}/p)^n$. Both H and the quotient $(\mathbf{Z}/p)^m =(\mathbf{Z}/p)^n/H$ are elementary abelian, and S is induced from the regular representation $\eta_m=(\mathbf{Z}/p)^m/1$ of $(\mathbf{Z}/p)^m$. Thus $\alpha(S)$ factors

(4.1) $$B(\mathbf{Z}/p)^n \to B(\mathbf{Z}/p)^m \xrightarrow{\alpha(\eta_m)} QS^0$$

Recall that the composition product in QS^0 corresponds to the product in $\coprod B\Sigma_n$ coming from the homomorphisms

$$\psi_{n,m}\colon \Sigma_n \times \Sigma_m \to \Sigma_{nm}$$

defined as

$$\psi_{n,m}(g,h)(i,j) = (g(i),h(j)) .$$

Here Σ_{nm} is regarded as the permutation group of pairs (i,j), $1 \leqq i \leqq n$, $1 \leqq j \leqq m$. This requires a linear ordering of the pairs; we use the lexicographic one.

We can express η_m inductively in terms of $\psi_{n,m}$. The first η_1 is just the inclusion $\mathbf{Z}/p \subset \Sigma_p$ as cyclic permutations. Then $\eta_2 = \psi_{p,p} \circ (\eta_1 \times \eta_1)$, and generally

$$\eta_n\colon (\mathbf{Z}/p)^n = \mathbf{Z}/p \times (\mathbf{Z}/p)^{n-1} \xrightarrow{\eta_1 \times \eta_{n-1}} \Sigma_p \times \Sigma_{p^{n-1}} \xrightarrow{\psi_{p,p^{n-1}}} \Sigma_{p^n} .$$

Hence $\alpha(\eta_n)$ can be written as the composition

(4.2) $$\alpha(\eta_n)\colon B(\mathbf{Z}/p)^n = (B\mathbf{Z}/p)^n \xrightarrow{(B\eta_1)^n} (B\Sigma_p)^n \xrightarrow{i^n} (QS^0)^n \xrightarrow{\cdot} QS^0 .$$

We shall need certain facts about the homology of QS^0 with $\mathbf{Z}/p$-coefficients. General references for this are [10] and [12] for $p=2$ and [6] for $p>2$. Here is a summary.

The space QS^0 has two products: the loop sum $*$ and the composition product $\cdot$. They induce products on $H_*(QS^0; \mathbf{Z}/p)$, denoted similarly. They are homomorphisms $Q^b\colon H_*(QS^0; \mathbf{Z}/p) \to H_*(QS^0; \mathbf{Z}/p)$ with the following properties [modifications for the case $p=2$ are stated inside square brackets]:

(4.3) Degree: Q^b raises degree by $2b(p-1)$ $[b]$

(4.4) Evaluation: $Q^b x = 0$ if $2b < \deg x$ $[b < \deg x]$
$Q^b x = x^{*p}$ if $2b = \deg x$ $[b = \deg x]$

(4.5) Cartan formula: $Q^b(x * y) = \sum_{i+j=b} Q^i x * Q^j y$.

(4.6) Adem relations: If $a > pb$ then

$$Q^a Q^b x = \sum (-1)^{a+t} \binom{(p-1)(t-b)-1}{pt-a} Q^{a+b-t} Q^t x ;$$

if $p>2$, $a \geqq pb$ and β denotes the mod p Bockstein, then

$$Q^a \beta Q^b x = \sum (-1)^{a+t} \binom{(p-1)(t-b)}{pt-a} \beta Q^{a+b-t} Q^t x$$

$$+\sum (-1)^{a+t}\binom{(p-1)(t-b)-1}{pt-a-1} Q^{a+b-t}\beta Q^t x .$$

In all cases the summation is over t such that $(p+1)t \geqq a+b$.

(4.7) Nishida relations: If P^r_* is dual to the reduced pth power P^r [the square Sq^r] then

$$P^a_* Q^b x = \sum_{t \geqq 0} (-1)^{a+t}\binom{(p-1)(b-a)}{a-pt} Q^{b-a+t} P^t_* x;$$

if $p>2$ then

$$P^a_* \beta Q^b x = \sum_{t \geqq 0} (-1)^{a+t}\binom{(p-1)(b-a)-1}{a-pt} \beta Q^{b-a+t} P^t_* x$$

$$+\sum_{t \geqq 0} (-1)^{a+t}\binom{(p-1)(b-a)-1}{a-pt-1} Q^{b-a+t} P^t_* \beta x .$$

Let $[k] \in H_0(QS^0; \mathbf{Z}/p)$ denote the component of maps of degree k. E. Dyer and R. Lashof showed that the homology ring $H_*(QS^0; \mathbf{Z}/p)$ was generated by successive operations of $Q^a, \beta Q^a$ on $[1]$ as an algebra under $*$. To make a precise statement, we introduce the Dyer–Lashof algebra $R(p)$.

Let $\mathscr{F}$ be the free graded associative algebra generated by the symbols Q^s, $s \geqq 0$ and βQ^s, $s>0$ with degrees $2s(p-1)$ and $2s(p-1)-1$ respectively [if $p=2$, $\mathscr{F}$ is generated by $Q^s, s \geqq 0$, with degree s]. The monomials in $\mathscr{F}$ can be written as

$$\beta^{\varepsilon_1} Q^{s_1} \dots \beta^{\varepsilon_k} Q^{s_k}$$

with $\varepsilon_i = 0$ or 1 and $s_i \geqq \varepsilon_i$. Denote them by Q^I, where $I = (\varepsilon_1, s_1, \dots, \varepsilon_k, s_k)$. We say that I is admissible if $s_1 \leqq ps_2 - \varepsilon_2, \dots, s_{k-1} \leqq ps_k - \varepsilon_k$, and we define the length and excess of I by $l(I)=k$ and

$$e(I) = (2s_1 - \varepsilon_1) - \sum_{j=2}^{k} (2s_j(p-1) - \varepsilon_j) \qquad (p>2)$$

$$e(I) = s_1 - \sum_{j=2}^{k} s_j \qquad (p=2)$$

The quotient of $\mathscr{F}$ by the ideal generated by the Adem relations and by monomials with $e(I)<0$ is the Dyer–Lashof algebra $R(p)$.

The formulas (4.3)–(4.6) tell that $R(p)$ acts on $H_*(QS^0; \mathbf{Z}/p)$. In fact the set

$$X = \{Q^I[1]\,,\ I \text{ admissible},\ e(I)+\varepsilon_1 > 0\} \tag{4.8}$$

forms a basis for the $*$-algebra $H_*(QS^0; \mathbf{Z}/p)$ up to component shift. Indeed, let $\mathbf{Z}/p[\mathbf{Z}]$ be the group ring of $\mathbf{Z} = \pi_0(QS^0)$. Then

$$H_*(QS^0; \mathbf{Z}/2) = PX \otimes \mathbf{Z}/2[\mathbf{Z}]$$
$$H_*(QS^0; \mathbf{Z}/p) = PX^+ \otimes EX^- \otimes \mathbf{Z}/p[Z], \quad \text{if } p>2,$$

where P and E denote the polynomial and exterior algebras, respectively, and X^+ (X^-) is the even (odd) degree part of X.

The composition product is related to the operations Q^b by May's formula:

$$(4.9) \quad \begin{aligned} Q^b(x)\cdot f &= \sum_{t\geq 0} Q^{b+t}(x\cdot P^t_* f) \quad \text{and, if } p>2 \\ \beta Q^b(x)\cdot f &= \sum \beta Q^{b+t}(x\cdot P^t_* f) - (-1)^{\deg x} \sum Q^{b+t}(x\cdot P^t_* \beta f) \end{aligned}$$

After these preparations we turn to the evaluation of (4.2) in homology. The map $i\colon B\Sigma_p \to QS^0$ is obtained from the Dyer–Lashof map θ_p as the composite

$$i\colon B\Sigma_p = E\Sigma_p \underset{\Sigma_p}{\times} (*)^p \to E\Sigma_p \underset{\Sigma_p}{\times} (QS^0)^p \xrightarrow{\theta_p} QS^0,$$

where $*$ goes to the identity map in the 1-component Q_1S^0. By (4.2), $\alpha(\eta_1)$ is of the form $B\mathbf{Z}/p \xrightarrow{B\eta_1} B\Sigma_p \to QS^0$. This is precisely the map used in the definition of Q^s [6, pp. 7–8]; if $e_m \in H_m(B\mathbf{Z}/p; \mathbf{Z}/p)$ denotes the standard generator then

$$(4.10) \quad \alpha(\eta_1)_*(e_m) = Q_m[1] = \begin{cases} (-1)^s Q^s[1] & \text{if } m=2s(p-1) \\ (-1)^s \beta Q^s[1], & \text{if } m=2s(p-1)-1 \\ 0 & \text{otherwise} \end{cases} \quad \text{for } p \text{ odd},$$
$$= Q^m[1] \qquad \text{for } p=2.$$

It follows from (4.2) and (4.10) that $\alpha(\eta_n)_*$ takes the generators $e_{i_1}\otimes\ldots\otimes e_{i_n}$ of $H_*(B(\mathbf{Z}/p)^n; \mathbf{Z}/p)$ to products of the form

$$\pm \beta^{\varepsilon_1}Q^{s_1}[1]\cdot\ldots\cdot\beta^{\varepsilon_n}Q^{s_n}[1].$$

We would like to express these elements in the $*$-product basis (4.8). A two-fold product, for example $Q^a[1]\cdot Q^b[1]$, becomes

$$(4.11) \quad Q^a[1]\cdot Q^b[1] = \sum_{t\geq 0} Q^{a+t}(P^t_* Q^b[1])$$
$$= \sum_{t\geq 0} (-1)^t \binom{(p-1)(b-t)}{t} Q^{a+t}Q^{b-t}[1]$$

by (4.9) and (4.7). Applying the Adem relations (4.6) it can be written as a linear combination of admissible terms $Q^sQ^t[1]$. If the excess is negative then $Q^I[1]=0$ by (4.4). Similarly it is shown by induction on the length of the product that

LEMMA 4.12.

$$\beta^{\varepsilon_1}Q^{s_1}[1]\cdot\ldots\cdot\beta^{\varepsilon_n}Q^{s_n}[1] = \sum \lambda_I Q^I[1]$$

where I ranges over admissible sequence of length n and excess $\geqq 0$.

*The terms $Q^I[1]$ with $e(I)+\varepsilon_1=0$ decompose as *-products of shorter $Q^J[1]$'s* (4.4).

We shall now find the special case of Lemma 4.12 in the lowest degree where we can get an admissible $Q^I[1]$ of length n and excess >0 involving no Bocksteins β. It is clearly $Q^{p^n}Q^{p^{n-1}}\ldots Q^1[1]$ in dimension $2(p^{n+1}-1)$ with excess 2 [if $p=2$ then $d=2^{n+1}-1$ and $e=1$]. We give the proofs of the next two lemmas only for $p>2$. The (easier) case $p=2$ follows by trivial modifications.

LEMMA 4.13. $Q^{p^n}[1]\cdot Q^{p^{n-1}}[1]\cdot\ldots\cdot Q^1[1]=Q^{p^n}Q^{p^{n-1}}\ldots Q^1[1]$.

PROOF. To begin with, $Q^p[1]\cdot Q^1[1]=Q^pQ^1[1]$ by (4.11). Suppose by induction that the claim holds for n. Since $x_n=Q^{p^n}Q^{p^{n-1}}\ldots Q^1[1]$ is primitive, so is also $P^t_* x_n$. If $t>0$, then according to (4.7) $P^t_* x_n$ is a linear combination of $Q^I[1]$'s of length n and degree $<2(p^{n+1}-1)$, without Bocksteins. By the minimality of x_n, $P^t_* x_n$ is *-decomposable. By a general theorem of Hopf algebras [15, Proposition 4.23] $P^t_* x_n$ must then be a $*-p$th power, especially

$$\deg P^t_* x_n = 2(p^{n+1}-1)-2t(p-1) \equiv -2+2t \equiv 0 \pmod p$$

so that $t\equiv 1 \pmod p$. Now we can apply May's formula (4.9) to get

$$Q^{p^{n+1}}[1]\cdot Q^{p^n}[1]\cdot\ldots\cdot Q^1[1] = Q^{p^{n+1}}[1]\cdot x_n$$
$$= \sum_{t\geqq 0} Q^{p^{n+1}+t}(P^t_* x_n) = Q^{p^{n+1}}x_n = Q^{p^{n+1}}Q^{p^n}\ldots Q^1[1]$$

since $Q^s(x^{*p})\neq 0$ only if $s\equiv 0 \pmod p$ in virtue of the Cartan formula (4.5).

Now we can prove that the map $\alpha\colon A((\mathbf{Z}/p)^n) \to \pi^0_S(B(\mathbf{Z}/p)^n)$ is injective on $\mathbf{Z}\eta_n$ and thereby on the whole of $A((\mathbf{Z}/p)^n)$.

PROPOSITION 4.14. *$\alpha(m\eta_n)$ is homologically non-trivial for all non-zero integers m.*

PROOF. Let first $m>0$. Then using the diagonal formula

$$\psi(e_{2i}) = \sum e_{2i_1}\otimes\ldots\otimes e_{2i_m}, \quad i_1+\ldots+i_m=i$$

for $B\mathbf{Z}/p \to (B\mathbf{Z}/p)^m$ and Lemmas 4.12 and 4.13 we obtain

$$\alpha(m\eta_n)_*(e_{2mp^n(p-1)}\otimes\ldots\otimes e_{2m(p-1)}) = (Q^{p^n}Q^{p^{n-1}}\ldots Q^1[1])^{*m}+\ldots$$

where the other terms are of the form $Q^{I_1}[1]* \ldots *Q^{I_m}[1]$ with $l(I_j)=n$ and $\deg(I_j)<2(p^{n+1}-1)$ for at least one j. Since $Q^{p^n}Q^{p^{n-1}} \ldots Q^1[1]$ is a polynomial generator, they cannot cancel the first term.

If $m<0$, then apply the loop inverse χ_*, and note that $\chi_*(x)=x*[-2\deg x]$ on primitive elements x.

THEOREM 4.15. *$\hat{\alpha}$: $\hat{A}((\mathbf{Z}/p)^n) \to \pi_s^0 p(B(\mathbf{Z}/p)^n)$ is injective for all primes p.*

PROOF. By Theorems 3.2 and 3.8 we are reduced to showing that α is injective on $\mathbf{Z}x$, where

$$x = p-\eta_1^{\mathrm{tot}}+\ldots+(-1)^n p^{\binom{n-1}{2}}\eta_n .$$

The argument of Proposition 4.14 applies also here, since the terms η_i^{tot} contribute in homology only by $*$-products of $Q^I[1]$ with $l(I)<n$ (cf. (4.1)).

This completes the proof of theorem 4.15.

Let Q_0S^0 be the 0-component of QS^0. Let X_p denote the localization of the space X at a prime p [21]. D. Sullivan has showed that the space Q_0S^0 splits locally

$$Q_0S_p^0 \cong J_p \times \operatorname{cok} J_p . \tag{4.16}$$

The space J_2 is defined as the fibre of ψ^3-1: $BO_2 \to B\mathrm{Spin}_2$. At odd primes J_p is the fibre of ψ^k-1: $BU_p \to BU_p$, where k is a prime power generating the group of units in $\mathbf{Z}/p^2$. The homotopy groups of J_p are essentially the p-primary part of the image of the J-homomorphism $O \to G$ in the stable homotopy of spheres. To describe the second factor $\operatorname{cok} J_p$ we recall the discrete models for J_p due to D. Quillen [14, chapter VIII].

First, let $p=2$. Let F_3 denote the finite field with 3 elements. Let $N_n(F_3)$ be the group of orthogonal transformations of the quadratic space $(F_3^n, x_1^2+\ldots+x_n^2)$ for which the determinant and the spinor norm [14, p. 164] agree. We encounter now a similar situation to the construction of QS^0 from the symmetric groups: there are sum and product maps on the disjoint union

$$\coprod_{n\geq 0} BN_n(F_3) \tag{4.17}$$

coming from direct sum and tensor product of quadratic spaces.

Let $\bar{F}_3$ be an algebraic closure of F_3 and choose an embedding μ: $\bar{F}_3^* \to \mathbf{C}^*$. If G is a finite group and ϱ: $G \to \mathrm{Gl}_n(\bar{F}_3)$ a representation of G, then the complex-valued function on G

$$\chi(g) = \sum_{i=1}^{n} \mu(\lambda_i)$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\varrho(g)$, is the character of a unique element in the complex representation ring $R(G)$. Moreover, Quillen has proved that if ϱ takes values in $O_n(\bar{F}_3)$, then χ is the character of an element in the real representation ring $RO(G)$.

We lift the standard representations of $N_n(F_3)$ in $\bar{F}_3^n$ in the above way to virtual representations in $RO(N_n(F_3))$ and apply $\alpha\colon RO(G) \to KO(BG)$ to get maps

$$v_n\colon BN_n(F_3) \to BO \times (n) .$$

They are compatible with the sum and product on (4.17), giving rise to an H-map

$$v\colon \Omega B\Big(\coprod_{n \geqq 0} BN_n(F_3)\Big) \to BO \times \mathbf{Z}$$

from its group completion. Now the Adams operation ψ^3 is characterized by its action on the characters $(\psi^3\chi)(g) = \chi(g^3)$, so $\psi^3 \circ v_n = v_n$ as the Frobenius map $\lambda \to \lambda^3$ just permutes the eigenvalues of any representation realizable over F_3.

Let J_2^δ denote the zero component of $\Omega B(\coprod_{n \geqq 0} BN_n(F_3))$ localized at 2. Then $v\colon J_2^\delta \to BO_2$ lifts to an H-map $J_2^\delta \to J_2$, which can be proved to be a homotopy equivalence e.g. by cohomological methods. From now on we identify J_2^δ and J_2.

Let

$$e\colon QS^0 \cong \Omega B\Big(\coprod_{n \geqq 0} B\Sigma_n\Big) \to \Omega B\Big(\coprod_{n \geqq 0} BN_n(F_3)\Big)$$

be induced from the functor which takes a finite set S to the vector space F_3^S. We restrict e to the zero component and localize to get $e\colon Q_0S_2^0 \to J_2$. We shall also use the analogous map $e\colon Q_0S_2^0 \to BO_2$, induced from the functor $S \mapsto \mathbf{R}^S$. Then the triangle

(4.18)
$$\begin{array}{ccc} & & J_2 \\ & \overset{e}{\nearrow} & \downarrow \\ Q_0S_2^0 & & \\ & \underset{e}{\searrow} & \\ & & BO_2 \end{array}$$

clearly commutes. The space $\operatorname{cok} J_2$ is defined as the fibre in

(4.19)
$$\operatorname{cok} J_2 \to Q_0S_2^0 \xrightarrow{e} J_2 .$$

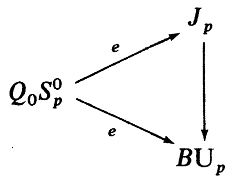
There exists a splitting $\alpha_2\colon J_2 \to Q_1S_2^0$, and $\alpha_2 * [-1]$ gives (4.16).

At odd primes p the model for J_p is constructed from general linear groups over the finite field F_k: J_p is equivalent to the zero component of the group completion of

(4.17′) $$\coprod_{n\geq 0} BGl_n(F_k),\ k \text{ a prime power generating } (Z/p^2)^* .$$

As before defines $e\colon Q_0S_p^0 \to J_p$ and gets a commutative diagram

(4.18′)



and cok J_p is defined as the fibre in

(4.19′) $$\operatorname{cok} J_p \to Q_0S_p^0 \xrightarrow{e} J_p .$$

Let now G be a p-group. We shall in the following consider the abelian groups $[BG, X_p]$, where $X = Q_0S^0$, BU, U for all p and in addition to these, $X = BO$ and SO for $p = 2$. We claim that in all cases

$$[BG, X_p] = [BG, X] .$$

If $X = Q_0S^0$ this holds because Q_0S^0 has finite homotopy groups, so $Q_0S^0 = \prod_p Q_0S_p^0$, and BG is p-local (even p-complete) [21, section 3].

For the other spaces we recall the results of Atiyah [1] and Atiyah–Segal [2]. Consider the representable K-theory and the theory $K^*(\,;Z_{(p)})$ defined by the unitary spectrum and its localization at p. For any finite CW-complex Y we have

$$K^*(Y;Z_{(p)}) \cong K^*(Y)\otimes Z_{(p)} .$$

The formula is valid also for $Y = BG$ since it follows from [1] and [2] that $\varprojlim^1$ of the inverse systems $K^*(B_nG)$ and $K^*(B_nG)\otimes Z_{(p)}$ vanishes, so that $K^*(BG) = \varprojlim K^*(B_nG)$ and $K^*(BG;Z_{(p)}) = \varprojlim K^*(B_nG)\otimes Z_{(p)}$. For any group G $K^0(BG) = \hat{R}(G)$ and $K^1(BG) = 0$ [1, p. 270] and in the case of p-groups the completion is the p-adic one: $\tilde{K}^0(BG) = I(G)\otimes\hat{Z}_p$ [4 p. 277]. Since these groups are clearly unaffected by $\otimes Z_{(p)}$, we get

(4.20) $$[BG, BU] = [BG, BU_p] = I(G)\otimes\hat{Z}_p,\ [BG, U_p] = 0$$

where $I(G)$ is the augmentation ideal of $R(G)$.

If $p = 2$, then using the Real K-theory KR^* instead of K^* and [2, p. 17] we obtain in the same fashion

(4.21) $$[BG, BO] = [BG, BO_2] = I(G)\otimes\hat{Z}_2 ,$$

$$[BG, SO_2] = \text{vector space over } \mathbf{Z}/2$$

for 2-groups G, where $I(G)$ is the augmentation ideal of $R_{\mathbf{R}}(G)$.

Let G be a p-group. After these preliminaries we turn to the question: when does a map $\hat{\alpha}(x)\colon BG \to Q_0S^0$ lift to $\operatorname{cok} J_p$ in the fibration (4.19), (4.19′). In order for $e \circ \alpha(x)\colon BG \to Q_0S^0 \to J_p$ to be nulhomotopic, it is necessary in the light of (4.18) and (4.18′) that the image of $\hat{\alpha}(x)$ under $e_*\colon \tilde{\pi}_s^0(BG) \to \tilde{K}^0(BG)$ is zero. From Proposition 3.1, this is equivalent to

$$x \in \hat{A}_0(G) = \operatorname{Ker}(\hat{A}(G) \to \hat{R}(G)) .$$

If p is odd, this condition is also sufficient, since in the mapping sequence of the fibration $J_p \to B\mathrm{U}_p \xrightarrow{\psi^k - 1} B\mathrm{U}_p$

$$[BG, \mathrm{U}_p] \to [BG, J_p] \to [BG, B\mathrm{U}_p]$$

the first group is trivial (4.20), so $x \in \hat{A}_0(G)$ maps to zero already in $[BG, J_p]$.

In particular, if G is an elementary abelian group $(\mathbf{Z}/p)^n$ with odd p, we know that all the maps $\hat{\alpha}(x)\colon BG \to Q_0S^0$, $x \in \hat{A}_0(G)$ are homotopically distinct (Theorem 4.15). Thus $\hat{\alpha}$ lifts to a monomorphism $\hat{\alpha}'$

$$\begin{array}{ccccc} & & \hat{A}_0(G) & & \\ & \swarrow^{\hat{\alpha}'} & \downarrow{\scriptstyle \hat{\alpha}} & & \\ [BG, \operatorname{cok} J_p] & \to & [BG, Q_0S^0] & \to & [BG, J_p] . \end{array}$$

THEOREM 4.22. *Let p be an odd prime and G the elementary abelian group $(\mathbf{Z}/p)^n$. Then the ideal*

$$\hat{A}_0(G) = \operatorname{Ker}(\hat{A}(G) \to \hat{R}(G))$$

maps monomorphically into $[BG, \operatorname{cok} J_p]$.

Let then G be a 2-group and $x \in \hat{A}_0(G)$. Then the image of $\hat{\alpha}(x)$ is 0 in $[BG, B\mathrm{U}_2]$. To see when $e \circ \hat{\alpha}(x)\colon BG \to J_2$ is non-trivial, we consider the maps $J_2 \to B\mathrm{O}_2 \xrightarrow{c} B\mathrm{U}_2$, where c is complexification. The map $[BG, B\mathrm{O}_2] \to [BG, B\mathrm{U}_2]$ corresponds to the completion of $R_{\mathbf{R}}(G) \subset R(G)$ by (4.20), (4.21) and [2, p. 17], which is injective. In the mapping sequence of $J_2 \to B\mathrm{O}_2 \xrightarrow{\psi^3 - 1} B\,\mathrm{Spin}_2$

$$[BG, \mathrm{Spin}_2] \to [BG, J_2] \to [BG, B\mathrm{O}_2]$$

the first group is a subgroup of $[BG, S\mathrm{O}_2]$ as $S\mathrm{O}_2 \cong \mathbf{R}P^\infty \times \mathrm{Spin}_2$, hence it is a vector space over $\mathbf{Z}/2$. Thus (at least) $2x$ maps to zero in $[BG, J_2]$. We have proved the first half of

THEOREM 4.23. *Let G be the elementary abelian 2-group* $(\mathbf{Z}/2)^n$. *Then the ideal* $2\hat{A}_0(G)$ *maps monomorphically into* $[BG, \operatorname{cok} J_2]$. *There are elements in* $\hat{A}_0(G)$ *which do not lift to* $\operatorname{cok} J_2$.

PROOF. Consider the critical element $x \in A_0(G)$ with $\chi_G(x)=2$ and $\chi_H(x)=0$ for all genuine subgroups $H<G$. We claim that $1-x$ can be written as a product in terms of the 2^n-1 quotients $\eta_1^i = G/(\mathbf{Z}/2)^{n-1}$:

$$1-x = \prod_{i=1}^{2^n-1} (\eta_1^i - 1)\,.$$

Indeed, check the characters. First $\chi_H(\eta_1^i)=2$ or 0 according to whether $H \leqq (\mathbf{Z}/2)^{n-1}$ or $H \nleqq (\mathbf{Z}/2)^{n-1}$, the hyperplane defining η_1^i. Therefore we get

$$\chi_G\left(\prod_{i=1}^{2^n-1} (\eta_1^i - 1)\right) = (-1)^{2^n-1} = -1\,.$$

On the other hand each hyperplane containing H corresponds to a line inside $H^\perp$. If $H<G$ the number of these, $|H|^\perp - 1$, is odd, so

$$\chi_H\left(\prod_{i=1}^{2^n-1} (\eta_1^i - 1)\right) = 1^{\text{odd}}(-1)^{\text{even}} = 1, \qquad H<G\,.$$

Thus $\alpha(1-x)$ is a composition product of maps of the form

$$BG \xrightarrow{B\eta_1^i} B\mathbf{Z}/2 \xrightarrow{i_2} Q_2S^0 \xrightarrow{*[-1]} SG\,.$$

But the map $i_2 * [-1]$: $B\mathbf{Z}/2 \to SG$ is homotopy equivalent to the composite

$$\mathrm{RP}^\infty \to \mathrm{SO} \xrightarrow{J} SG$$

[6, p. 120] so that $\alpha(1-x)$ factors through J: SO $\to SG$. Let e_1: $SG \to J^\otimes$ be the 1-component of the map e defined just before (4.18). We showed in 4.14 that $\alpha(-x)$, hence $\alpha(1-x)$ induces a non-trivial map in homology. It is well-known that the composite

$$H_*(\mathrm{SO}) \xrightarrow{J_*} H_*(SG) \xrightarrow{e_{1*}} H_*(J)$$

is injective [6, p. 120 and Theorem 12.5 p. 185]. Then $e_1 \circ \alpha(1-x)$, hence $e \circ \alpha$ $(-x)$ must be homologically non-trivial.

This completes the proof of Theorem 4.23.

REMARK 4.24. Theorems 4.22 and 4.23 enable us to get hold of elements in $H_*(\operatorname{cok} J_p; \mathbf{Z}/p)$. Let us consider the first case $A_0(\mathbf{Z}/2 \oplus \mathbf{Z}/2) = \mathbf{Z}x$ ($A_0(\mathbf{Z}/2) = 0$!). It is most convenient to evaluate $f = \alpha(1-x)$, since from the preceding proof

$$1-x = (\eta_1^1 - 1)(\eta_1^2 - 1)(\eta_1^3 - 1)$$

where η_1^i: $\mathbf{Z}/2\oplus\mathbf{Z}/2 \to \mathbf{Z}/2$ are projections to the first and second factor for $i=1,2$, and η_1^3 takes the quotient modulo the diagonal subgroup $\Delta Z/2 \subset \mathbf{Z}/2\oplus\mathbf{Z}/2$.

The maps $f_i=\alpha(\eta_1^i-1)$: $\mathbf{R}P^\infty\times\mathbf{R}P^\infty \to \mathbf{R}P^\infty$, $i=1,2,3$, have the effect

$$f_{1*}(e_m\otimes e_n) = \delta_{n0}x_m,\ f_{2*}(e_m\otimes e_n) = \delta_{m0}x_n$$

and

$$f_{3*}(e_m\otimes e_n) = \binom{m+n}{m}x_{m+n}$$

on homology (cf. 4.10).Here $x_k=Q^k[1]*[-1] \in H_*(SG;\ \mathbf{Z}/2)$, and adding up we get

$$f_*(e_m\otimes e_n) = \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m+n-i-j}{m-i}x_i\cdot x_j\cdot x_{m+n-i-j}\,. \tag{4.25}$$

As a special case of this formula $f_*(e_{2n}\otimes e_1)=p_{2n+1}$, where the polynomial

$$p_{2n+1} = x_{2n+1}+\sum_{i=1}^{n} x_ix_{2n+1-i}$$

is the standard primitive element of degree $2n+1$ in the subalgebra $E(x_1,x_2,\ldots)\subset H_*(SG)$.

Let $\bar f$ denote $\alpha(-x)=f*[-1]$. We know from theorem 4.23 that $2\bar f$: $(\mathbf{R}P^\infty)^2 \to Q_0S^0$ lifts to $\operatorname{cok} J_2$. Therefore the elements

$$\begin{aligned} C_{4n+2} &= (2\bar f)_*(e_{4n}\otimes e_2) = \bar f_*(e_{2n}\otimes e_1)*\bar f_*(e_{2n}\otimes e_1)\\ &= p_{2n+1}*p_{2n+1}*[-2] \end{aligned}$$

$n\geqq 1$, lie in $\operatorname{Ker} e_*$. Since they are primitive, the lie in $H_*(\operatorname{cok} J_2, \mathbf{Z}/2)$. (The elements C_{2^i-2} have a connection with the Arf invariant conjecture: they are spherical if and only if there are stable homotopy classes in $\pi^S_{2^i-2}(S^0)$ of Arf invariant one [9].)

REMARK 4.26. We succeeded in proving that $\hat\alpha_G$ is injective for elementary abelian G by evaluating the maps $\alpha_G(nx)$ in homology. Let us indicate briefly where this program fails for more complicated groups. The smallest ones we have not covered are the following three groups of order 8:

$$\begin{aligned} \mathbf{Z}/4\oplus\mathbf{Z}/2 &= \langle x,y \mid x^4=y^2=1,\ xy=yx\rangle\\ D8 &= \langle x,y \mid x^4=y^2=1,\ y^{-1}xy=x^3\rangle\\ Q8 &= \langle x,y \mid x^4=1,\ y^2=x^2,\ y^{-1}xy=x^3\rangle \end{aligned}$$

(D8 and Q8 are the dihedral and the quaternion groups). In all cases the Frattini subgroup $\Phi(G)=G^2$ is $\mathbf{Z}/2$ generated by x^2. The cohomology of G (with $\mathbf{Z}/2$ coefficients) can be computed from the spectral sequence of the central extension

$$1 \to \Phi(G) \to G \xrightarrow{\pi} \mathbf{Z}/2\oplus\mathbf{Z}/2 \to 1 .$$

The E^2-term is

$$H^*(\mathbf{Z}/2)\otimes H^*(\mathbf{Z}/2\oplus\mathbf{Z}/2) = P(t)\otimes P(t_1,t_2) .$$

We choose t_1 and t_2 as the generators of the cohomology of $\langle\pi(x)\rangle$ and $\langle\pi(y)\rangle$. The differentials are determined by the characteristic class $d_2(t) \in H^2(\mathbf{Z}/2\oplus\mathbf{Z}/2)$, which is

$$t_1^2, t_1^2+t_1t_2 \quad \text{and} \quad t_1^2+t_1t_2+t_2^2 ,$$

respectively.

The critical elements $BG \to QS^0$ are compositions of

$$BG \xrightarrow{B\pi} B(\mathbf{Z}/2\oplus\mathbf{Z}/2) = (\mathbf{RP}^\infty)^2$$

with the maps $\alpha(nx)$: $(\mathbf{RP}^\infty)^2 \to QS^0$. We evaluated $\bar{f}=\alpha(-x)$ in the preceding remark. From (4.25) we get

$$\text{(4.27)} \qquad \bar{f}_*(e_n\otimes e_m) = \bar{f}_*(e_m\otimes e_n), \quad \bar{f}_*(e_n\otimes e_0) = 0 \quad (n>0) .$$

Consider now e.g. the cohomology of $G=\mathrm{D8}$. In its spectral sequence

$$d_3(t^2) = d_3(\mathrm{Sq}^1 t) = \mathrm{Sq}^1 d_2 t = \mathrm{Sq}^1 (t_1^2+t_1t_2) = t_1^2t_2+t_1t_2^2 = t_2d_2t = 0$$

so that $E^3=E^\infty$ and

$$H^*(\mathrm{D8}) = P(s)\otimes P(t_1,t_2)/(t_1^2+t_1t_2)$$

where $s \in H^2(\mathrm{D8})$ is any element whose image is $t^2 \in H^2(\mathbf{Z}/2)$, and t_1 and t_2 come from $H^*(\mathbf{Z}/2\oplus\mathbf{Z}/2)$. Thus the image of $B\pi^*$ in $H^n(\mathrm{D8})$ is generated by the elements $t_1^n=t_1^{n-1}t_2=\ldots=t_1t_2^{n-1}$ and t_2^n. Dually the image of $B\pi_*$ in $H_n(\mathbf{Z}/2\oplus\mathbf{Z}/2)$ is generated by

$$e_n\otimes e_0+e_{n-1}\otimes e_1+\ldots+e_1\otimes e_{n-1} \quad \text{and} \quad e_0\otimes e_n .$$

From (4.27) $\bar{f}_*(\mathrm{Im}\, B\pi_*)=0$. Hence all maps $\alpha_{\mathrm{D8}}(nx)=(-n\bar{f})\circ B\pi$ vanish in $\mathbf{Z}/2$-homology.

A similar computation shows that $\bar{f}\circ B\pi$ induces the zero map $H_*(\mathrm{Q8}) \to H_*(Q_0S^0)$. In fact here $\mathrm{Im}\, B\pi^*=0$ from dimension 4 on. Finally for $G=\mathbf{Z}/4\oplus\mathbf{Z}/2$ we get that $(\bar{f}\circ B\pi)_*$ is non-trivial precisely in dimension 3. But then $(2\bar{f}\circ B\pi)_*$ vanishes.

By Proposition 3.1 these maps induce 0 also in K-theory. We pose the

QUESTION. Are the maps

$$f_n: BG \xrightarrow{B\pi} \mathbf{R}P^\infty \times \mathbf{R}P^\infty \xrightarrow{\alpha(nx)} Q_0S^0 ,$$

where $G = \mathbf{Z}/4 \oplus \mathbf{Z}/2, \mathrm{D8}, \mathrm{Q8}$ and $x = 2 - \eta_1^1 - \eta_1^2 - \eta_1^3 + \eta_2 \in A_0(\mathbf{Z}/2 \oplus \mathbf{Z}/2)$ homotopic to zero?

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