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ON A CARTAN FORMULA FOR EXOTIC CHARACTERISTIC CLASSES I

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It is our object to prove Cartan formulae for exotic characteristic classes which will be given in part II. In particular we will be concerned with the exotic characteristic classes for spherical fibrations as defined by D. Ravenel [9]. We deal with the case of Z_2-coefficients. These are elements $e_k \in H^{2^k-1}(BG; Z_2)$, where BG is the classifying space for stable spherical fibrations. The e_k are defined by twisted secondary cohomology operations based on the relation

$$\sum_{j=0}^{k-1} \mathrm{Sq}^{2^j}\mathrm{Sq}^{2^k-2^j} = 0 .$$

What we need is therefore a Cartan formula for twisted secondary cohomology operations. The untwisted case was done by L. Kristensen [6]. We will carry over his method to the twisted case. This means first of all to use cochain operations for the definition of cohomology operations. In particular we prove two exact sequences on which the definition is based (see Thm. 1 and 2 of section 1). The proofs rely on a very general theorem about cochain functors [4, part II, Thm. 3.1] and are given in section 1 and section 2. In section 3 we define twisted secondary cohomology operations. Section 4 is devoted to the classes $e_k \in H^{2^k-1}(BG; Z_2)$. In particular we will see fairly easy that the e_k are welldefined modulo ordinary characteristic classes. This is one of the main points in Ravenels paper (compare Thm. 3.1.1 of [9]). I would like to thank L. Kristensen and I. Madsen for their interest and discussions as well as the Matematisk Institut of the Aarhus University for partial support.

1. Twisted secondary cochain operations.

This section is devoted to the proof of two exact sequences on which the definition of twisted secondary operation is based.

If X is a space let $C^*(X)$ denote the singular cochain complex of X over Z_2. Let Y be a fixed space. We consider the two functors from the category of pairs of spaces to the category of chain complexes, ∂Ab

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$$C(X) = C^*(X) \quad \text{and} \quad C'(X) = C^*(Y\times X)\,.$$

Let $O(Y)$ be the set of all natural transformations $\theta=\{\theta_n\}_{n=0,1,2,\ldots}: C \to C'$ such that θ preserves zero. So for each n

$$\theta_n\colon C^n \to C'^{n+q}$$

is a natural transformation. q is fixed for all n and is called the degree of θ. We write

$$\theta_n(X) = \theta_n^X\colon C^n(X) \to C^{n+q}(Y\times X)\,.$$

If $\theta' \in O(Y)$ has degree q', we define $\theta+\theta' \in O(Y)$ by

$$(\theta^X+\theta'^X)(c) = \theta^X(c)+\theta'^X(c)$$

for each X and $c \in C(X)$. This is welldefined and makes $O(Y)$ into an $\mathbf{Z}_2$-module, i.e. if $O^q(Y)$ denotes the natural transformations of degree q, $O^q(Y)$ is a $\mathbf{Z}_2$-module and $O(Y)$ is the direct product of the $O^q(Y)$.

One also can define the composition $\theta\circ\theta'$ by $(\theta\circ\theta')^X$ to be the composition

$$C(X) \xrightarrow{\theta'^X} C(Y\times X) \xrightarrow{\theta^{Y\times X}} C(Y\times(Y\times X)) \to C(Y\times X)$$

where the last map is induced by

$$Y\times X \to Y\times(Y\times X)$$

$$(y,x) \to (y,y,x)$$

It is easily checked that $\theta\circ\theta'$ is an element of $O(Y)$ and that $\deg(\theta\circ\theta')=\deg\theta+\deg\theta'$.

Because we do not assume additivity of the θ, $\theta\circ(\theta'+\theta'')\neq\theta\circ\theta'+\theta\circ\theta''$. But $(\theta+\theta')\circ\theta''=\theta\circ\theta''+\theta'\circ\theta''$ is true. In $O(Y)$ we define a differential

$$\nabla\colon O^n(Y) \to O^{n+1}(Y)$$

by

$$\nabla(\theta) = \delta\theta+\theta\delta$$

where δ is the boundary operator in $C^*(X)$ respectively $C^*(Y\times X)$.

That is, $\nabla\circ\nabla=0$. Then we have

THEOREM 1. *Let $ZO(Y)=\operatorname{Ker}\nabla$. Then there is a homomorphism $\varepsilon\colon ZO(Y) \to H^*(Y;Z_2)\otimes A$ such that*

$$O(Y) \xrightarrow{\nabla} ZO(Y) \xrightarrow{\varepsilon} H^*(Y,\mathbf{Z}_2)\otimes A \to 0$$

is exact. Here A is the Steenrod algebra mod 2.

We call the elements of $O(Y)$ twisted cochain operations of the first kind. A twisted cochain operation of the second kind is a natural transformation $H=\{H_{m,n}\}$ of two variables

$$H_{m,n}\colon C^m \oplus C^n \to C'^{m+n+q}$$

satisfying $H(c,0)=H(0,c)=0$ for all c. q is again fixed and called the degree of H.

With $Q(Y)$ we denote the set of all cochain operation of the second kind. Again we can define the sum of $H, H' \in Q(Y)$ by $(H+H')(c,d)=H(c,d) + H'(c,d)$ which gives $Q(Y)$ a $\mathbf{Z}_2$-module structure.

If $Q^q(Y)$ are the elements of $Q(Y)$ of degree q, there is a homomorphism

$$\nabla\colon Q^q(Y) \to Q^{q+1}(Y)$$

defined by

$$(\nabla H)(c,d) = \delta H(x,y)+H(\delta x,y)+H(x,\delta y)$$

for $(c,d) \in C^m(X) \oplus C^n(X)$. We will prove the following

THEOREM 2. *Let $ZQ(Y)=\operatorname{Ker}\nabla$. There is a homomorphism $\varepsilon\colon ZQ(Y) \to H^*(Y)\otimes A\otimes A$ such that the sequence*

$$Q(Y) \xrightarrow{\nabla} ZQ(Y) \xrightarrow{\varepsilon} H^*(Y)\otimes A \otimes A \to 0$$

is exact.

Both theorems are consequences of Theorem 3.1 in [4]. We will make this more explicit.

To the cochain functors C and C' are cohomology functors H and H' associated, namely

$$H(X) = H^*(X;\mathbf{Z}_2) \quad \text{and} \quad H'(X) = H^*(Y\times X;\mathbf{Z}_2)$$

Note that if $U \subset X$,

$$H'(X,U) = H^*(Y\times(X,U)) = H^*(Y\times X, Y\times U)\,.$$

So we have coboundary operators $\delta^*\colon H'^n(U) \to H'^{n+1}(X,U)$.

A stable cohomology operation is a natural transformation $\lambda\colon H \to H'$ which commutes with the coboundary operators, i.e.

$$\lambda\delta^*(u) = \delta^*\lambda(u)$$

where $u \in H^*(U)$. Let $A(Y)$ be the set of all stable cohomology operations. $A(Y)$ has a $\mathbf{Z}_2$-module structure and if $\lambda, \lambda' \in A(Y)$ then

$$H^*(X) \xrightarrow{\lambda'X} H^*(Y\times X) \xrightarrow{\lambda Y\times X} H^*(Y\times Y\times X) \to H^*(Y\times X)$$

defines a composition product $\lambda\circ\lambda'$. (The last map is induced by the map $Y \times X \to Y\times Y\times X$ given above). From Thm. 3.1 in [4] we have the following result:

LEMMA 1.

$$O(Y) \xrightarrow{\nabla} ZO(Y) \xrightarrow{\varepsilon} A(Y) \to 0$$

is an exact sequence.

So it remains to prove $A(Y)=H^*(Y)\otimes A$. Note that $A(\mathrm{pt})=A$. An element $y\otimes a \in H^*(Y)\otimes A$ is identified with the element $\theta \in A(Y)$, where

$$\theta(x) = y\otimes a(x)$$

for $x \in H^*(X)$. This defines a map

$$\Phi: H^*(Y)\otimes A \to A(Y)\,.$$

PROOF OF THEOREM 1. We will construct a map

$$\Psi: A(Y) \to H^*(Y)\otimes A$$

which is inverse to Φ.

Let $\lambda \in A(Y)$ be given and set $K_n=K(\mathbf{Z}_2,n)$, the Eilenberg–Mac Lane-space of type $(\mathbf{Z}_2,n)$. Let $\iota_n \in H^n(K_n;\ \mathbf{Z}_2)$ be the generator.

Then

$$\lambda^{K_n}(\iota_n) \in \sum_{i=0}^{q} H^{q-i}(Y)\otimes H^{n+i}(K_n)+H^{n+q}(Y)\otimes \mathbf{Z}_2$$

where $\deg\lambda=q$. If $\{a_1^i,\ldots,a_{r_i}^i\}$ is a basis for $H^i(Y)$ we can write

$$\lambda(\iota_n) = \sum_{i=0}^{q}\sum_{j=1}^{r_i} a_j^{q-i}\otimes\iota_i^j(n)+y_n\otimes 1$$

where $a_j^{q-i}\otimes\iota_i^j(n) \in H^{q-i}(Y)\otimes H^{n+i}(K_n)$ and $y_n\otimes 1 \in H^{n+q}(Y)\otimes H^0(K_n)$.

Similarly, we have

$$\lambda(\iota_{n+1}) = \sum_{i=0}^{q}\sum_{j=1}^{r_i} a_j^{q-i}\otimes\iota_i^j(n+1)+y_{n+1}\otimes 1\,.$$

The stability of λ implies

$$\lambda\cdot\sigma(\iota_{n+1}) = (1\otimes\sigma)\lambda(\iota_{n+1})\,,$$

where $\sigma: H^{n+1}(K_{n+1}) \to H^n(K_n)$ is the cohomology suspension. Because $\sigma(\iota_{n+1})=\iota_n$, this implies

$$\lambda(\iota_n) = (1 \oplus \sigma)\lambda(\iota_{n+1})$$

and

$$y_n = y_{n+1} = 0 .$$

So we have for each i, j a sequence $\{\iota_i^j(n) \in H^{n+i}(K_n)\}_{n=1,2,3,\dots}$ with

$$\sigma(\iota_i^j(n+1)) = \iota_i^j(n)$$

which corresponds to an element λ_i^j of A. This correspondance is given by

$$\lambda_i^j(\iota_n) = \iota_i^j(n) .$$

We define

$$\Psi(\lambda) = \sum_{i=0}^{q} \sum_{j=0}^{r_i} a_j^{q-i} \otimes \lambda_i^j .$$

It is easy to check that Ψ is inverse to Φ.

Before we prove Theorem 2 we note that the composition product in $A(Y)$ corresponds to the product defined by Massey–Peterson, see e.g. [8]: If $\varkappa \otimes a$, $y \otimes b$ are elements of $H^*(Y) \otimes A$, then this product is defined by

$$(\varkappa \otimes a) \circ (y \otimes b) = \sum \varkappa \cdot a'(y) \otimes a'' b ,$$

where $\sum a' \otimes a''$ is the diagonal of a. The map

$$\Phi \colon H^*(Y) \otimes A \to A(Y)$$

as defined above then satisfies

$$\Phi(x \otimes a) \cdot (y \otimes b) = \Phi(x \otimes a) \cdot \Phi(y \otimes b) ,$$

where in $A(Y)$ is taken the composition product.

2. Proof of Theorem 2.

To prove theorem 2 it is convenient to give $Q(Y)$ another interpretation.

Let X be a space. We will consider the category χ of "spaces over X", i.e. an object is a pair (V, f) with V a space and $f \colon V \to X$ a continuous map. A morphism $\xi \colon (V', f') \to (V, f)$ is a continuous map $\xi \colon V' \to V$ such that

$$\begin{tikzcd} V' \arrow[rr, "\xi"] \arrow[dr, "f'"'] & & V \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

commutes. Note that any object (V, f) can be viewed as a morphism, namely

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[dr, "f"'] & X \arrow[d, equal] \\
& X
\end{tikzcd}$$

The functors C and C' are defined on χ:

$$C(V,f) = C^*(V) \qquad C'(V,f) = C^*(Y\times V)\,.$$

By the remark above f induces chain maps

$$f^\#\colon C^*(X) \to C^*(V)$$

and

$$(1\times f)^\#\colon C'^*(X) \to C'^*(V)\,.$$

We define now $S(X)$ to be the set of natural transformations from C to C'. An element $T\in S(X)$ is a family $(T^{(V,f)})$,

$$T^{(V,f)}\colon C^*(V,f) \to C'^*(V,f)$$

of maps which increases dimension by a fixed number q, the degree of T. We therefore have for each n a map

$$T_n^{(V,f)}\colon C^n(V,f) \to C'^{n+q}(V,f)\,.$$

If $\xi\colon (V',f') \to (V,f)$ is a morphism in χ we obtain the following commutative diagram

$$\begin{tikzcd}
C^n(V,f) \arrow[r, "T_n^{(V,f)}"] \arrow[dd, bend right, "\xi^\#"'] & C'^{n+q}(V,f) \arrow[dd, bend left, "(1\times\xi)^\#"] \\
Cn(X) \arrow[u, "f^\#"] \arrow[r, "T_n^{(X,\mathrm{Id})}"] \arrow[d, "(f')^\#"] & C'^{n+q}(X) \arrow[u, "(1\times f)^\#"'] \arrow[d, "(1\times f')^\#"'] \\
C^n(V',f') \arrow[r, "T_n^{(V',f')}"'] & C'^{n+q}(V',f')
\end{tikzcd}$$

We further require that the zero elements are preserved, i.e.

$$T_n^{(V,f)}(0) = 0\,.$$

As above we can define $T+T'$ by

$$(T^{(V,f)}+T'^{(V,f)})(x) = T^{(V,f)}(x)+T'^{(V,f)}(x)\,.$$

We denote by $S^q(X)$ the elements of degree q. A differential

$$\nabla\colon S^q(X) \to S^{q+1}(X)$$

is defined by $\nabla(T)=\delta T+T\delta$. Then $(S(X),\nabla)$ is a chain complex. A continuous map $g\colon X'\to X$ induces a chain map $S(g)\colon S(X)\to S(X')$. If $V\xrightarrow{f}X'$ is a "space over X'" and $T\in S(X)$, then

$$(S(g)(T))^{(V,f)} = T^{(V,g\cdot f)}\,.$$

Therefore S is a functor (contravariant) from the category of spaces to the category of chaincomplexes.

Next we consider $O(C,S)$, the set of natural transformations between C and S which preserve zero. We use the notation of [4]. So for example $O(Y)=O(C,C')$.

$O(C,S)$ is a chain complex. The differential, also denoted by ∇, is given by

$$\nabla(F) = \nabla F+F\delta\,.$$

LEMMA 2. *The chain complex $Q(Y)$ can be identified with* $(O(C,S),\nabla)$.

PROOF. We define chain maps

$$\hat{\ }\colon Q(Y)\to O(C,S)$$

and $\check{\ }\colon O(C,S)\to Q(Y)$ which are inverse to each other.

First let $F\in Q(Y)$ be of degree q, i.e.

$$F^X\colon C^m(X)\oplus C^n(X)\to C'^{n+m+q}(X)\,.$$

For a fixed $x\in C^m(X)$ we have

$$\hat{F}^X(x) = F^X(x,\cdot)\colon C^n(X)\to C'^{n+m+q}(X)$$

and one defines $\hat{F}^X(x)\in S(X)$ by

$$(\hat{F}^X(x))^{(V,g)} = \hat{F}^V(g^\#(x))\,.$$

Then $\hat{F}\colon C\to S$ can be defined by

$$\hat{H}^X\colon C(X)\to S(X)\,.$$

It is easily checked that $\hat{F}$ is a natural transformation. Conversely let $T\in O(C,S)$ be given. For any space X then

$$T^X\colon C(X)\to S(X)\,.$$

If $(x,y)\in C^n(X)\oplus C^m(X)$ we define

$$\check{T}^X(x,y) = ((T^X(x))^{(X,\mathrm{Id})})(y)$$

We will check naturality of $\check{T}$. Let $f\colon X'\to X$ be given. We note first that by naturality of T

$$\left(S(f)(T^X(x))\right)^{(W,\varphi)} = \left(T^{X'}(f^\#(x))\right)^{(W,\varphi)}$$

for any map $\varphi\colon W \to X'$. By definition we have

$$\left(S(f)(T^X(x))\right)^{(W,\varphi)} = \left(T^X(x)\right)^{(W,f\circ\varphi)} .$$

Taking $\varphi = \mathrm{Id}\colon X' \to X'$ we have

$$\left(T^X(x)\right)^{(X',f)} = \left(T^{X'}(f^\#(x))\right)^{(X',\mathrm{Id})} .$$

Therefore

$$\begin{aligned}(1\times f)^\#(\check{T}^X(x,y)) &= (1\times f)^\#\left((T^X(x)^{(X,\mathrm{Id})}(y)\right)\\ &= \left((T^X(x))^{(X',f)}\right)(f^\#(y))\\ &= \left(T^{X'}(f^\#(x))\right)^{(X',\mathrm{Id})}(f^\#(y))\\ &= \check{T}^{X'}(f^\#(x), f^\#(y)) .\end{aligned}$$

The second equality is the commutativity of the diagram

$$\begin{tikzcd} C(X) \arrow[r, "(T^X(x))^{(X,\mathrm{Id})}"] \arrow[d, "f^\#"] & C'(X) \arrow[d, "(1\times f)^\#"] \\ C(X') \arrow[r, "(T^X(x))^{(X',f)}"] & C'(X') \end{tikzcd}$$

because $T^X(x) \in S(X)$.

It is easy to check that $\hat{}$ and $\check{}$ are chain maps.

Remember that we have fixed a space Y at the beginning of the section. To prove theorem 2 we need also the following

Lemma 3. *For any space X the chain complex $S(X)$ can be identified with $O(Y\times X)$.*

Proof. We define

$$\Phi\colon O(Y\times X) \to S(X)$$

by the rule: Let $f\colon V \to X$ be a map, i.e. (V,f) is an object in χ and let $\theta \in O(Y\times X)$. With

$$\begin{aligned}\bar{f}\colon V &\to X\times V\\ v &\to (f(v), v)\end{aligned}$$

we denote the graph of f. Then $\Phi(\theta)^{(V,f)}$ is the composition

$$C(V) \xrightarrow{\theta^V} C(Y \times X \times V) \xrightarrow{(1 \times \bar{f})^{\#}} C(Y \times V) .$$

We define $\Psi: S(X) \to O(Y \times X)$ by the rule: if $T \in S(X)$ we set $\Psi(T)^V$ to be the composition

$$C(V) \xrightarrow{\pi_2^{\#}} C(X \times V) \xrightarrow{T^{(X \times V, \pi_1)}} C(Y \times X \times V) ,$$

$\pi_2: X \times V \to V$, $\pi_1: X \times V \to X$ are the projections. To prove $\Psi\Phi(\theta)^V = \theta^V$ we note that

$$\begin{tikzcd}
C(V) \arrow[rr, "\pi_2^{\#}"] \arrow[d, "\theta^V"'] & & C(X \times V) \arrow[d, "\theta^{X \times V}"] \\
C(Y \times X \times V) \arrow[rr, "\pi_2'^{\#}"] \arrow[dr, equal] & & C(Y \times X \times X \times V) \arrow[dl, "(1 \times \bar{\pi}_1)^{\#}"] \\
& C(Y \times X \times V) &
\end{tikzcd}$$

is commutative. Here

$$\pi_2': Y \times X \times X \times V \to Y \times X \times V \text{ maps}$$
$$(y, x, x', v) \to (y, x, v) .$$

To prove $\Phi(\Psi(T))^{(V,f)}$ we note that the following diagram commutes:

$$\begin{tikzcd}
C(V) \arrow[r, "\pi_2^{\#}"] \arrow[dr, equal] & C(X \times V) \arrow[r, "T^{(X \times V, \pi_1)}"] \arrow[d, "\bar{f}^{\#}"] & C(Y \times X \times V) \arrow[d, "(1 \times \bar{f})^{\#}"] \\
& C(V) \arrow[r, "T^{(V,f)}"] & C(Y \times V)
\end{tikzcd}$$

From theorem 1 and lemma 3 we have immediately

COROLLARY 3. *The cohomology $H^*(S(X), \nabla)$ of the complex $S(X)$ is isomorphic to $H^*(Y \times X) \otimes A$.*

We are now ready to prove theorem 2. We will again apply theorem 3.1 of [4]. C and S are cohomology generating functors and satisfy the required assumptions. Here

$$H^S(X) = H^*(Y \times X) \otimes A$$

is the cohomology associated to S.

Let $A(H, H^S)$ denote the set of all stable cohomology operations from H to H^S. Then we have from theorem 3.1 the following exact sequence:

$$(S) \quad O(C,S) \xrightarrow{\nabla} ZO(C,S) \xrightarrow{\varepsilon} A(H,S^S) \to 0$$

$(ZO(C,S) = \operatorname{Ker} \nabla)$.

To compute $A(H, H^S)$ we have to identify a natural transformation

$$\theta^X \colon H^*(X) \to H^*(Y \times X) \otimes A$$

or

$$\theta^X \colon H^*(X) \to (H^*(Y) \otimes A) \otimes H^*(X) .$$

$H^*(Y) \otimes A = M$ is a fixed module and by the same method as in the proof of theorem 1 one obtains

$$A(H, H^S) \cong H^*(Y) \otimes A \otimes A .$$

By lemma 2, the sequence (S) can be identified with

$$Q(Y) \xrightarrow{\nabla} ZQ(Y) \to H^*(Y) \otimes A \otimes A \to 0 .$$

This proves theorem 2.

To define secondary operations we will need a slight generalization of theorem 1. We will have to use cochain operations of m variables. To be more precise, we consider natural transformations

$$\theta_n \colon C^n \times \ldots \times C^n \to C'^{n+q} .$$

So for any space X,

$$\theta_n^X \colon C^n(X) \times \ldots \times C^n(X) \to C^{n+q}(Y \times X) .$$

It is required that $\theta_n^X(x_1, \ldots, x_m) = 0$ if $x_1 = x_2 = \ldots = x_m = 0$. $C \times \ldots \times C$ and C' are cohomology generating functors. The differential in $C \times \ldots \times C$ is given by

$$\delta(x_1, \ldots, x_m) = (\delta x_1, \ldots, \delta x_m) .$$

From theorem 3.1 of [4] one obtains the following generalization of theorem 1.

THEOREM 1'.. *Let* $\nabla \colon O(C \times \ldots \times C, C') \to O(C \times \ldots \times C, C')$ be defined by

$$\nabla\theta(x_1, \ldots, x_m) = \delta\theta(x_1, \ldots, x_m) + \theta(\delta x_1, \ldots, \delta x_m) .$$

Then there is an exact sequence

$$O(C \times \ldots \times C, C') \xrightarrow{\nabla} ZO(C \times \ldots \times C, C) \xrightarrow{\varepsilon} H^*(Y) \otimes (A \oplus \ldots \oplus A) \to 0 .$$

3. Definition and properties of twisted secondary cohomology operations.

Let (D,S) be a pair of spaces and $\pi\colon D \to Y$ a map. We consider $H^*(Y)\otimes A$ as the Massey–Peterson algebra, i.e. the multiplication is that given at the end of section 1. To keep this in mind we follow the convention and write $H^*(Y)\odot A$ for it. So $H^*(Y)\odot A$ can be considered as the algebra of stable cohomology operations from $H^*(X)$ to $H^*(Y\times X)$.

In this section we will associate to a relation in $H^*(Y)\odot A$ a cohomology operation $H^*(D,S) \to H^*(D,S)$.

Let $a_1,\ldots,a_r,b_1,\ldots,b_r \in ZO(Y)$ with $\deg a_i+\deg b_i=q+1$ and set $c=\sum_{i=1}^r a_ib_i$. Then

$$\nabla c = \delta c+c\delta = \sum \delta a_ib_i+a_ib_i\delta = 0$$

since $\delta a_i=a_i\delta$ and $\delta b_i=b_i\delta$.

We have the map $\varepsilon\colon ZO(Y) \to H^*(Y)\odot A$ and we will abbreviate $\varepsilon(x)=\hat{x}$. Assume $\varepsilon(c)=\hat{c}=\sum_{i=1}^r \hat{a}_i\hat{b}_i=0$.

By theorem 1 there exists a $\theta \in O(Y)$ with

$$\nabla\theta = c = \sum a_ib_i .$$

Let $[u] \in H^n(D,S)$ have the property

$$\hat{b}_i[u] = 0 \qquad \imath=1,2,\ldots,r$$

and let $b_iu=\delta v_i$ for some $v_i \in C^{n-1}(Y\times (D,S))$. Consider $\theta(u)+\sum_{i=1}^r a_i(v_i) \in C^{n+q}(Y\times (D,S))$:

$$\delta(\theta(u)+\sum a_i(v_i)) = \delta\theta(u)+\sum \delta a_i(v_i) =$$

$$\nabla\theta(u)+\theta(\delta u)+\sum a_i(\delta v_i) = c(u)+\sum a_ib_i(u) = 0 .$$

Therefore $[\theta(u)+\sum_{i=1}^r a_i(v_i)] \in H^{n+q}(Y\times (D,S))$.

Remark. Strictly speaking $a_i(v_i) \in C^{n+q}(Y\times Y\times (D,S))$ but we think of it as an element of $C^{n+q}(Y\times (D,S))$ under the induced map of $Y\times (D,S) \to Y\times Y \times (D,S)$ sending (y,x) to (y,y,x). This is consistent with the product in $O(Y)$.

The class $[\theta(u)+\sum a_i(v_i)]$ depends on various choices. Let $\theta' \in O(Y)$ also satisfy $\nabla\theta'=c$. Then $w=\theta'-\theta \in ZO(Y)$ and

$$[\theta(u)+\sum a_i(v_i)]-[\theta'(u)+\sum a_i(v_i)] = \hat{w}([u]) .$$

Before we determine the dependence of the v_i we need the following

Lemma 4 (see L. Kristensen [5]). *There exists to each $a \in O(Y)$ a $d(a) \in O(C\times \ldots \times C)$ (t factors, t arbitrary) such that*

(1) $a(x_1+\ldots+x_t)+a(x_1)+\ldots+a(x_t)=\delta d(a;\, x_1,\ldots,x_t)+d(a;\, \delta x_1,\ldots,\delta x_t)$

and

(2) $d(a;\, 0,\ldots,x_i,0,\ldots,0)=0.$

PROOF (as in [4]), $a(\sum x_i)+\sum a(x_i)$ can be considered as an element of $ZO(C\times\ldots\times C)$. Because $\hat{a}$ is additive it is mapped to zero under ε. By theorem 1′ there exists $s'(a)\in O(C\times\ldots\times C)$ with

$$\nabla d'(a)(x_1,\ldots,x_t) = a(\textstyle\sum x_i)+\sum a(x_i)\,.$$

From this we get

$$\nabla d'(a)(0,\ldots,x_i,0\ldots 0) = a(x_i)+a(x_i) = 0\,,$$

or

$$\delta d'(a)(0,\ldots,x_i,0\ldots 0) = d'(a)(0,\ldots,\delta d_i,0\ldots 0)\,.$$

Therefore we may replace $d'(a)(x_1,\ldots,x_t)$ by

$$d(a;\, x_1,\ldots,x_t) = d'(a)(x_1,\ldots,x_t)+\sum_{i=1}^{t} d'(a)(0,\ldots,x_i,0,\ldots,0)$$

which has the desired properties.

Let now $v_i'\in C^{n-1}(Y\times(D,S))$ also satisfy

$$b_i u = \delta v_i'\,.$$

Then $v_i'=v_i+z_i$ with $\delta z_i=0$ and using lemma 4 we have

$$\begin{aligned}
&\theta(u)+\textstyle\sum a_i(v_i')+\theta(u)+\sum a_i(v_i)\\
&= \textstyle\sum a_i(v_i+z_i)+\sum a_i(v_i)\\
&= \textstyle\sum a_i(v_i)+a_i(z_i)+\delta d(a_i;\, v_i,z_i)+d(a_i;\, \delta v_i,0)+\sum a_i(v_i)\\
&= \textstyle\sum a_i(z_i)+\delta\sum d(a_i;\, v_i,z_i)\,.
\end{aligned}$$

So we have

$$[\theta(u)+\textstyle\sum a_i(v_i')]-[\theta(u)+\sum a_i(v_i)] = \sum \hat{a}_i([z_i])\,.$$

At last let u' be another cocycle representing $[u]$. Then the same proof as in [5, p. 73] gives

$$[\theta(u')+\textstyle\sum a_i(v_i')] = [\theta(u)+\sum a_i(v_i)]$$

where $\delta v_i'=b_i(u')$.

Summarizing we obtain an element in

$$H^{n+q}(Y\times (D,S))/\sum \hat{a}_i H^{n-1+\deg a_i}(Y\times (D,S)) .$$

Here $\hat{a}_i H^*(Y\times (D,S))$ is understood to be the image of

$$H^*(Y\times (D,S)) \xrightarrow{\hat{a}_i} H^*(Y\times Y\times (D,S)) \xrightarrow{(\Delta\times 1)^*} H^*(Y\times (D,S)) ,$$

with Δ: Y $\to$ Y $\times$ Y the diagonal.

Using the map π: $(D,S) \to Y$, $H^*(D,S)$ is a $H^*(Y)\odot A$-module. If $\hat{a} \in H^*(Y)\odot A$ and $w \in H^*(D,S)$ then $\hat{a}\cdot w$ is the image of w under the composition

$$H^*(D,S) \xrightarrow{\hat{a}} H^*(Y\times (D,S)) \xrightarrow{\bar{\pi}^*} H^*(D,S)$$

with $\bar{\pi}$: $(D,S) \to Y\times (D,S)$, $\bar{\pi}(x)=(\pi(x),x)$ the graph of π. The class of $[\theta(u) + \sum a_i(v_i)]$ in

$$H^{n+q}(Y\times (D,S))/\sum \hat{a}_i H^{n-1+\deg a_i}(Y\times (D,S))$$

therefore maps under $\bar{\pi}^*$ to a class in

$$H^{n+q}(D,S)/\sum \hat{a}_i H^{n-1+\deg a_i}(D,S) .$$

Denote this element by $\varphi(c,\theta)([u])$. The following theorem lists properties of $\varphi(c,\theta)$. We proved only (b) of the theorem. For the rest we refer to Kristensen [5].

THEOREM 3. *Let* $c=\sum_{i=1}^r a_i b_i$ *with* $a_i, b_i \in ZO(Y)$ *be given and assume* c *is a relation, i.e.* $\hat{c}=\sum \hat{a}_i\hat{b}_i=0$. *Let*

$$D(n,c,(D,S)) = \{[u] \in H^n(D,S) \mid \hat{b}_1[u] \ldots = \hat{b}_r[u]=0\} .$$

If $\theta \in O(Y)$ *with* $\nabla\theta=c$, *then*

(a) $\varphi(c,\theta)$: $D(n,c,(D,S)) \to H^{n+q}(D,S)/\sum \hat{a}_i H^{n-1+\deg \hat{a}_i}(D,S)$

is a homomorphism.

(b) *If* $\theta' \in O(Y)$ *also satisfies* $\nabla\theta'=c$, *then there exists* $\hat{w} \in H^*(Y)\odot$ *such that*

$$\varphi(c,\theta)[u]-\varphi(c,\theta')[u] = \{\hat{w}[u]\}$$

where $\{\cdot\}$ *denotes the class in the quotient*

$$H^{n+q}(D,S)/\sum \hat{a}_i H^{n-1+\deg \hat{a}_i}(D,S) .$$

(c) *Let*

$$\begin{tikzcd} (D',S') \arrow[rr, "f"] \arrow[dr, "\pi'"'] & & (D,S) \arrow[dl, "\pi"] \\ & Y & \end{tikzcd}$$

be commutative then

$$\begin{tikzcd} D(n,c,(D,S)) \arrow[r, "\varphi(c,\theta)"] \arrow[d, "f^*"] & H^{n+q}(D,S)/\sum \hat{a}_i H^{n-1+\deg \hat{a}_i}(D,S) \arrow[d, "\{f^*\}"'] \\ D(n,c,(D',S')) \arrow[r, "\varphi(c,\theta)"] & H^{n+q}(D',S')/\sum \hat{a}_i H^{n-1+\deg \hat{a}_i}(D',S') \end{tikzcd}$$

is commutative.

(d) *The boundary operator* $\delta^*\colon H^*(S) \to H^{*+1}(D,S)$ *induces the following commutative diagram*:

$$\begin{tikzcd} D(n,c,S) \arrow[r, "\varphi(c,\theta)"] \arrow[d, "\delta"] & H^{n+q}(S)/\sum \hat{a}_i H^{n-1+\deg \hat{a}_i}(S) \arrow[d, "\{\delta^*\}"'] \\ D(n+1,c,(D,S)) \arrow[r, "\varphi(c,\theta)"] & H^{n+q+1}(D,S)/\sum a_i H^{n+\deg \hat{a}_i}(D,S)\,. \end{tikzcd}$$

4. Ravenels exotic characteristic classes of spherical fibrations.

In [9] Ravenel defined exotic characteristic classes. We will define these classes using our construction of twisted secondary cohomology operation. One of the main points in [9] was to show that these classes are welldefined modulo ordinary characteristic classes. We use a simple structure property of $H^*(BSG)$ (see [1]) to prove it.

Let $q_i \in H^*(BSG)$ denote the ith *Wu*-class of the universal spherical fibration. We recall from [8] the following fact: There is an injection

$$j\colon A \to H^*(BSG) \odot A$$

as algebras such that

$$j(\mathrm{Sq}^n) = \sum_{i=1}^{n} q_i \otimes \mathrm{Sq}^{n-i}\,.$$

We recall that the product in $H^*(BSG)\odot$**A** is the composition product. For $x\otimes a$, $y \otimes b \in H^*(BSG)\odot A$ it is

$$(x\otimes a)\circ(y\otimes b) = \sum xa'(y)\otimes a''b$$

where $\sum a'\otimes a''$ is the diagonal of a.

LEMMA 5. *There is a map of algebras*

$$h\colon H^*(BSO)\odot A \to H^*(BSG)\odot A\,.$$

PROOF. By lemma 3.2 of [1] there is a map $f\colon H^*(BSO) \to H^*(BSG)$ of Hopf algebras, and f is a map of left A-modules. Then we may take $h=f\otimes\mathrm{Id}$, because

$$\begin{aligned} h((x\otimes a)\cdot(y\otimes b)) &= h(\textstyle\sum xa'(y)\otimes a''b) \\ &= \textstyle\sum f(xa'(y))\otimes a''b \\ &= \textstyle\sum f(x)a'f(y)\otimes a''b \\ &= (h(x\otimes a))\cdot(h(y\otimes b))\,. \end{aligned}$$

Now there is also an algebra injection

$$j'\colon A \to H^*(BSO)\odot A$$

with $j'(\mathrm{Sq}^n)=\sum q_i\otimes \mathrm{Sq}^{n-i}$. The following lemma is then clear

LEMMA 6.

$$\begin{tikzcd} & H^*(BSG)\odot A \arrow[d, "h"] \\ A \arrow[ur, "j"] \arrow[r, "j'"'] & H^*(BSO)\odot A \end{tikzcd}$$

is commutative.

By a wellknown theorem of homological algebra (see f.e. [3, p. 169, Corollary 10.13]) there exists a chain map

$$\bar{h}\colon O(BSO) \to O(BSG)$$

which induces h. $\bar{h}$ is unique up to chainhomotopy. So there is a commutative diagram:

$$\begin{tikzcd} O(BSG) \arrow[r, "\nabla"] & ZO(BSG) \arrow[r, "\varepsilon"] & H^*(BSG)\odot A \arrow[r] & 0 \\ O(BSO) \arrow[r, "\nabla"] \arrow[u, "\bar{h}"] & ZO(BSO) \arrow[r, "\varepsilon"] \arrow[u, "\bar{h}"] & H^*(BSO)\odot A \arrow[r] \arrow[u, "h"] & 0 \end{tikzcd}$$

Let BSG_n be the classyfing space for $(n-1)$-spherical fibrations and

$$k\colon BSG_n \to BSG$$

the natural map. For convenience we write down the following commutative diagram:

$$\begin{tikzcd}
O(BSG_n) \arrow[r, "\nabla"] & ZO(BSG_n) \arrow[r, "\varepsilon"] & H^*(BSG_n)\odot A \arrow[r] & 0 \\
O(BSG) \arrow[r, "\nabla"] \arrow[u, "\bar{k}"] & ZO(BSG) \arrow[r, "\varepsilon"] \arrow[u, "\bar{k}"] & H^*(BSG)\odot A \arrow[r] \arrow[u, "k^*\otimes \mathrm{Id}"] & 0 \\
O(BSO) \arrow[r, "\nabla"] \arrow[u, "\bar{h}"] & ZO(BSO) \arrow[r, "\varepsilon"] \arrow[u, "\bar{h}"] & H^*(BSO)\odot A \arrow[r] \arrow[u, "h"] & 0
\end{tikzcd}$$

$\bar{k}$: $O(BSG) \to O(BSG_n)$ is defined by

$$\bar{k}(\theta)\colon C(X) \xrightarrow{\theta} C(BSG \times X) \xrightarrow{(k\times \mathrm{Id})^\#} C(BSG_n \times X) .$$

Note that $\bar{h}, \bar{k}$ have the following properties:

$$\bar{h}\nabla = \nabla\bar{h} \quad \text{but} \quad \bar{h}\delta \neq \delta\bar{h}$$

$$\bar{k}(\delta\theta) = \delta\bar{k}(\theta), \qquad \bar{k}(\theta\delta) = \bar{k}(\theta)\delta .$$

Let $\gamma_n \to BSG_n$ be the universal $(n-1)$ spherical fibration, $MSG_n = T(\gamma_n)$ its Thom space and $[u] \in H^n(MSG_n)$ the Thom class. We consider $T(\gamma_n)$ as a pair of spaces over BSG_n, π: $(D, S) \to BSG_n$. Suppose given a relation $\sum \hat{a}_i \hat{b}_i = 0$ in A of degree $q+1$ such that $\deg \hat{b}_i > 0$. Then by [8, theorem 4.1],

$$(((k^*\otimes \mathrm{Id})hj)(\hat{b}_i))[u] = 0 .$$

We write $\hat{\alpha}_i = j(\hat{a}_i)$, $\hat{\beta}_i = j(\hat{b}_i)$.

$c = \sum \alpha_i \cdot \beta_i \in ZO(BSO)$ is a relation by lemma 6.

Choose $\theta \in O(BSO)$ with $\nabla\theta = c$ and $v_i \in C^{n-1}(BSG_n \times (D, S))$ with

$$\bar{k}\bar{h}(\beta_i)(u) = \delta v_i .$$

Then we claim that

$$(\bar{k}\bar{h}(\theta))(u) + \sum (\bar{k}\bar{h}(\alpha_i))(v_i) \in C^{n+q}(D, S)$$

is a cocycle.

Before we prove it we note that the above cochain is in $C^{n+q}(BSG_n \times (D, S))$ but maps under $\bar{\pi}$ to the above element in $C^{n+q}(D, S)$.

Now

$$\delta(\bar{k}\bar{h}(\theta)(u)) = \bar{k}\delta\bar{h}(\theta)(u) = \bar{k}(\nabla\bar{h}(\theta)(u) + \bar{h}(\theta)\delta(u))$$

$$= \bar{k}\nabla\bar{h}(\theta)(u) = \bar{k}\bar{h}(\nabla\theta)(u) = \bar{k}\bar{h}(c)(u)$$

$$\begin{aligned}\delta(\sum \bar{k}\bar{h}(\alpha_i)(v_i) &= \sum \bar{k}\delta\bar{h}(\alpha_i)(v_i)\\ &= \sum \bar{k}(\nabla\bar{h}(\alpha_i)(v_i)+\bar{h}(\alpha_i)(\delta v_i))\\ &= \sum \bar{k}\bar{h}(\nabla\alpha_i)(v_i)+\sum \bar{k}(\bar{h}(\alpha_i)\bar{h}(\beta_i))(v_i)\\ &= \sum \bar{k}\bar{h}(\alpha_i\beta_i)(v_i)\end{aligned}$$

This proves that it is a cocycle. We used $\bar{h}(\alpha_i\beta_i)=\bar{h}(\alpha_i)\bar{h}(\beta_i)$. We will prove below that we can find a cochain map $\bar{h}\colon O(BSO)\to O(BSG)$ with this property.

The class $[\bar{k}\bar{h}(\theta)(u)+\sum\bar{k}\bar{h}(\alpha_i)(v_i)]\in H^{n+q}(MSG_n)$ is welldefined (see [8, Proposition 4.2]. Under the Thom-isomorphism we obtain an element in $H^q(BSG_n)$. If we take another θ our construction shows that the resulting class in $H^{n+q}(BSG_n)$ is changed by ordinary characteristic classes.

Ravenels classes $e_{2^i-1}\in H^{2^i-1}(BSG_n)$ are then defined by the above procedure with

$$\sum_{j=0}^{i-1} c(\mathrm{Sq}^{2^i-2^j})c(\mathrm{Sq}^{2^j}) = 0$$

as relation.

To construct a chain map $\bar{h}$ with $\bar{h}(\alpha_i\circ\beta_i)=\bar{h}(\alpha_i)\circ\bar{h}(\beta_i)$ we proceed as follows. Let

$$Y = \prod_{n>0} K_n$$

and

$$q\colon BSO\to Y,\qquad q'\colon BSG\to Y$$

be the total *Wu*-classes. The diagram

$$\begin{tikzcd} H^*(BSO) \arrow[rr, "h"] \arrow[dr, "q^*"'] & & H^*(BSG) \arrow[dl, "(q')^*"] \\ & H^*(Y) & \end{tikzcd}$$

commutes.

q and q' induce $\bar{q}\colon O(Y)\to O(BSO)$ and $\bar{q}'\colon O(Y)\to O(BSG)$. Note that $\bar{q}$ and $\bar{q}'$ preserve compositions, i.e. if $\theta,\eta\in O(Y)$, then

$$\bar{q}(\theta\circ\eta) = \bar{q}(\theta)\bar{q}(\eta) \quad\text{and}\quad \bar{q}'(\theta\circ\eta) = \bar{q}'(\theta)\circ\bar{q}'(\eta)\,.$$

Consider then the diagram:

$$\begin{tikzcd}
O(BSO) \arrow[r, "\nabla"] \arrow[dd, bend left, "\bar{h}"] & ZO(BSO) \arrow[r, "\varepsilon"] \arrow[dd, bend left, "\bar{h}"] & H^*(BSO)\otimes A \arrow[dd, bend left, "h"] \\
O(Y) \arrow[r, "\nabla"] \arrow[u, "\bar{q}"] \arrow[d, "\bar{q}'"'] & ZO(Y) \arrow[r, "\varepsilon"] \arrow[u, "\bar{q}"] \arrow[d, "\bar{q}'"'] & H^*(Y)\otimes A \arrow[u, "q^*\otimes \mathrm{Id}"'] \arrow[d, "q'^*\otimes \mathrm{Id}"'] \\
O(BSG) \arrow[r, "\nabla"] & ZO(BSG) \arrow[r, "\varepsilon"] & H^*(BSG)\otimes A
\end{tikzcd}$$

It is easy to construct a chain map $\bar{h}$ which makes it commutative. The relation under consideration $c=\sum a_i \circ b_i \in ZO(BSG)$ comes from an element $z=\sum x_i \circ y_i \in ZO(Y)$. Let for instance $a_i \circ b_i = (q'_r \otimes \mathrm{sq}^m) \circ (q'_s \otimes \mathrm{sq}^n)$, then

$$x_i \circ y_i = (\iota_r \otimes \mathrm{sq}^m) \circ (\iota_s \otimes \mathrm{sq}^n) .$$

(For any space Y and $c\otimes a \in C^*(Y)\otimes O$, $c\otimes a$: $C^*(X) \to C^*(Y)\otimes C^*(X) \cong C^*(Y\times X)$ defined by $(c\otimes a)(x) = c\otimes a(x)$, can be considered as an element of $O(Y)$. Here $O=O(\mathrm{pt})$). For the definition of sq^n see [5].

Note that $z\in ZO(Y)$ is not necessarily a relation, i.e. $\varepsilon(z)=0$. In fact, Ravenel claims that the only indecomposable relation which can be lifted to a relation in $ZO(Y)$ comes from

$$\sum_{j=0}^{i-1} \mathrm{Sq}^{2^j}\mathrm{Sq}^{2^i-2^j} = 0 ,$$

see [9]. But $\bar{q}(z) \in ZO(BSO)$ is a relation because h is injective.

We see that $\bar{q}(z) = \sum \alpha_i \circ \beta_i$ and therefore

$$\bar{h}(\alpha_i \circ \beta_i) = \bar{q}'(x_i \circ y_i) = \bar{q}'(x_i) \circ \bar{q}'(y_i) = a_i \circ b_i .$$

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