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AN INEQUALITY FOR MEASURES ON A HALF-SPACE

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Abstract.

We give a new and simple way of estimating a maximal function $T_{\lambda r}$ in harmonic analysis which was studied recently by B. Muckenhoupt and R. Wheeden. This is achieved by generalising an inequality of L. Carleson for measures on a half-space. We apply this method also to the theory of an operator g_λ^* of Littlewood–Paley–Zygmund. The paper ends with a note on restriction theorems for Bessel potentials.

Notation.

Q always denotes a cube in R^n with sides parallel to the axes, KQ denotes the cube with the same centre as Q but with sides K times as long, and side Q denotes the side length of Q.

$A \approx B$ menas $C_1 < A/B < C_2$ for some positive constants C_1, C_2.

$|E|$ denotes the Lebesgue measure of the set E.

It is to be understood that constants e.g. A, C, K, etc. may change from line to line.

$X^{\complement}$ or $\complement X$ denotes the complement of the set X.

For $f \in L^1_{\mathrm{loc}}(\mathsf{R}^n)$, define $(Mf)(x)$ by

$$(Mf)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f|$$

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Our first lemma is a generalisation of an inequality of Carleson [2] and Hörmander [8]. See also [6].

LEMMA 1. *Let $\theta \geqq 1$, and let μ be a positive measure on the upper half-space*

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$\mathsf{R}^{n+1}_{+} = \mathsf{R}^n \times [0, \infty)$ *which has the property that if* Q *is a cube in* R^n, *and* Q^* *denotes the cube* $Q \times [0, \text{side } Q]$ *in* R^{n+1}_{+}, *then* $\mu(Q^*) \leqq C|Q|^\theta$ *for some constant* C *independent of* Q. *Let* Ψ *be any continuous function in* R^{n+1}, *and let* $\Psi^*(x) = \sup|\Psi(y,t)|$ *the supremum being taken over the cone* $\{(y,t) : |x-y| < t\}$. *Then*

(i) $\mu\{|\Psi| > \alpha\} \leqq C|\{\Psi^* > \alpha\}|^\theta$ *for each* $\alpha > 0$.
(ii) $\iint_{\mathsf{R}^{n+1}_{+}} |\Psi|^\theta \, d\mu \leqq C(\int_{\mathsf{R}^n} \Psi^*)^\theta$.
(iii) *For each cube* Q *in* R^n,

$$\iint_{Q^*} |\Psi|^\theta \, d\mu \leqq C\left(\int_{KQ} \Psi^*\right)^\theta$$

PROOF. If we apply Whitney's lemma ([12, p. 167]) to the open set $\{\Psi^* > \alpha\}$, we obtain a pairwise disjoint family of cubes $\{Q_j\}$ with $\sum|Q_j| = \{\Psi^* > \alpha\}$. Moreover, these cubes have the property that for an appropriate choice of the absolute constant k, each cube kQ_j intersects the set $\{\Psi^* \leqq \alpha\}$.

We now assert that for an appropriate constant D, we have:

$$\{|\Psi| > \alpha\} \subseteqq \bigcup_j (Q_j \times [0, D \text{ side } Q_j])$$

To see this, note that if $(y,t) \in \mathsf{R}^{n+1}_{+}$ is such that $y \notin \bigcup Q_j$ then

$$|\Psi(y,t)| \leqq \Psi^*(y) \leqq \alpha\,.$$

On the other hand if $y \in Q_j$, there exists $x \notin \bigcup Q_j$ with $|y-x| \leqq D$ side Q_j (since kQ_j intersects $(\bigcup Q_j)^c$). It then follows that $|\Psi(y,t)| \leqq \Psi^*(x) \leqq \alpha$, provided that $t \geqq D$ side Q_j. So

$$\begin{aligned}
\mu\{|\Psi| > \alpha\} &\leqq \sum_j \mu(Q_j \times [0, D \text{ side } Q_j]) \\
&\leqq C \sum |Q_j|^\theta \\
&\leqq C(\sum |Q_j|)^\theta \\
&= C|\{\Psi^* > \alpha\}|^\theta
\end{aligned}$$

which proves (i).

To obtain (ii), multiply each side of (i) by $\alpha^{\theta-1}$ and integrate:

$$\frac{1}{\theta}\iint_{\mathsf{R}^{n+1}_{+}} |\Psi|^\theta \, d\mu = \int_0^\infty \alpha^{\theta-1} \mu\{|\Psi| > \alpha\} \, d\alpha$$

$$\leqq C \int_0^\infty \alpha^{\theta-1} |\Psi^* > \alpha|^\theta \, d\alpha$$

$$\leqq C \sup_{\alpha>0} (\alpha|\Psi^* > \alpha|)^{\theta-1} \int_0^\infty |\Psi^* > \alpha| \, d\alpha$$

$$\leqq C\left(\int_{\mathbf{R}^n} \Psi^*\right)^{\theta-1} \left(\int_{\mathbf{R}^n} \Psi^*\right)$$

$$= C\left(\int_{\mathbf{R}^n} \Psi^*\right)^{\theta}$$

which proves (ii).

To obtain (iii), select a function b on $\mathbf{R}^{n+1}_+$ such that b is continuous, $0 \leqq b \leqq 1$, $b \equiv 1$ on Q^* and $b \equiv 0$ on $((\frac{11}{10}Q)^*)^C$. Then

$$\iint_{Q^*} |\Psi|^\theta \, d\mu \leqq \iint_{\mathbf{R}^{n+1}_+} |b\Psi|^\theta \, d\mu \leqq C\left(\int_{\mathbf{R}^n} (b\Psi)^*\right)^\theta$$

However, it is easy to see that $(b\Psi)^*$ vanishes on $(KQ)^C$ for suitable choice of the absolute constant K, and also that $(b\Psi)^* \leqq \Psi^*$ everywhere. This gives:

$$\int_{\mathbf{R}^n} (b\Psi)^* \leqq \int_{KQ} \Psi^* .$$

Insertion of this in the above proves (iii).

REMARKS. (a) We could have used e.g. the cones $\{|x-y| < \frac{1}{2}t\}$ rather than those of the form $\{|x-y| < t\}$.

(b) Note that by applying (ii) to $\Psi = |\Phi|^p$, we have

$$\|\Phi\|_{L^{p\theta}(d\mu)} \leqq C\|\Phi^*\|_p$$

In fact, by using Lorentz spaces (see [13]) one may obtain a stronger conclusion:

If $p > 1$, $\quad \|\Phi\|_{L^{p\theta}(d\mu)} \leqq C\|\Phi^*\|_{L(p,\, p\theta)}$.

Now let us consider the maximal operator studied by Muckenhoupt and Wheeden.

When u is harmonic on $\mathbf{R}^{n+1}_+$, define $T_{\lambda r}(u)$ as follows: For $\lambda > 0$, $r > 0$, and $x \in \mathbf{R}^n$,

$$T_{\lambda r}(u)(x) = \sup_{Q:\, x \in Q} \left(\frac{1}{|Q|^{1+\lambda}} \iint_{Q^*} t^{\lambda n - 1} |u|^r \, dy \, dt\right)^{1/r}$$

PROPOSITION 2. *$T_{\lambda r}(u) \leqq C_{\lambda r}(M(u^*)^{p_0})^{1/p_0}$ where $p_0 = r/(1+\lambda)$.*

PROOF. The measure $d\mu = t^{\lambda n - 1} \, dy \, dt$ satisfies the hypotheses of lemma 1 with $\theta = 1 + \lambda$. We now apply lemma 1 part (iii) with $\Psi = |u|^{p_0}$ to obtain

$$T_{\lambda r}(u)(x) \leqq C \sup_{Q: X \in Q} \left(\frac{1}{|Q|} \int_{KQ} |u|^{p_0} \right)^{1/p_0}$$
$$\leqq C(M(u^*)^{p_0})^{1/p_0}$$

as required.

Let us now specialise to the case where u is the Poisson integral of some boundary function f.

PROPOSITION 3. *For appropriate f on $\mathbf{R}^n$, let $u(x,t) = (P_t * f)$ where P_t is the Poisson kernel*:

$$P_t(x) = \frac{c_n t}{(|x|^2 + t^2)^{(n+1)/2}}$$

Then

$$T_{\lambda r}(u) \leqq C(M(Mf)^{p_0})^{1/p_0} \quad \text{where } p_0 = r/(1+\lambda)\,.$$

PROOF. It is well known that $u^* \leqq C(Mf)$ (e.g. [12, p. 197]). Proposition 2 now gives the result.

The next lemma enables us to simplify proposition 3:

LEMMA 4. *Let $1 < p \leqq q < \infty$, and $f \in L^q(\mathbf{R}^n)$. Then $M(Mf)^p \leqq C_p M(|f|^p)$ pointwise.*

PROOF. Let $x \in Q \subseteqq \mathbf{R}^n$. Write

$$f = f_1 + f_2 \quad \text{where } f_1 = f\chi_{2Q},\ f_2 = f\chi_{(2Q)^c}$$

Then

$$\frac{1}{|Q|} \int_Q |Mf|^p \leqq \frac{C_p}{|Q|} \int_Q |Mf_1|^p + |Mf_2|^p$$

But

$$\int_Q |Mf_1|^p \leqq \int_{\mathbf{R}^n} |Mf_1|^p \leqq C_p \int_{\mathbf{R}^n} |f_1|^p = C_p \int_{2Q} |f|^p$$

by the maximal theorem.

Also $(Mf_2)(y) \approx (Mf_2)(x)$ for $y \in Q$ since f_2 is supported outside $2Q$. This means that

$$\int_Q |Mf_2|^p \leqq C_p|Q||Mf_2(x)|^p$$
$$\leqq C_p|Q||Mf(x)|^p$$
$$\leqq C_p|Q|M(|f|^p)(x)$$

by Hölder's inequality. So

$$\frac{1}{|Q|}\int_Q |Mf|^p \leqq C_pM(|f|^p)(x)$$

Taking the sup over all Q with $x \in Q$ gives the result.

COROLLARY 5. *Let u be the Poisson integral of f. Let $p_0>1$, $\lambda>0$, $p \geqq p_0$, $r=p_0(1+\lambda)$. Then for $p=p_0$, the map*

$$f \to T_{\lambda r}(u)$$

is of weak-type (p,p).
When $p>p_0$, it is of strong-type (p,p).

PROOF. By proposition 3 and lemma 4, we have $T_{\lambda r}(u) \leqq CM(|f|^{p_0}$. The result now follows from the maximal theorem.

The reader should note that weighted inequalities for $T_{\lambda r}$ may also be deduced from proposition 2 simply by applying the weighted maximal theorem ([3], [9]).

Let us now turn to Littlewood–Paley theory.

When u is harmonic on $\mathbf{R}_+^{n+1}$, we define the following:

$$u^*(x) = \sup_{|x-y|<t} |u(y,t)| \qquad (x \in \mathbf{R}^n)$$

$$S(u)(x) = \left(\iint_{|x-y|<t} t^{1-n}|\nabla u(y,t)|^2\,dy\,dt\right)^{\frac{1}{2}}$$

$$g_\lambda^*(u)(x) = \left(\iint_{\mathbf{R}_+^{n+1}} t^{1-n}\left(\frac{t}{|x-y|+t}\right)^{\lambda n}|\nabla u(y,t)|^2\,dy\,dt\right)^{\frac{1}{2}}$$

Muckenhoupt and Wheeden [10] used the function $T_{\lambda r}$ to study g_λ^*. We shall see how our generalisation of Carleson's inequality may be used instead. We shall in fact show how to adapt the method for the S-function in [5] to apply to g_λ^*. See also [1] and [7] for the theory of the S-function.

PROPOSITION 6. *Let $\lambda>1$, $p=2/\lambda$, and u be the Poisson integral of $f \in L^2(\mathbf{R}^n)$. Then*

$$|\{g_\lambda^*(u)>\alpha\}| \leqq C_p\alpha^{-2}\int_0^\alpha \beta|u^*>\beta|\,d\beta + C_p\alpha^{-p}\int_{k\alpha}^\infty \beta^{p-1}|u^*>\beta|\,d\beta$$

Proof. Let $\alpha>0$. By Whitney's lemma ([12, p. 167]), there is a family of cubes $\{Q_j\}$ so that

(i) $\bigcup Q_j=\{M((u^*)^p)>\alpha^p\}$
(ii) $1/|Q_j|\int_{Q_j}(u^*)^p\approx\alpha^p$
(iii) $\sum|Q_j|\leqq C\alpha^{-p}\|u^*\|_p^p$
(iv) When $K>1$ is a fixed constant,

$$\frac{1}{|Q_j|}\int_{KQ_j}(u^*)^p \leqq C\alpha^p$$

(v) Each $4Q_j$ intersects $\{M((u^*)^p)\leqq\alpha^p\}$.

Now for $x\in\mathbf{R}^n$, let $T_1(x)=\{(y,t)\colon |x-y|<\frac{1}{2}t\}$. Write

$$(g_\lambda^*(u))^2 = \iint_B + \iint_{B^\complement} \equiv h_1+h_2$$

where $B=\bigcup_{x\notin \cup Q_j}T_1(x)$. It is h_2 that is the interesting term for us.

$$\int_{(\cup Q_j)^\complement} h_1 \leqq C\iint_B t|\nabla u(y,t)|^2\,dy\,dt$$

$$\leqq C\alpha^2|u^*>\alpha| + \int_0^\alpha t|u^*>t|\,dt$$

as in [5, pp. 162–163]. Now decompose $h_2=\sum_j h_2^j$ being the contribution from $\{(y,t)\colon y\in Q_j\}$.

Geometrical considerations and (v) show that

$$h_2^j(x) \leqq \iint_{Q_j\times[0,K\text{ side }Q_j]} t^{1-n}\left(\frac{t}{|x-y|+t}\right)^{\lambda n}|\nabla u(y,t)|^2\,dy\,dt$$

Suppose now $x\in(2Q_j)^\complement$, and y^j is the centre of Q_j, and $(t\nabla u)_{\frac{1}{2}}^*$ denotes $\sup_{|x-y|<\frac{1}{2}t}|t\nabla u|$. Then

$$h_2^j(x) \leqq \frac{C}{|x-y^j|^{\lambda n}}\iint_{Q_j\times[0,K\text{ side }Q_j]} t^{\lambda n+1-n}|\nabla u(y,t)|^2\,dy\,dt$$

Apply lemma 1 part (iii) with $\Psi=|t\nabla u|^{2/\lambda}$ and $d\mu=t^{\lambda n-1-n}dydt$, and $\theta=\lambda$.

$$h_2^j(x) \leqq \frac{C}{|x-y^j|^{\lambda n}}\left(\int_{K^1Q_j}(|t\nabla u|_{\frac{1}{2}}^*)^{2/\lambda}\right)^\lambda$$

However $(t\nabla u)_{\frac{1}{2}}^*\leqq Cu^*$ ([12, p. 207]). Inserting this in the above, and using (iv) gives

$$h_2^j(x) \leqq \frac{C}{|x-y^j|^{\lambda n}} |Q_j|^\lambda \alpha^2 .$$

So

$$\int_{(2Q_j)^c} h_2^j(x) \leqq C|Q_j|^{1-\lambda}|Q_j|^\lambda \alpha^2 = C\alpha^2|Q_j|$$

Summation over j yields

$$\int_{(\bigcup 2Q_j)^c} h_2 \leqq C\alpha^2 \sum |Q_j| = C\alpha^2|M(u^*)^p > \alpha^p|$$

All in all,

$$\int_{(\bigcup 2Q_j)^c} (g_\lambda^*(u)) \leqq C\left[\alpha^2|u^* > \alpha| + \int_0^\alpha \beta|u^* > \beta|\, d\beta\right]$$

Note

$$\{u^* > \alpha\} \subseteqq \{M(u^*)^p > \alpha^p\} \quad \text{and} \quad |\bigcup 2Q_j| \leqq C \sum |Q_j|$$

However

$$|M\Phi > \alpha| \leqq C\alpha^{-1} \int_{|\Phi| > \frac{1}{2}\alpha} |\Phi|$$

by standard estimates for the maximal function. So

$$|M(u^*)^p > \alpha^p| \leqq C\alpha^{-p} \int_{|u^*|^p > \frac{1}{2}\alpha^p} |u^*|^p \leqq C\alpha^{-p} \int_{\frac{1}{2}\alpha}^\infty \beta^{p-1}|u^* > \beta|\, d\beta$$

This, Chebyshev's inequality, and (iii) gives

$$|g_\lambda^*(u) > \alpha| \leqq C\left[\alpha^{-2} \int_0^\infty \beta|u^* > \beta|\, d\beta + \alpha^{-p} \int_{\frac{1}{2}\alpha}^\infty \beta^{p-1}|u^* > \beta|\, d\beta\right]$$

as required.

Simple integration shows that proposition 6 includes all the standard results for g_λ^*, e.g. $u \in H^p$ $(0<p<2)$ and $\lambda = 2/p$ implies $g_\lambda^*(u)$ is of weak-type L^p. It can also be used to prove some new results, e.g. let $\lambda = 2/p$, $1<p<2$, and suppose $f \in L^p \log^+ L$. Then $g_\lambda^*(u)$ is locally L^p. Note the air on the g-function of [4] demonstrates that $g_\lambda^*(u)$ could *not* be globally L^p.

The final part of this paper consists of a brief note on "restriction" phenomena.

It is well known that many of the results of classical harmonic analysis carry over to appropriate weighted function spaces. We indicate a weighted version of one of the results in [11].

Recall that $w>0$ is in A_1 if $Mw \leqq Cw$.
$w>0$ is in A_p $(1<p<\infty)$ if

$$\left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{-q/p}\right)^{p/q} \leqq C \quad \text{for all cubes } Q\,,$$

where $q=p/(p-1)$. See [3] or [9] for information.

Notice that a function $w(x)$ on R is in A_p if and only if $w(x,y)\equiv w(x)$ is in A_p on R^2.

PROPOSITION 7. *Let*

$$\mathsf{R}^2 = \{(x,y) : x,y \in \mathsf{R}\}, \quad \mathsf{R}^1 = \{(x,0) : x \in \mathsf{R}\} \subseteqq \mathsf{R}^2\,.$$

Let $1<p<\infty$. Suppose $w(x,y)>0$ is a weight independent of y and w belongs to A_p. Let $f \in L^p(R^2,w)$ and $g=G_\alpha * f$ where G_α is the Bessel potential of order α, (see [12, p. 132]). *Suppose $\alpha>1/p$. Let*

$$B(x,h) = \{Z \in R^2 : |x-Z|<h\}$$

Then

$$(M_1 g)(x) = \sup_{0<h<1} \frac{1}{|B(x,h)|}\int_{B(x,h)} |g|$$

is finite almost everywhere in R^1 and

$$\|M_1 g\|_{L^p(\mathsf{R},w)} \leqq C\|f\|_{L^p(\mathsf{R}^2,w)}$$

DEFINITION. A function g is *strictly defined* at $Z \in \mathsf{R}^2$ if

$$\lim_{\varepsilon\to 0}\frac{1}{|B(Z,\varepsilon)|}\int_{B(Z,\varepsilon)} g$$

exists.

PROPOSITION 8. *Under the hypotheses of proposition 7, g is strictly defined almost everywhere in R^1, and the "restriction" $R(g)$ satisfies*

$$\|R(g)\|_{L^p(\mathsf{R},w)} \leqq C\|f\|_{L^p(\mathsf{R}^2,w)}$$

PROOF OF 7. Assume $\alpha<1$. The inequalites for G_α in [12, p. 132] show

$$M_1 G = O(e^{-c|x|}) \quad \text{as } |x| \to \infty$$

$$= O\left(\frac{1}{|x|^{2-\alpha}}\right) \quad \text{as } |x| \to 0$$

Let $(x, 0) \in \mathbf{R}^1$. We have

$$(M_1 g)(x) \leqq ((M_1 G_\alpha) * |f|(x, 0)$$

But

$$((M_1 G_\alpha) * f)(x, 0) = \int \left(\int (M_1 G_\alpha (x - x^1, -y) f(x^1, y)\, dx^1 \right) dy$$

Let M denote the maximal operator acting in the x variable, and let f_y denote the map $x \to f(x, y)$. Then the above estimates for $M_1 G_\alpha$ as $|x| \to 0$ or ∞ give

$$(M_1 g)(x) \leqq C \int M(f_y)(x) h(y)\, dy$$

where $h \geqq 0$ satisfies

$$h(y) = O(e^{-c|y|}) \quad \text{as } |y| \to \infty$$

$$h(y) = O(|y|^{\alpha - 1}) \quad \text{as } |y| \to 0$$

Minkowski's integral inequality yields

$$\|M_1 g\|_{L^p(\mathbf{R}, w)} \leqq C \int \|M(f_y)\|_{L^p(\mathbf{R}, w)} h(y)\, dy$$

Set $q = p/(p-1)$ and apply Hölder's inequality:

$$\|M_1 g\|_{L^p(\mathbf{R}, w)} \leqq C \|h\|_{L^q(\mathbf{R})} \left(\int \left(\int |Mf_y(x)|^p w\, dx \right) dy \right)$$

However $\|h\|_{L^q(\mathbf{R})} < \infty$ if $\alpha > 1/p$. Also, by Muckenhoupt's weighted maximal theorem ([3], [9]), we have

$$\int |Mf_y|^p w\, dx \leqq C \int |f_y|^p w\, dx \, .$$

All in all,

$$\|M_1 g\|_{L^p(\mathbf{R}, w)} \leqq C \left(\iint |f_y(x)|^p w(x)\, dx\, dy \right.$$

$$= C \|f\|_{L^p(\mathbf{R}^2, w)}$$

All that remains to be done is to remove the restriction $\alpha < 1$. To perform this in general, suppose $\alpha \geqq 1$. Then choose α' so that $\alpha' < 1$ and $\alpha' > 1/p$. Now

$$G_\alpha = G_{\alpha'} * G_{\alpha - \alpha'}$$

by the semi-group property of the Bessel potentials. Further

$$(G_{\alpha-\alpha'} * f) \leqq C(f^*)$$

where f^* is the usual maximal function in two variables. Remember $\alpha' < 1$. So

$$\begin{aligned} \|G_\alpha * f\|_{L^p(\mathbf{R}, w)} &\leqq C\|G_{\alpha-\alpha'} * f\|_{L^p(\mathbf{R}^2, w)} \\ &\leqq C\|f^*\|_{L^p(\mathbf{R}^2, w)} \\ &\leqq C\|f\|_{L^p(\mathbf{R}^2, w)} \end{aligned}$$

by the weighted maximal theorem.

Proof of 8. This is now straightforward. Let $\varepsilon > 0$, and write $f = f_1 + f_2$ where f_1 is smooth and of compact support, and $\|f_2\|_{L^p(\mathbf{R}^2, w)} < \varepsilon$.

$G_\alpha * f$ is clearly smooth and so strictly defined everywhere. So

$$\begin{aligned} \Lambda(x) &= \limsup_{\delta, \delta^1 \to 0} \left| \frac{1}{|B(x,\delta)|} \int_{B(x,\delta)} g - \frac{1}{|B(x,\delta^1)|} \int_{B(x,\delta^1)} g \right| \\ &\leqq 2M_1(G_\alpha * f_2) \end{aligned}$$

which has $L^p(\mathbf{R}^1, w)$ norm less than $C\varepsilon$ by proposition 7. Therefore

$$m_w\{\Lambda(x) > \alpha\} \leqq C \epsilon \alpha^{-p} \qquad (\alpha > 0)$$

$\varepsilon > 0$ being arbitrary we have $m_w\{\Lambda(x) > \alpha\} = 0$, which means $\Lambda = 0$ almost everywhere in $\mathbf{R}^1$.

Now clearly $|Rg| \leqq M_1 g$ where R is the restriction. So

$$\|Rg\|_{L^p(\mathbf{R}, w)} \leqq C\|f\|_{L^p(\mathbf{R}^2, w)}$$

by proposition 7.

Of course analogous results may be formulated for restriction to $\mathbf{R}^m$ of functions on $\mathbf{R}^n$, where the weight depends only on $x_1, \ldots, x_m$. The critical relation then is $\alpha > (n-m)/p$.

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