MATH. SCAND. 44 (1979), 103–134

ON FOURIER–STIELTJES TRANSFORMS, INTEGER-VALUED ON A GIVEN SUBSET

ERIK SVENSSON*

0. Introduction.

Throughout this paper standard notations of harmonic analysis as found e.g. in Rudin [10] will be used. In particular, for every locally compact abelian group (LCAG) Γ with dual $G=\hat{\Gamma}$, $B(\Gamma)$ is the Banach algebra of all functions on Γ which are Fourier–Stieltjes transforms (FSTs) of elements in $M(G)$, the Banach algebra of all bounded regular complex-valued Borel measures on G. Also if Γ is an LCAG, then Γ_d is the group Γ with the discrete topology.

The Cohen Idempotent Theorem (Cohen [2]) and the Kessler Semi-Idempotent Theorem (Kessler [7] and [8]) made us interested in the following problem.

(*) Let Γ be a given LCAG and let Ω be a given open subset of Γ. If we suppose that $\hat{\mu} \in B(\Gamma_d)$ is integer-valued on Ω, then how can the restriction $\hat{\mu}|_\Omega$ of $\hat{\mu}$ to Ω look.

The first of two main results of this paper gives a partial solution to problem (*) in the general case when Γ and Ω are arbitrary. The second completely solves problem (*) in many cases when $\Gamma=\mathbf{R}^n$. Before stating these two results explicitly we shall recall the Idempotent and the Semi-Idempotent Theorem and explain our interest in problem (*).

The version, suitable for this paper, of the Idempotent Theorem which we shall give is a slight modification of Cohen's original one.

Definition. Let Γ be an arbitrary abelian group. Let E be a subset of Γ. A function φ is called canonical on E (in Γ) if φ satisfies

1°. φ is defined and integer-valued on $E-E+E$.

2°. The set $\{\varphi(\cdot+\gamma)|_{(E-E)} \mid \gamma \in E \text{ and } \varphi(\gamma)\neq 0\}$ of restrictions of translates of φ is a finite set.

Definition. Let G be an arbitrary LCAG with dual Γ. A measure $\mu \in M(G)$ is called canonical (on G) if μ can be written $d\mu(\cdot)=(n_1(\cdot,\gamma_1)+\ldots$

* Partially supported by the Swedish Natural Science Research Council, contract no F2220-100.

Received September 20, 1978.

$+n_N(\cdot,\gamma_N))dm_H(\cdot)$, where $n_1,\ldots,n_N$ are integers, $\gamma_1,\ldots,\gamma_N \in \Gamma$, H is a compact subgroup of G, and m_H is the normalized $(m_H(H)=1)$ Haar measure of H.

The following is our version of the Cohen Idempotent Theorem.

THEOREM. *Let Γ be an arbitrary* LCAG *with dual G.*

(a). *Let φ be a function defined on Γ. Then the following two conditions are equivalent.*

1°. *φ is integer-valued and $\varphi \in B(\Gamma)$.*

2°. *φ is equal to a finite sum of functions, each being continuous and canonical on Γ.*

(b). *If the function ψ is continuous and canonical on Γ, then ψ is the* FST *of a canonical measure in $M(G)$.*

Amemiya and Ito [1] pointed out and used in their proof of the Cohen Idempotent Theorem the fact that if $\psi \in B(\Gamma)$ is canonical on Γ, then ψ is the FST of a canonical measure in $M(G)$ $(G=\hat{\Gamma})$. But as stated in (b) above the same conclusion still holds even if ψ is only continuous and canonical on Γ. We shall later prove this observation (Lemma 1.2.1). The above theorem can be considered as the FST version of the Idempotent Theorem in the formulation of Amemiya and Ito combined with this observation and the trivial fact that the FST of a canonical measure in $M(G)$ is canonical on $\Gamma=\hat{G}$. The Cohen Idempotent Theorem applied to the discrete version of an arbitrary LCAG solves problem (*) for an arbitrary LCAG when the open subset is the whole group.

In his Semi-Idempotent Theorem Kessler proved

THEOREM. *Let Γ be an arbitrary totally ordered discrete abelian group. Suppose $\hat{\mu} \in B(\Gamma)$ is integer-valued on $\Gamma_+=\{\gamma \in \Gamma \mid \gamma>0\}$. Then there exists an integer-valued $\hat{\nu} \in B(\Gamma)$ such that $\hat{\mu}=\hat{\nu}$ on Γ_+.*

The special case $\Gamma=\mathbf{R}_d$ of Kessler's theorem gives, when combined with Cohen's theorem, a solution to problem (*) when the group is $\mathbf{R}$ and the open subset is $\{\gamma \in \mathbf{R} \mid \gamma>0\}$.

DEFINITION. An open subset Ω of $\mathbf{R}^n$ is called an open half-space of $\mathbf{R}^n$ if there exists an affine mapping T of $\mathbf{R}^n$ onto $\mathbf{R}^n$ such that

$$T(\Omega) = \{(\gamma_1,\ldots,\gamma_n) \in \mathbf{R}^n \mid \gamma_1>0\}\ .$$

Using Kessler's theorem, it is moreover not hard to see that we have

THEOREM. *Let Ω be an open half-space of* $\mathbf{R}^n$. *Suppose $\hat{\mu} \in B(\mathbf{R}^n_d)$ is integer-valued on Ω. Then there exists an integer-valued $\hat{\nu} \in B(\mathbf{R}^n_d)$ such that $\hat{\mu}=\hat{\nu}$ on Ω.*

Here is the easy proof: Let T be an affine mapping of $\mathbf{R}^n$ onto $\mathbf{R}^n$ such that

$$T(\Omega) = \{(\gamma_1,\ldots,\gamma_n) \in \mathbf{R}^n \mid \gamma_1>0\} .$$

Define a total order $>$ on $\mathbf{R}^n$, by saying that $\alpha>\beta$ ($\alpha,\beta \in \mathbf{R}^n$) if $\delta_{i_{\alpha\beta}}>0$ where $T\alpha - T\beta = (\delta_1,\ldots,\delta_n)$ and $i_{\alpha\beta}=\min\{i \mid \delta_i \neq 0\}$. Then $\Omega \subset (\mathbf{R}^n_d)_+ \subset \bar{\Omega}$, where $(\mathbf{R}^n_d)_+ = \{\gamma \in \mathbf{R}^n \mid \gamma>0\}$ and $\bar{\Omega}$ is the closure of Ω in $\mathbf{R}^n$. Since $\partial\Omega=\bar{\Omega}\setminus\Omega$ is a coset of a subgroup of $\mathbf{R}^n$ (it is a hyperplane in $\mathbf{R}^n$), the characteristic function $\chi_{\partial\Omega}$ of $\partial\Omega$ is in $B(\mathbf{R}^n_d)$. Consequently the function $\hat{\mu}-\hat{\mu}\chi_{\partial\Omega}$ is in $B(\mathbf{R}^n_d)$ and since it is also integer-valued on $\bar{\Omega} \supset (\mathbf{R}^n_d)_+$ we get applying Kessler's theorem an integer-valued $\hat{\nu} \in B(\mathbf{R}^n_d)$ such that $\hat{\mu}-\hat{\mu}\chi_{\partial\Omega}=\hat{\nu}$ on $(\mathbf{R}^n_d)_+$. In particular $\hat{\mu}=\hat{\nu}$ on Ω for this $\hat{\nu}$.

This last theorem and the Cohen Idempotent Theorem are our main reasons for considering problem (*).

The first main result of this paper is

THEOREM A. *Let Γ be an arbitrary* LCAG. *Let φ be a function defined on a neighborhood of a point γ in Γ. Then the following two conditions are equivalent.*

1°. *There exists a neighborhood Ω_1 of γ such that φ on Ω_1 is integer-valued and coincides with a function in $B(\Gamma_d)$.*

2°. *There exists a neighborhood Ω_2 of γ such that φ on Ω_2 coincides with a finite sum of functions, each being canonical on Ω_2.*

Theorem A can be considered as a local version of Cohen's theorem. However, for an arbitrary LCAG Γ we have no simple explicit characterization of the functions which are canonical only on a neighborhood Ω of Γ. The set of all such functions clearly contains the restrictions to $\Omega-\Omega+\Omega$ of all functions which are canonical on the whole of Γ, but is in general larger. We shall later give an example (Example 4.1.2) showing this. The example also shows that "each being canonical on Ω_2" in 2° of Theorem A cannot in general be replaced by "each being canonical on Γ". For an arbitrary LCAG Γ and an arbitrary open subset Ω of Γ, Theorem A determines the local behavior on Ω of a function in $B(\Gamma_d)$ which is integer-valued on Ω. Theorem A therefore gives at least a partial solution to problem (*) in the general case.

When the group in problem (*) is $\mathbf{R}^n$, we have in many cases results similar to Kessler's theorem.

DEFINITION. An open convex subset Ω of $\mathbf{R}^n$ is called an open convex slice of

$\mathbf{R}^n$ if there exist an affine mapping T of $\mathbf{R}^n$ onto $\mathbf{R}^n$, non-negative integers k and l with $k+l=n$, and a bounded open convex subset Δ of $\mathbf{R}^k$, such that $T(\Omega)=\Delta\times\mathbf{R}^l$.

The second main result of this paper is

THEOREM B. *Let Ω be an open convex slice of* $\mathbf{R}^n$. *Suppose $\hat{\mu}\in B(\mathbf{R}_d^n)$ is integer-valued on Ω. Then there exists an integer-valued $\hat{\nu}\in B(\mathbf{R}_d^n)$ such that $\hat{\mu}=\hat{\nu}$ on Ω.*

It should be observed that Ω in Theorem B can in particular be allowed to be an arbitrary bounded open convex subset of $\mathbf{R}^n$, for every such set is an open convex slice of $\mathbf{R}^n$. However, no open half-space of $\mathbf{R}^n$ is an open convex slice of $\mathbf{R}^n$. The corresponding theorem for an open half-space of $\mathbf{R}^n$ given above is therefore not implied by Theorem B. It is conceivable that Ω in Theorem B can be allowed to be an arbitrary open convex subset of $\mathbf{R}^n$, but we cannot prove that. An example and three remarks given at the end of this paper (Section 4.2) will further illustrate the situation on $\mathbf{R}^n$.

Both the Theorems A and B are consequences of the following theorem.

THEOREM C. *Let Ω be a bounded open subset of* $\mathbf{R}^n$. *Let Γ be an arbitrary* LCAG. *Suppose $\hat{\mu}\in B(\mathbf{R}_d^n\times\Gamma)$ is integer-valued on $\Omega\times\Gamma$. Then for each point $\gamma'\in\Omega$ there exist a neighborhood Ω' of γ' and an integer-valued $\hat{\nu}\in \mathrm{B}(\mathbf{R}_d^n\times\Gamma)$ such that $\hat{\mu}=\hat{\nu}$ on $\Omega'\times\Gamma$.*

Theorems A and B follow from Theorem C using respectively structure theorems for LCAGs and a patching argument partly based on Davenport [3]. The proof we shall give of Theorem C is inspired by Amemiya's and Ito's [1] short and elegant proof of Cohen's theorem. A detailed description of their method can be found in Meyer [9, p. 199]. Here also note that Theorem C (with $\Gamma=\{0\}$) shows, when combined with Cohen's theorem, that if $\Gamma=\mathbf{R}^n$ in Theorem A, then "each being canonical on Ω_2 in 2° of Theorem A can be replaced by "each being canonical on $\mathbf{R}^n$".

In what follows the following conventions will be made. Γ and G (with corresponding indices) is a pair of dual LCAGs, and if not otherwise is maid clear, the dual pair is arbitrary. G is regarded as a subgroup of $\bar{G}=(\Gamma_d)\hat{}$, the Bohr compactification of G. A $\gamma\in\Gamma$ is sometime thought of as simply a point in Γ, sometimes as a character on G, and sometimes as a character on $\bar{G}$. It will be clear from the context what is meant. The Haar measure of a compact group is always normalized so that the whole group has measure one. The empty set is considered as a linear subspace of $\mathbf{R}^n$. The circle group $\mathbf{T}$ is identified with $\mathbf{R}/\mathbf{Z}$.

The following facts will tacitly be used. If $K\subset G$ is compact in the topology of G, it is also compact in the topology of $\bar{G}$ (this follows e.g. from Rudin [10, p.

30, Theorem 1.8.2]). $M(G)$ can be naturally embedded in $M(\bar{G})$ (see e.g. Hewitt and Ross [5, p. 303, Theorem 33.19]).

Finally, I wish to thank professor Yngve Domar for his support and kind interest in my work.

1. Theorem C.

1.1. *A convergence theorem* (*Theorem* 1.1.4).

In their proof of Cohen's theorem Amemiya and Ito [1] used the following consequence of Helson's translation lemma (for Helson's translation lemma, see e.g. Rudin [10, p. 66, Lemma 3.5.1]). If $\mu, \nu \in M(G)$, if ν is canonical, and if the net $\{\gamma_\alpha \mu\}$ converges to ν in the weak* topology of $M(G)$ (the $\sigma(M(G), C_0(G))$ topology), then $\gamma_\alpha \mu|_H = \nu$ eventually, where H is the support group of ν. Theorem 1.1.4 is our substitute for this. It is what we need in our proof of Theorem C. We begin with some measure theoretic considerations.

LEMMA 1.1.1. *Suppose G is a* LCAG, *F is a σ-compact and H a closed subgroup of G, and $F \cap H = \{0\}$. Then $F+E$ is a Borel subset of G whenever E is a Borel subset of H. (H is endowed with the restriction topology from G.)*

PROOF. The Borel subsets of a topological set is the σ-algebra generated by its closed subsets. Since $F \cap H = \{0\}$, and since H is a closed subgroup of G, it is therefore enough to prove that $F+E$ is a Borel subset of G whenever E is a closed subset of G and $E \subset H$. But, since F is σ-compact, this is evident, using the fact that the sum of a compact and a closed subset of a LCAG is closed.

Now, let G, F and H be as in Lemma 1.1.1 and let $\tau \in M(G)$. Define the set function $\pi\tau$ on the Borel subsets of H by $\pi\tau(E) = \tau(F+E)$ whenever E is a Borel subset of H. Lemma 1.1.1 shows that this definition is consistent with τ being defined only on the Borel subsets of G. With these notations we have

LEMMA 1.1.2. $\pi\tau \in M(H)$.

PROOF. Since $F \cap H = \{0\}$, $\pi\tau$ is countably additive on the Borel subsets of H, and therefore a Borel measure on H. Obviously, $\|\pi\tau\| \leqq \|\tau\| < \infty$. We now prove that $\pi\tau$ is inner regular. Let E be an arbitrary Borel subset of H and choose an arbitrary $\varepsilon > 0$. Since τ is regular there is a compact subset K of G such that $K \subset F+E$ and $|\tau|(F+E) < |\tau|(K) + \varepsilon$. Clearly, $K \subset F+K \subset F+E$. It follows that $|\tau|(F+E) < |\tau|(F+K) + \varepsilon$. Since $F \cap H = \{0\}$, we have $F+K = F+(H \cap (F+K))$. We conclude that $H \cap (F+K) \subset E$. But $F = \bigcup_{i=1}^{\infty} C_i$ for certain compact subsets $C_1, C_2, \ldots$ of G. Thus

$$H \cap (F+K) = H \cap \left(\left(\bigcup_{i=1}^{\infty} C_i\right)+K\right) = \bigcup_{i=1}^{\infty} H \cap (C_i+K) .$$

Choose N so large that

$$|\tau|(F+K) = |\tau|\left(F+\bigcup_{i=1}^{\infty} H \cap (C_i+K)\right) < |\tau|\left(F+\bigcup_{i=1}^{N} H \cap (C_i+K)\right)+\varepsilon .$$

Set $D=\bigcup_{i=1}^{N} H \cap (C_i+K)$. Then D is a compact subset of H and $D \subset E$. Also

$$|\pi\tau|(E) = |\tau|(F+E) < |\tau|(F+K)+\varepsilon < |\tau|(F+D)+2\varepsilon = |\pi\tau|(D)+2\varepsilon .$$

Hence $\pi\tau$ is inner regular. Since $\|\pi\tau\| < \infty$, outer regularity follows from inner regularity.

DEFINITION 1.1.3. If E is a subset of $\mathsf{R}^n \times \Gamma$ let $\boldsymbol{P}(E)=\boldsymbol{P}_{n,\Gamma}(E)$ be the projection of E onto R^n and let $\boldsymbol{CLP}\,(E)=\boldsymbol{CLP}_{n,\Gamma}\,(E)$ be the closure of $\boldsymbol{P}(E)$ in R^n.

We can now prove

THEOREM 1.1.4. *Let* $\mu \in M(\bar{\hat{\mathsf{R}}}^n \times G)$. *Let* $\nu \in M(\bar{\hat{\mathsf{R}}}^n \times G)$ *be a canonical measure such that* $\boldsymbol{CLP}\,(\hat{\nu}^{-1}(\mathsf{Z} \setminus \{0\}))$ *is a non-empty linear subspace of* R^n. *Let* $\varrho \in M(\hat{\mathsf{R}}^n) \subset M(\bar{\hat{\mathsf{R}}}^n \times G)$ *be such that* $\hat{\varrho}(0,0) \neq 0$. *Let* $\{\gamma_\alpha = (\gamma'_\alpha, \gamma''_\alpha),\ \alpha \in A\}$ *be a net in* $\mathsf{R}^n_d \times \Gamma$ *such that* $\lim_\alpha \gamma'_\alpha = 0$ *in the topology of* R^n. *Suppose* $\lim_\alpha \{\gamma_\alpha(\varrho * \mu)\} = \varrho * \nu$ *in the weak* topology of* $M(\bar{\hat{\mathsf{R}}}^n \times G)$. *Then there exists a* σ*-compact subgroup* F *of* $\bar{\hat{\mathsf{R}}}^n \times G$ *such that*

$$\lim_\alpha \|\gamma_\alpha(\varrho * \mu)|_F - \varrho * \nu\| = 0 .$$

PROOF. Set $\Lambda=\boldsymbol{CLP}\,(\hat{\nu}^{-1}(\mathsf{Z} \setminus \{0\}))$ and let Σ be the orthogonal complement of Λ in R^n. Then $\mathsf{R}^n=\Lambda+\Sigma$, and $\Lambda \cap \Sigma = \{0\}$. Let Q be the orthogonal projection of R^n onto Λ. Q is a continuous homomorphism of R^n into R^n. Define the function φ on $\mathsf{R}^n \times \Gamma$ by

$$\varphi(\gamma', \gamma'') = \hat{\varrho}(Q(\gamma'), \gamma'') \qquad ((\gamma', \gamma'') \in \mathsf{R}^n \times \Gamma) .$$

Then $\varphi \in B(\mathsf{R}^n \times \Gamma)$ (see e.g. Rudin [10, p. 79, Lemma 4.2.1]), i.e. $\varphi = \hat{\varrho'}$ for some $\varrho' \in M(\hat{\mathsf{R}}^n \times G) \subset M(\bar{\hat{\mathsf{R}}}^n \times G)$. Note that ϱ' is constructed so that $\hat{\varrho'} = \hat{\varrho}$ whenever $\hat{\nu} \neq 0$. Thus $\varrho' * \nu = \varrho * \nu$. We also have $\hat{\varrho'}(0,0) = \hat{\varrho}(0,0)$ $(\neq 0)$. Moreover,

$$\hat{\varrho'}(\gamma + \Sigma \times \Gamma) = \hat{\varrho'}(\gamma) \qquad \text{for all } \gamma \in \mathsf{R}^n \times \Gamma .$$

Hence supp (ϱ') is included in the annihilator of $\Sigma \times \Gamma$ in $\bar{\mathbf{R}}^n \times G$ (see e.g. Rudin [10, p. 53, Theorem 2.7.1]), i.e. supp $(\varrho') \subset \hat{\Lambda} \times \{0\}$ $(\bar{\mathbf{R}}^n = \hat{\Lambda} + \hat{\Sigma},\ \hat{\Lambda} \cap \hat{\Sigma} = \{0\})$. Let $H \subset \bar{\mathbf{R}}^n \times G$ be the support group of ν. We claim that the theorem is true with $F = (\hat{\Lambda} \times \{0\}) + H$. Note that $\hat{\Lambda} \times \{0\}$ is a σ-compact and H a compact subgroup of $\bar{\mathbf{R}}^n \times G$. Therefore F is a σ-compact subgroup of $\bar{\mathbf{R}}^n \times G$. Now suppose $x \in (\hat{\Lambda} \times \{0\}) \cap H$. Since $x \in H$, $(x, \gamma) = 1$ for all $\gamma \in H^{\perp}$, where $H^{\perp}$ is the annihilator of H in $\mathbf{R}_d^n \times \Gamma$ (1). Set $\Delta = P(H^{\perp})$. Since ν is canonical, $P(\hat{\nu}^{-1}(\mathbf{Z} \setminus \{0\}))$ is the union of finitely many translates of Δ. We conclude that Δ is dense in Λ in the topology of $\mathbf{R}^n$ (2). Since $x \in \hat{\Lambda} \times \{0\}$, x considered as a character on $\mathbf{R}_d^n \times \Gamma$ is continuous in the topology of $\mathbf{R}^n \times \Gamma$ and independent of $\Sigma \times \Gamma$ (3). Combining (1), (2), and (3) we see that $((x, \gamma) = 1$ for all $\gamma \in \mathbf{R}^n \times \Gamma$, and consequently $x = 0$. We have shown that $(\hat{\Lambda} \times \{0\}) \cap H = \{0\}$. If $\tau \in M(\bar{\mathbf{R}}^n \times G)$, we define the set function $\pi\tau$ on the Borel subsets of H by setting

$$\pi\tau(E) = \tau((\hat{\Lambda} \times \{0\}) + E)$$

whenever E is a Borel subset of H. Lemmas 1.1.1 and 1.1.2 show respectively that this definition is justified and that $\pi\tau \in M(H)$. Set $\mu' = (\varrho * \mu)|_F$. Let the functions k: $(\hat{\Lambda} \times \{0\}) + H \to \hat{\Lambda} \times \{0\}$ and h: $(\hat{\Lambda} \times \{0\}) + H \to H$ be such that $x = k(x) + h(x)$ when $x \in (\hat{\Lambda} \times \{0\}) + H$. Since $(\hat{\Lambda} \times \{0\}) \cap H = \{0\}$, k and h are uniquely defined. The rest of the proof is divided into five steps.

STEP (i). Let $K \subset \hat{\Lambda} \times \{0\}$ be compact. Then $k|_{K+H}$ and $h|_{K+H}$ are continuous as functions of $K+H$ (endowed with the restriction topology from $\bar{\mathbf{R}}^n \times G$) into $\bar{\mathbf{R}}^n \times G$.

Endow K, H and $K+H$ with their respective restriction topologies from $\bar{\mathbf{R}}^n \times G$. Using $(\hat{\Lambda} \times \{0\}) \cap H = \{0\}$ it is easy to see that the function $K \times H \ni (x, y) \to x + y \in K + H$ is a continuous injection of a compact space onto a Hausdorff space. But as is well-known such a mapping has a continuous inverse. Since the inverse is the mapping $(k|_{K+H}, h|_{K+H})$: $K + H \to K \times H$, this proves Step i.

Step i will tacitly be used in the remaining parts of the proof.

STEP ii. $\lim_\alpha \pi(\gamma_\alpha \mu') = \hat{\varrho}'(0)\nu$ in the weak* topology of $M(H)$.

Let E be an arbitrary Borel subset of H. Using supp $(\varrho') \subset \hat{\Lambda} \times \{0\}$, supp $(\nu) \subset H$, and $(\hat{\Lambda} \times \{0\}) \cap H = \{0\}$, we get

$$\pi(\varrho' * \nu)(E) = (\varrho' * \nu)((\hat{\Lambda} \times \{0\}) + E) = \int_{\bar{\mathbf{R}}^n \times G} \varrho'((\hat{\Lambda} \times \{0\}) + E - x)\, d\nu(x)$$

$$= \int_H \varrho'((\hat{\Lambda} \times \{0\}) + E - x)\, d\nu(x) = \hat{\varrho}'(0) \int_E d\nu(x) = \hat{\varrho}'(0)\nu(E).$$

Hence $\pi(\varrho' * \nu) = \hat{\varrho}'(0)\nu$. Now fix an arbitrary $f \in C(H)$ and choose an arbitrary $\varepsilon > 0$. Since both μ' and $\varrho' * \nu$ are concentrated on $(\hat{\Lambda} \times \{0\}) + H$, and since $\hat{\Lambda} \times \{0\}$ is σ-compact in the topology of $\hat{\mathbf{R}}^n \times G$, there is a subset K of $\hat{\Lambda} \times \{0\}$ such that K is compact in the topology of $\hat{\mathbf{R}}^n \times G$, and

$$\|\mu' - \mu'|_{K+H}\| < \varepsilon \quad \text{and} \quad \|\varrho' * \nu - (\varrho' * \nu)|_{K+H}\| < \varepsilon .$$

Set $\lambda = \mu'|_{K+H}$ and $\sigma = (\varrho' * \nu)|_{K+H}$. Define the function g on $K+H$ by $g(x) = f(h(x))$, $x \in K+H$. g is a continuous complex-valued function on $K+H$, and $K+H$ is a compact subset of $\bar{\hat{\mathbf{R}}}^n \times G$. Tietze's extension theorem shows that g can be extended to a continuous complex-valued function which is defined on the entire $\bar{\hat{\mathbf{R}}}^n \times G$, vanishes at infinity, and has supremum norm equal to $\sup_{x \in K+H} |g(x)| = \sup_{x \in H} |f(x)|$. Fix such an extension and call it g, too. We have

$$\left| \int_{\bar{\hat{\mathbf{R}}}^n \times G} g\gamma_\alpha \, d\mu' - \int_H f \, d(\pi(\gamma_\alpha \mu')) \right| \leqq \left| \int_{K+H} g\gamma_\alpha \, d\lambda - \int_H f \, d(\pi(\gamma_\alpha \lambda)) \right| + 2\|f\|_\infty \varepsilon .$$

But

$$\int_{K+H} g\gamma_\alpha \, d\lambda = \int_H f \, d(\pi(\gamma_\alpha \lambda)) ,$$

as is easy to see. Thus

$$\left| \int_{\bar{\hat{\mathbf{R}}}^n \times G} g\gamma_\alpha \, d\mu' - \int_H f \, d(\pi(\gamma_\alpha \mu')) \right| \leqq 2\|f\|_\infty \varepsilon . \tag{1}$$

Similarly we get

$$\begin{aligned} \left| \int_{\bar{\hat{\mathbf{R}}}^n \times G} g \, d(\varrho' * \nu) - \hat{\varrho}'(0) \int_H f \, d\nu \right| &= \left| \int_{\bar{\hat{\mathbf{R}}}^n \times G} g \, d(\varrho' * \nu) - \int_H f \, d(\pi(\varrho' * \nu)) \right| \\ &\leqq \left| \int_{K+H} g \, d\sigma - \int_H f \, d\pi\sigma \right| + 2\|f\|_\infty \varepsilon = 2\|f\|_\infty \varepsilon . \end{aligned} \tag{2}$$

Since $\{\gamma_\alpha(\varrho * \mu)\}$ converges to $\varrho * \nu = \varrho' * \nu$ in the weak* topology of $M(G)$, it is easy to see that $\{\gamma_\alpha \mu'\}$ converges to $\varrho * \nu$ in the weak* topology of $M(G)$. (Recall that $\mu' = (\varrho * \mu)|_F$ and $\operatorname{supp}(\varrho' * \nu) \subset F$.) Hence there exists an $\alpha_0 \in A$ such that

$$\left| \int_{\bar{\hat{\mathbf{R}}}^n \times G} g\gamma_\alpha \, d\mu' - \int_{\bar{\hat{\mathbf{R}}}^n \times G} g \, d(\varrho' * \nu) \right| < \varepsilon \quad \text{if } \alpha \geqq \alpha_0 . \tag{3}$$

Combining (1), (2), and (3) we see that

$$\left| \int_H f \, d(\gamma_\alpha \mu')) - \hat{\varrho}'(0) \int_H f \, d\nu \right| < 4\|f\|_\infty \varepsilon + \varepsilon \quad \text{if } \alpha \geqq \alpha_0 .$$

This proves Step ii.

STEP iii. $\lim_\alpha \|\pi(\gamma_\alpha\mu') - \gamma_\alpha\pi\mu'\| = 0$.

Choose an arbitrary $\varepsilon > 0$. As in Step ii, there is a subset K of $\hat{\Lambda} \times \{0\}$ such that K is compact in the topology of $\hat{\mathbf{R}}^n \times G$ and $\|\mu' - \mu'|_{K+H}\| < \varepsilon$. Set $\lambda = \mu'|_{K+H}$. Since $K \subset \hat{\Lambda} \times \{0\}$ and K is compact in the topology of $\hat{\mathbf{R}}^n \times G$, and since $\lim_\alpha \gamma'_\alpha = 0$ in the topology of $\mathbf{R}^n$, there is an $\alpha_0 \in A$ such that $|(x, \gamma_\alpha) - 1| < \varepsilon$ if $x \in K$ and $\alpha \geqq \alpha_0$. We have for $\alpha \geqq \alpha_0$

$$
\begin{aligned}
\|\pi(\gamma_\alpha\mu') - \gamma_\alpha\pi\mu'\| &= \|\pi(\gamma_\alpha\mu') - \pi((\gamma_\alpha \circ h)\mu')\| \\
\leqq \|\gamma_\alpha\mu' - (\gamma_\alpha \circ h)\mu'\| &= \|((\gamma_\alpha \circ k) - 1)\mu'\| \\
\leqq \|((\gamma_\alpha \circ k) - 1)(\mu' - \lambda)\| &+ \|((\gamma_\alpha \circ k) - 1)\lambda\| < 2\varepsilon + \|\lambda\|\varepsilon < 2\varepsilon + \|\mu'\|\varepsilon .
\end{aligned}
$$

This proves Step iii.

STEP iv. $\{\gamma_\alpha|_H, \alpha \in A\}$ is eventually in a finite set of $\hat{H}$.

Write $\gamma_\alpha|_H = \gamma_\alpha^H$. Suppose $\{\gamma_\alpha^H\}$ is not eventually in any finite subset of $\hat{H}$. Then there is a subnet $\{\gamma_\beta^H\}$ of $\{\gamma_\alpha^H\}$ such such that $\{\gamma_\beta^H\}$ is eventually in the complement of each finite subset of $\hat{H}$ (see e.g. Kelley [6, p. 70 Lemma 5]). Steps ii and iii together show that $\lim_\alpha \{\gamma_\alpha^H \pi\mu'\} = \hat{\varrho}'(0)\nu$ in the weak* topology of $M(H)$. Since $\{\gamma_\beta^H\}$ is a subnet of $\{\gamma_\alpha^H\}$, also $\lim_\beta \{\gamma_\beta^H \pi\mu'\} = \hat{\varrho}'(0)\nu$ in the weak* topology of $M(H)$. Using that the trigonometric polynomials are dense in $C(H)$, we conclude that $\hat{\varrho}'(0)\nu$ is singular with respect to m_H, the Haar measure of H. (Compare Lemma 3.5.1 (Helson's translation lemma) in Rudin [10, p. 66].) But we have chosen H to be the support group of the not identically zero canonical measure ν, and therefore $\nu \neq 0$ and ν is absolutely continuous with respect to m_H. Also $\hat{\varrho}'(0) \neq 0$. It follows that our assumption that $\{\gamma_\alpha^H\}$ is not eventually in any finite subset of $\hat{H}$ is false.

This proves Step iv.

STEP v. $\lim_\alpha \|\gamma_\alpha\mu' - \varrho * \nu\| = 0$.

Choose an arbitrary $\varepsilon > 0$. As in step ii, there is a subset of K of $\hat{\Lambda} \times \{0\}$ such that K is compact in the topology of $\hat{\mathbf{R}}^n \times G$ and $\|\mu' - \mu'|_{K+H}\| < \varepsilon$. Set $\lambda = \mu'|_{K+H}$. As in Step iii, there is an $\alpha_0 \in A$ such that $|(x, \gamma_\alpha) - 1| < \varepsilon$ if $x \in K$ and $\alpha \geqq \alpha_0$. By Step iv $\{\gamma_\alpha|_H \mid \alpha \in A\}$ is eventually in a finite subset, say $\{\theta_1, \ldots, \theta_N\}$, of $\hat{H}$. After a renumbering if necessary we may suppose that $\{\gamma_\alpha|_H \mid \alpha \in A\}$ is frequently in each of the sets $\{\theta_1\}, \ldots, \{\theta_M\}$, and not frequently in any of the sets $\{\theta_{M+1}\}, \ldots, \{\theta_N\}$ (for some M, $1 \leqq M \leqq N$). For each $i \in \{1, \ldots, M\}$, set $A_i = \{\alpha \in A \mid \gamma_\alpha|_H = \theta_i\}$. Then each $A_1, \ldots, A_M$ is a cofinal subset of A, and $\bigcup_{i=1}^M A_i$ is a residual subset of A.

Now fix an arbitrary $i \in \{1, \ldots, M\}$. Choose $\alpha_{1i} \in A_i$ so that $\alpha_{1i} \geqq \alpha_0$. This is possible since A_i is a cofinal subset of A. If $\alpha \in A_i$ and $\alpha \geqq \alpha_{1i}$ we have

$$(1) \qquad \|\gamma_\alpha\mu' - \gamma_{\alpha_{1i}}\mu'\| < 2\varepsilon + \|\gamma_\alpha\lambda - \gamma_{\alpha_{1i}}\lambda\|$$

$$= 2\varepsilon + \int_{K+H} |(k(x),\gamma_\alpha)(h(x),\gamma_\alpha) - (k(x),\gamma_{\alpha_{1i}})(h(x),\gamma_{\alpha_{1i}})|\, d|\lambda|(x)$$

$$= 2\varepsilon + \int_{K+H} |(k(x),\gamma_\alpha) - (k(x),\gamma_{\alpha_{1i}})|\, d|\lambda|(x) < 2\varepsilon + 2\|\lambda\|\varepsilon \leqq 2\varepsilon + 2\|\mu'\|\varepsilon .$$

Choose $f_i \in C_0(\bar{\mathbf{R}}^n \times G)$ with $\|f_i\|_\infty \leqq 1$ such that

$$(2) \qquad \left| \|\gamma_{\alpha_{1i}}\mu' - \varrho' * \nu\| - \int_{\bar{\mathbf{R}}^n \times G} f_i(\gamma_{\alpha_{1i}}\, d\mu' - d(\varrho' * \nu)) \right| \leqq \varepsilon .$$

Since $\lim_\alpha \gamma_\alpha\mu' = \varrho' * \nu$ in the weak* topology of $M(\bar{\mathbf{R}}^n \times G)$, there is an $\alpha_{2i} \in A_i$ such that for $\alpha \geqq \alpha_{2i}$

$$(3) \qquad \left| \int_{\bar{\mathbf{R}}^n \times G} f_i(\gamma_\alpha d\mu' - d(\varrho' * \nu)) \right| < \varepsilon .$$

Choose $\alpha_{3i} \in A_i$ so that $\alpha_{3i} \geqq \alpha_{1i}$ and $\alpha_{3i} \geqq \alpha_{2i}$. Using (1), (2), and (3) we get if $\alpha \in A_i$ and $\alpha \geqq \alpha_{3i}$

$$\|\gamma_\alpha\mu' - \varrho' * \nu\| < 2\varepsilon + 2\|\mu'\|\varepsilon + \|\gamma_{\alpha_{1i}}\mu' - \varrho' * \nu\|$$

$$< 3\varepsilon + 2\|\mu'\|\varepsilon + \left| \int_{\bar{\mathbf{R}}^n \times G} f_i(\gamma_{\alpha_{1i}} d\mu' - d(\varrho' * \nu)) \right|$$

$$\leqq 3\varepsilon + 2\|\mu'\|\varepsilon + \left| \int_{\bar{\mathbf{R}}^n \times G} f_i(\gamma_{\alpha_{3i}} d\mu' - d(\varrho' * \nu)) \right| + \left| \int_{\bar{\mathbf{R}}^n \times G} f_i(\gamma_{\alpha_{1i}} - \gamma_{\alpha_{3i}})\, d\mu' \right|$$

$$< 4\varepsilon + 2\|\mu'\|\varepsilon + \left| \int_{(\bar{\mathbf{R}}^n \times G) \setminus (K+H)} f_i(\gamma_{\alpha_{1i}} - \gamma_{\alpha_{3i}})\, d\mu' \right| + \left| \int_{K+H} f_i(\gamma_{\alpha_{1i}} - \gamma_{\alpha_{3i}})\, d\mu' \right|$$

$$< 6\varepsilon + 2\|\mu'\|\varepsilon + \left| \int_{K+H} f_i(x)((k(x),\gamma_{\alpha_{1i}})(h(x),\gamma_{\alpha_{1i}}) - (k(x),\gamma_{\alpha_{3i}})(h(x),\gamma_{\alpha_{3i}}))\, d\mu'(x) \right|$$

$$= 6\varepsilon + 2\|\mu'\|\varepsilon + \left| \int_{K+H} f_i(x)((k(x),\gamma_{\alpha_{1i}}) - (k(x),\gamma_{\alpha_{3i}}))\, d\mu'(x) \right| < 6\varepsilon + 4\|\mu'\|\varepsilon .$$

But we had fixed an arbitrary $i \in \{1,\ldots,M\}$. Thus, for each $i \in \{1,\ldots,M\}$ we have proved that there exists an $\alpha_{3i} \in A_i$ such that if $\alpha \in A_i$ and $\alpha \geqq \alpha_{3i}$, then $\|\gamma_\alpha\mu' - \varrho' * \nu\| < 6\varepsilon + 4\|\mu'\|\varepsilon$. Choose $\alpha_4 \in A$ so that $\alpha_4 \geqq \alpha_{31},\ldots,\alpha_4 \geqq \alpha_{3M}$. Then, if $\alpha \in \bigcup_{i=1}^M A_i$ and $\alpha \geqq \alpha_4$, we have

$$\|\gamma_\alpha\mu' - \varrho' * \mu\| < 6\varepsilon + 4\|\mu'\|\varepsilon .$$

Since $\bigcup_{i=1}^{M} A_i$ is a residual subset of A, this proves Step v.

Since $\mu' = (\varrho * \mu)|_F$ and $\varrho' * \nu = \varrho * \nu$ the theorem is proved, too.

1.2. *A theorem for an integer-valued function* (*Theorem* 1.2.3).

In this section we essentially show that if a function is canonical on a vertical strip in $\mathbf{R}^n \times \Gamma$, then on a narrower vertical strip it coincides with a function which is canonical on the whole of $\mathbf{R}^n \times \Gamma$. Theorem 1.2.3 of this section gives a somewhat more precise result, necessary for our proof of Theorem C. We begin with two lemmas. The first of which we referred to when we stated our version of Cohen's theorem.

LEMMA 1.2.1. *Suppose the function φ is continuous and canonical on Γ. Then φ is the* FST *of a canonical measure belonging to $M(G)$.*

PROOF. If $\varphi \equiv 0$ the lemma is trivial. We suppose

$$A = \{\varphi(\cdot + \gamma) \mid \gamma \in \Gamma \text{ and } \varphi(\gamma) \neq 0\}$$

has N $(N>0)$ distinct elements. We can write $A = \{\varphi(\cdot + \gamma_1), \ldots, \varphi(\cdot + \gamma_N)\}$ for certain $\gamma_1, \ldots, \gamma_N \in \Gamma$. Set $A_i = \{\gamma \in \Gamma \mid \varphi(\cdot + \gamma) = \varphi(\cdot + \gamma_i)\}$ $(i = 1, \ldots, N)$. Now note that

1°. $A_1 - \gamma_1 = \ldots = A_N - \gamma_N$.

2°. Since φ is continuous, each of the sets $A_1 - \gamma_1, \ldots, A_N - \gamma_N$ is clopen.

3°. If $\gamma_i', \gamma_i'' \in A_i - \gamma_i$, then $\varphi(\gamma_i + \gamma_i' - \gamma_i'' + \gamma) = \varphi(\gamma_i - \gamma_i'' + \gamma) =$
$= \varphi(\gamma_i + \gamma_i'' - \gamma_i'' + \gamma) = \varphi(\gamma_i + \gamma)$, i.e. $\gamma_i' - \gamma_i'' \in A_i - \gamma_i$.

1°, 2°, and 3° together imply that there is a clopen subgroup Λ of Γ such that $A_1 = \gamma_1 + \Lambda, \ldots, A_N = \gamma_N + \Lambda$. Let H be the annihilator of Λ in G. Since Λ is open, H is compact (see e.g. Rudin [10, p. 35, Theorem 2.1.2]). Set $\mu = (\varphi(\gamma_1)\bar{\gamma}_1 + \ldots + \varphi(\gamma_N)\bar{\gamma}_N)m_H$, where m_H is the Haar measure of H. Then $\mu \in M(G)$, μ is canonical, and as is easy to see $\varphi = \hat{\mu}$.

LEMMA 1.2.2. *Suppose $\mu \in M(\bar{\hat{\mathbf{R}}}^n \times G)$ is canonical. Then there exist a canonical measure $\nu \in M(\bar{\hat{\mathbf{R}}}^n \times G)$ with* $\boldsymbol{CLP}$ $(\hat{\nu}^{-1}(\mathbf{Z} \setminus \{0\}))$ *equal to a linear subspace of* $\mathbf{R}^n$ *and a neighborhood Ω of the origin in* $\mathbf{R}^n$ *such that $\hat{\mu} = \hat{\nu}$ on $\Omega \times \Gamma$.*

PROOF. Set $\Delta = \boldsymbol{CLP}(\hat{\mu}^{-1}(\mathbf{Z} \setminus \{0\}))$. Since μ is canonical, Δ is either the empty set or is the union of finitely many translates of a closed subgroup of $\mathbf{R}^n$. Hence there is a unique linear subspace Λ of $\mathbf{R}^n$ such that $\Delta \cap \Omega = \Lambda \cap \Omega$ whenever Ω is a sufficiently small neighborhood of the origin in $\mathbf{R}^n$. Let H be the obviously compact subgroup of $\bar{\hat{\mathbf{R}}}^n \times G$ which is the annihilator of $\Lambda \times \Gamma$ in $\bar{\hat{\mathbf{R}}}^n \times G$ (H should be interpreted as $\bar{\hat{\mathbf{R}}}^n \times \{0\}$ if $\Lambda = \emptyset$.) Set $\nu = \mu * m_H$, where m_H is the Haar measure of H. Then, as is easy to see, ν is canonical,

$\boldsymbol{CLP}\,(\hat{\nu}^{-1}(\mathsf{Z}\setminus\{0\}))$ equals the linear subspace Λ of R^n, and $\hat{\mu}=\hat{\nu}$ on $\Omega\times\Gamma$ whenever Ω is a sufficiently small neighborhood of the origin in R^n.

We can now prove

THEOREM 1.2.3. *Let Ω be a neighborhood of the origin in R^n. Let $\varphi(\gamma',\gamma'')$ be an integer-valued function on $(\Omega+\Omega)\times\Gamma$ which for each fixed $\gamma'\in\Omega+\Omega$ is continuous on Γ $(\gamma=(\gamma',\gamma'')\in\mathsf{R}^n\times\Gamma)$. Suppose the set $\{\varphi(\,.\,+\gamma)|_{\Omega\times\Gamma}\mid\gamma\in\Omega\times\Gamma$ and $\varphi(\gamma)\neq 0\}$ is finite. Then there exist a canonical measure $\mu\in M(\bar{\mathsf{R}}^n\times G)$ with $\boldsymbol{CLP}\,(\hat{\mu}^{-1}(\mathsf{Z}\setminus\{0\}))$ equal to a linear subspace of R^n and a neighborhood Ω' of the origin in R^n such that $\varphi=\hat{\mu}$ on $\Omega'\times\Gamma$.*

PROOF. For $h>0$ set $\Omega_h=\{\gamma'\in\mathsf{R}^n\mid|\gamma'|\leqq h\}$. Choose $h_0>0$ so that $\Omega_{h_0}+\Omega_{h_0/2n}\subset\Omega$. Set $\Omega_0=\Omega_{h_0}$, $\Omega_1=\Omega_{h_0/n}$ and $\Omega_2=\Omega_{h_0/2n}$. Let Δ be the set of all points $\gamma\in\Omega_1\times\Gamma$ such that $\varphi(\cdot)=\varphi(\cdot+\gamma)$ on $\Omega_0\times\Gamma$. Let Λ be the linear span of $\boldsymbol{P}(\Delta)$ in R^n. Now suppose there exists an infinite sequence $(\gamma_i=(\gamma_i',\gamma_i''))_{i=1}^{\infty}\subset\Omega_2\times\Gamma$ such that each $\varphi(\gamma_i)\neq 0$, each $\gamma_i'\notin\Lambda$, and $\lim_i\gamma_i'=0$. Since each $\varphi(\gamma_i)\neq 0$, and since $\{\varphi(\cdot+\gamma)|_{\Omega\times\Gamma}\mid\gamma\in\Omega\times\Gamma$ and $\varphi(\gamma)\neq 0\}$ is finite, it is easy to see that there is an infinite subsequence $(\gamma_{i_j})_{j=1}^{\infty}$ of $(\gamma_i)_{i=1}^{\infty}$ such that

$$(1)\qquad \varphi(\cdot+\gamma_{i_1}) = \varphi(\cdot+\gamma_{i_2}) = \dots \quad \text{on } \Omega\times\Gamma\,.$$

Since $\Omega\supset\Omega_0-\Omega_2$, (1) implies that

$$(2)\qquad \varphi(\cdot) = \varphi(\cdot+\gamma_{i_1}-\gamma_{i_2}) = \varphi(\cdot+\gamma_{i_1}-\gamma_{i_3}) = \dots \quad \text{on } \Omega_0\times\Gamma\,.$$

Since $\Omega_2-\Omega_2\subset\Omega_1$, (2) shows that each $\gamma_{i_1}-\gamma_{i_j}\in\Delta$. In particular each $\gamma_{i_1}'-\gamma_{i_j}'\in\Lambda$. Since $\gamma_{i_1}'\notin\Lambda$ and $\lim_j\gamma_{i_j}=0$, this is a contradiction. We conclude that there is a neighborhood Ω_3 of the origin in R^n such that if $\gamma=(\gamma',\gamma'')\in\Omega_3\times\Gamma$ and $\varphi(\gamma)\neq 0$, then $\gamma'\in\Lambda$. Let m be the dimension of Λ. Choose $\alpha_1=(\alpha_1',\alpha_1''),\dots,\alpha_m=(\alpha_m',\alpha_m'')\in\Omega_1\times\Gamma$ so that each $\varphi(\cdot+\alpha_i)=\varphi(\cdot)$ on $\Omega_0\times\Gamma$, and so that $\alpha_1',\dots,\alpha_m'$ span Λ. This is possible because of the definition of Λ. Let E be the set of all $\gamma'\in\mathsf{R}^n$ which can be written $\gamma'=\varepsilon_1\alpha_1+\dots+\varepsilon_m\alpha_m$, where each $\varepsilon_i\in\,]-\frac{1}{2},\frac{1}{2}]$. Note that

$$\gamma'\in E+E \;\Rightarrow\; |\gamma'| \leqq 2\cdot m\cdot\frac{1}{2}\cdot\frac{h_0}{n} \leqq h_0 \;\Rightarrow\; \gamma'\in\Omega_0\,.$$

It follows that each $\varphi(\cdot+\alpha_i)=\varphi(\cdot)$ on $(E+E)\times\Gamma$.

Define the function ξ on $\mathsf{R}^n\times\Gamma$ by

$$\xi(\gamma) = \begin{cases}\varphi(\gamma) & \text{if } \gamma\in E\times\Gamma\\ 0 & \text{elsewhere}\,.\end{cases}$$

Then define the function η on $\mathbf{R}^n \times \Gamma$ by

$$\eta(\gamma) = \sum_{(i_1,\ldots,i_m)\in \mathbf{Z}^m} \xi(\gamma + i_1\alpha_1 + \ldots + i_m\alpha_m), \qquad \gamma \in \mathbf{R}^m \times \Gamma .$$

η is a well-defined function, because for each $\gamma \in \mathbf{R}^n \times \Gamma$ we have $\xi(\gamma + i_1\alpha_1 + \ldots + i_m\alpha_m) \neq 0$ for at most one m-tuple $(i_1, \ldots, i_m) \in \mathbf{Z}^m$. It is also clear that η is integer-valued and continuous in the topology of $\mathbf{R}^n_d \times \Gamma$. We now prove that

$$T_1 = \{\eta(\cdot + \gamma) \mid \gamma \in \mathbf{R}^n \times \Gamma \text{ and } \eta(\gamma) \neq 0\}$$

is finite. By the construction of η the cardinality of T_1 is equal to the cardinality of

$$T_2 = \{\eta(\cdot + \gamma)|_{E\times\Gamma} \mid \gamma \in E \times \Gamma \text{ and } \eta(\gamma) \neq 0\} .$$

Now observe that since each $\eta(\cdot + \alpha_i) = \eta(\cdot)$ on $(E+E)\times\Gamma$, the construction of η also shows that $\eta|_{(E+E)\times\Gamma} = \varphi|_{(E+E)\times\Gamma}$. Thus

$$T_2 = \{\varphi(\cdot + \gamma)|_{E\times\Gamma} \mid \gamma \in E \times \Gamma \text{ and } \varphi(\gamma) \neq 0\} ,$$

and the cardinality of this last set is finite, since it is obviously less than the cardinality of $\{\varphi(\cdot + \gamma)|_{\Omega\times\Gamma} \mid \gamma \in \Omega \times \Gamma \text{ and } \varphi(\gamma) \neq 0\}$, which is finite by assumption. We have proved that T_1 is finite, or in other words that η is canonical on $\mathbf{R}^n \times \Gamma$. Let Σ be the orthogonal complement of Λ in $\mathbf{R}^n$ and set $\Omega_4 = (E+\Sigma)\cap\Omega_3$. Then Ω_4 is a neighborhood of the origin in $\mathbf{R}^n$. Since $\varphi = \eta$ on $\Omega_4 \times \Gamma$, Theorem 1.2.3 now follows from Lemmas 1.2.1 and 1.2.2.

1.3. *A theorem for a set of measures* (*Theorem* 1.3.4).

In Amemiya's and Ito's proof of Cohen's theorem, for a canonical measure $\mu \in M(G)$, properties of the weak* closure of $\{\bar{\gamma}\mu \mid \hat{\mu}(\gamma) \neq 0\}$ in $M(G)$ played a crucial role. In our proof of Theorem C, for a $\mu \in M(\hat{\mathbf{R}}^n \times G)$ whose FST $\hat{\mu}$ is integer-valued on a set $\Omega \times \Gamma$ where Ω is a neighborhood of the origin in $\mathbf{R}^n$, similar properties of the set $S_{\hat{\mathbf{R}}^n, G}(\mu)$ (see Definition 1.3.1 below) play a corresponding role. Theorem 1.3.4 (with $G' = \hat{\mathbf{R}}^n$, $G'' = G$) gives the relevant propertie of $S_{\hat{\mathbf{R}}^n, G}(\mu)$.

DEFINITION 1.3.1. Let $\mu \in M(\bar{G}', G'')$. We define $S_{G', G''}(\mu)$ to be the set of all measures $\nu \in M(\bar{G}' \times G'')$ for which there exists a net $\{\gamma_\alpha = (\gamma'_\alpha, \gamma''_\alpha)\}$ in $\Gamma'_d \times \Gamma''$ such that $\lim_\alpha \gamma'_\alpha = 0$ in the topology of Γ', each $\hat{\mu}(\gamma_\alpha) \neq 0$, and $\lim_\alpha \bar{\gamma}_\alpha \mu = \nu$ in the weak* topology of $M(\bar{G}' \times G'')$.

DEFINITION 1.3.2. Let $\mu \in M(\bar{G}' \times G'')$. We say that μ has property $L_{G', G''}$ if there exists an infinite net $\{\gamma_\alpha = (\gamma'_\alpha, \gamma''_\alpha)\}$ in $\Gamma'_d \times \Gamma''$ such that $\lim_\alpha \gamma'_\alpha = 0$ in the topology of Γ', each $\hat{\mu}(\gamma_\alpha) \neq 0$, and $\|\gamma_{\alpha'}\mu - \gamma_{\alpha''}\mu\| \geqq 1$ for all α', α'' with $\alpha' \neq \alpha''$.

The following lemma (due to Ito) is given in Glicksberg [4].

LEMMA 1.3.3. *Let X be a locally compact Hausdorff space. Let $\mu \in M(X)$. Let E be a family of unimodular Borel functions on X for which $(E\mu)^-$, the weak* closure of $E\mu$ in $M(X)$, consists of measures all of norm $= \|\mu\|$. Then the weak* and norm topologies coincide on $(E\mu)^-$.*

We can now prove

THEOREM 1.3.4. *Let $\mu \in M(\bar{G}' \times G'')$. Then the following is true for $S_{G',G''}(\mu)$*

1°. *$S_{G',G''}(\mu)$ is a weak* compact subset of $M(\bar{G}' \times G'')$,*
2°. *If $\sigma \in S_{G',G''}(\mu)$, then $S_{G',G''}(\sigma) \subset S_{G',G''}(\mu)$,*
3°. *If μ has property $L_{G',G''}(\mu)$, then $S_{G',G''}(\mu)$ contains a measure σ such that $\|\sigma\| < \|\mu\|$ (strict inequality).*

PROOF. 1°. Since $S_{G',G''}(\mu)$ clearly is a subset of the weak* compact set of all measures in $M(\bar{G}' \times G'')$ with norm less than or equal to $\|\mu\|$, it is enough to prove that $S_{G',G''}(\mu)$ is weak* closed. We now do that. Let $\{\lambda_\alpha, \alpha \in A\}$ be an arbitrary net in $S_{G',G''}(\mu)$ and suppose $\lim_\alpha \lambda_\alpha = \lambda$ w^* (in the weak* topology of $M(\bar{G}' \times G'')$). We have to show that $\lambda \in S_{G',G''}(\mu)$. Since each $\lambda_\alpha \in S_{G',G''}(\mu)$, there is for each fixed $\alpha \in A$ a net $\{\gamma_{\alpha\beta} = (\gamma'_{\alpha\beta}, \gamma''_{\alpha\beta}), \beta \in B_\alpha\}$ in $\Gamma'_d \times \Gamma''$ such that $\lim \{\gamma'_{\alpha\beta}, \beta \in B_\alpha\} = 0$ in the topology of Γ', $\hat{\mu}(\gamma_{\alpha\beta}) \neq 0$ for all $\alpha \in A$ and $\beta \in B_\alpha$, and $\lim \{\bar{\gamma}_{\alpha\beta}\mu, \beta \in B_\alpha\} = \lambda_\alpha$ w^*. Now note that $\lim_\alpha \lim_\beta \gamma'_{\alpha\beta} = 0$ in the topology of Γ', $\hat{\mu}(\gamma_{\alpha\beta}) \neq 0$ for all $\alpha \in A$ and $\beta \in B_\alpha$, and $\lim_\alpha \lim_\beta \bar{\gamma}_{\alpha\beta}\mu = \nu$ w^*. Thus, by a well-known result for iterated limits of nets (see e.g. Kelley [6, p. 69, Theorem 4]), there is a net $\{\gamma_\delta = (\gamma'_\delta, \gamma''_\delta)\}$ in $\Gamma'_d \times \Gamma''$ such that $\lim_\delta \gamma'_\delta = 0$ in the topology of Γ', each $\hat{\mu}(\gamma_\delta) \neq 0$, and $\lim_\delta \bar{\gamma}_\delta \mu = \lambda$ w^*, and therefore $\lambda \in S_{G',G''}(\mu)$.

2°. Let $\sigma \in S_{G',G''}(\mu)$. Choose an arbitrary $\varrho \in S_{G',G''}(\sigma)$. We have to show that $\varrho \in S_{G',G''}(\mu)$. Since $\varrho \in S_{G',G''}(\sigma)$, there is a net $\{\gamma_\alpha = (\gamma'_\alpha, \gamma''_\alpha), \alpha \in A\}$ in $\Gamma'_d \times \Gamma''$ such that $\lim \{\gamma'_\alpha, \alpha \in A\} = 0$ in the topology of Γ', each $\hat{\sigma}(\gamma_\alpha) \neq 0$, and $\lim \{\bar{\gamma}_\alpha \sigma, \alpha \in A\} = \varrho$ w^*. Furthermore, since $\sigma \in S_{G',G''}(\mu)$, there is a net $\{\gamma_\beta = (\gamma'_\beta, \gamma''_\beta), \beta \in B\}$ in $\Gamma'_d \times \Gamma''$ such that $\lim \{\gamma'_\beta, \beta \in B\} = 0$ in the topology of Γ', each $\hat{\mu}(\gamma_\beta) \neq 0$, and $\lim \{\bar{\gamma}_\beta \mu, \beta \in B\} = \sigma$ w^*. Now note that for each $\alpha \in A$ we have $\lim_\beta \hat{\mu}(\gamma_\alpha + \gamma_\beta) = \hat{\sigma}(\gamma_\alpha) \neq 0$. Thus for each $\alpha \in A$ there exists a $\beta_\alpha \in B$ such that $\hat{\mu}(\gamma_\alpha + \gamma_\beta) \neq 0$ if $\beta \geqq \beta_\alpha$. For each $\alpha \in A$ set $B_\alpha = \{\beta \in B \mid \beta \geqq \beta_\alpha\}$ and give each B_α the induced order from B. This makes each B_α a directed set. Also set $\gamma_{\alpha\beta} = \gamma_\alpha + \gamma_\beta$ when $\alpha \in A$ and $\beta \in B_\alpha$. Then it is easy to see that $\lim_{\alpha \in A} \lim_{\beta \in B_\alpha} \gamma'_{\alpha\beta} = 0$ in the topology of Γ', $\hat{\mu}(\gamma_{\alpha\beta}) \neq 0$ for all $\alpha \in A$ and $\beta \in B_\alpha$, and $\lim_{\alpha \in A} \lim_{\beta \in B_\alpha} \bar{\gamma}_{\alpha\beta}\mu = \varrho$ w^*. Repeating an argument from the proof of 1° we see that $\varrho \in S_{G',G''}(\mu)$.

3°. Let $\{\gamma_\alpha = (\gamma'_\alpha, \gamma''_\alpha)\}$ be an infinite net in $\Gamma'_d \times \Gamma''$ such that $\lim_\alpha \gamma'_\alpha = 0$ in the topology of Γ', each $\hat{\mu}(\gamma_\alpha) \neq 0$, and $\|\bar{\gamma}_{\alpha'}\mu - \bar{\gamma}_{\alpha''}\mu\| \geqq 1$ for all α', α'' with $\alpha' \neq \alpha''$. Set $E = \{\bar{\gamma}_\alpha \mu\}$. Since $\{\gamma_\alpha\}$ is an infinite net, and since $\|\bar{\gamma}_{\alpha'}\mu - \bar{\gamma}_{\alpha''}\mu\| \geqq 1$ if $\alpha' \neq \alpha''$, E is infinite. Let $\bar{E}$ be the weak* closure of E in $M(\bar{G}' \times G'')$. $\bar{E}$ is weak* compact and infinite. Hence $\bar{E}$ has at least one cluster point. It is easy to see that each cluster point belongs to $S_{G', G''}(\mu)$. Let σ be a cluster point. There is a subnet $\{\bar{\gamma}_\beta \mu\}$ of $\{\bar{\gamma}_\alpha \mu\}$ such that $\lim_\beta \bar{\gamma}_\beta \mu = \sigma$ w^*. Clearly $\|\sigma\| \leqq \|\mu\|$. Now suppose that $\|\sigma\| = \|\mu\|$. Then by Lemma 1.3.3 applied to the net $\{\bar{\gamma}_\beta \mu\}$ we have $\lim_\beta \|\bar{\gamma}_\beta \mu - \sigma\| = 0$, and therefore there is a β_0 such that $\|\bar{\gamma}_\beta \mu - \bar{\gamma}_{\beta_0}\mu\| \leqq \frac{1}{2}$ if $\beta \geqq \beta_0$. But, since $\{\bar{\gamma}_\beta \mu\}$ is a subnet of $\{\bar{\gamma}_\alpha \mu\}$, this contradicts $\|\bar{\gamma}_{\alpha'}\mu - \bar{\gamma}_{\alpha''}\mu\| \geqq 1$ for all α', α'' with $\alpha' \neq \alpha''$. We conclude that $\|\sigma\| < \|\mu\|$.

1.4. Proof of Theorem C. After the above preparations we are now in a position to prove Theorem C.

Theorem C. *Let Ω be an open subset of $\mathbf{R}^n$. Suppose $\hat{\mu} \in B(\mathbf{R}^n_d \times \Gamma)$ is integer-valued on $\Omega \times \Gamma$. Then for each point $\gamma'_0 \in \Omega$ there exist a neighborhood Ω_0 of γ'_0 and an integer-valued $\hat{\nu} \in B(\mathbf{R}^n_d \times \Gamma)$ such that $\hat{\mu} = \hat{\nu}$ on $\Omega_0 \times \Gamma$.*

Proof. After a translation if necessary we may suppose $\gamma'_0 = 0$. For $h > 0$ set

$$\Omega_h = \{\gamma' \in \mathbf{R}^n \mid |\gamma'| < h\} .$$

Choose $h_0 > 0$ so that $\Omega_{h_0} + \Omega_{h_0} \subset \Omega$. For $h \in]0, h_0]$ set

$$B_h = \{\hat{\mu}(\cdot + \gamma)|_{\Omega_h \times \Gamma} \mid \gamma \in \Omega_h \times \Gamma \text{ and } \hat{\mu}(\gamma) \neq 0\} .$$

We get two cases.

Case i. There is an $h \in]0, h_0]$ such that B_h is finite. Then it follows from Theorem 1.2.3 that there exist a canonical measure $\nu \in M(\bar{\mathbf{R}}^n \times G)$ (with $\boldsymbol{CLP}$ $(\hat{\nu}^{-1}(\mathbf{Z} \setminus \{0\}))$ equal to a linear subspace of $\mathbf{R}^n$) and a neighborhood Ω_1 of the origin in $\mathbf{R}^n$ such that $\hat{\mu} = \hat{\nu}$ on $\Omega_1 \times \Gamma$. The theorem is proved in Case i.

Case ii. B_h is infinite for all $h \in]0, h_0]$. For $h \in]0, h_0]$ set

$$A_h = \{\gamma \in \Omega_h \times \Gamma \mid \hat{\mu}(\gamma) \neq 0\} .$$

Since B_h is infinite for all $h \in]0, h_0]$, so is A_h. We now construct an infinite sequence $(\gamma_k)_{k=0}^\infty$. Choose $\gamma_0 \in A_{h_0}$. Then choose $\gamma_1 \in A_{h_0/2}$ such that $\hat{\mu}(\gamma + \gamma_1) \neq \hat{\mu}(\gamma + \gamma_0)$ for some $\Omega_{h_0} \times \Gamma$. Then choose $\gamma_2 \in A_{h_0/2^2}$ such that $\hat{\mu}(\gamma + \gamma_2) \neq \hat{\mu}(\gamma + \gamma_0)$ for some $\gamma \in \Omega_{h_0} \times \Gamma$, and $\hat{\mu}(\gamma + \gamma_2) \neq \hat{\mu}(\gamma + \gamma_1)$ for some $\gamma \in \Omega_{h_0} \times \Gamma$. It is obvious how the construction should be continued. Since A_h ($h \in]0, h_0]$) and B_{h_0} are infinite, the construction can always be carried out. Note that since $\hat{\mu}$ is integer-valued on $(\Omega_{h_0} + \Omega_{h_0}) \times \Gamma$ we have

$$\|\hat{\mu}(\cdot + \gamma_k) - \hat{\mu}(\cdot + \gamma_l)\|_\infty \geqq 1 \quad \text{if } k \neq l .$$

Thus in Case ii there is an infinite sequence $(\gamma_k=(\gamma_k',\gamma_k''))_{k=0}^{\infty}$ such that $\lim_k \gamma_k'=0$ in the topology of $\mathbf{R}^n$, each $\hat{\mu}(\gamma_k)\neq 0$, and $\|\bar{\gamma}_k\mu-\bar{\gamma}_l\mu\|\geqq 1$ if $k\neq l$, i.e. μ has property $L_{\hat{\mathbf{R}}^n,G}$ in Case ii. By Theorem 1.3.4 the set $S_{\hat{\mathbf{R}}^n,G}(\mu)$ is weak* compact and contains a measure with norm strictly less than $\|\mu\|$. Now recall that the function norm is a lower semi-continuous function in the weak* topology. Thus there is a measure with minimal norm in $S_{\hat{\mathbf{R}}^n,G}(\mu)$ and this minimal norm is strictly less than $\|\mu\|$. Let $\sigma\in S_{\hat{\mathbf{R}}^n,G}(\mu)$ be a measure with minimal norm. Note that

$$\sigma\in S_{\hat{\mathbf{R}}^n,G}(\mu) \;\Rightarrow\; |\hat{\sigma}(0)|\geqq 1\,.$$

Also note that

$$\hat{\mu}(\gamma)\neq 0\;\forall\gamma\in\Omega\times\Gamma \;\Rightarrow\; \sigma(\gamma)\neq 0\;\forall\gamma\in\Omega\times\Gamma \qquad \text{(we suppose } \gamma_0'=0)\,.$$

Now restart from the beginning with μ replaced with σ. There are two possibilities.

(a) We end up in Case i with σ. Then there are a canonical measure $\nu\in M(\bar{\hat{\mathbf{R}}}^n\times G)$ with $\boldsymbol{CLP}(\hat{\nu}^{-1}(\mathbf{Z}\setminus\{0\}))$ equal to a linear subspace of $\mathbf{R}^n$ and a neighborhood Ω_1 of the origin in $\mathbf{R}^n$ such that $\hat{\sigma}=\hat{\nu}$ on $\Omega_1\times\Gamma$.

(b) We end up in Case ii with σ. But since $S_{\hat{\mathbf{R}}^n,G}(\sigma)\subset S_{\hat{\mathbf{R}}^n,G}(\mu)$ (Theorem 1.3.4) this would imply that there is a measure in $S_{\hat{\mathbf{R}}^n,G}(\mu)$ with norm strictly less than $\|\sigma\|$. A contradiction. Hence (b) is impossible.

Taking into account that $\sigma\in S_{\hat{\mathbf{R}}^n,G}(\mu)$ we get summing up Case ii, a net $\{\gamma_\alpha=(\gamma_\alpha',\gamma_\alpha'')\}$, a measure $\sigma\in M(\bar{\hat{\mathbf{R}}}^n\times G)$ with $|\hat{\sigma}(0)|\geqq 1$, a canonical measure $\nu\in M(\bar{\hat{\mathbf{R}}}^n\times G)$ with $\boldsymbol{CLP}\,(\hat{\nu}^{-1}(\mathbf{Z}\setminus\{0\}))$ equal to a linear subspace of $\mathbf{R}^n$, and a neighborhood Ω_1 of the origin in $\mathbf{R}^n$ such that: $\lim_\alpha\gamma_\alpha'=0$ in the topology of $\mathbf{R}^n$, (each $\hat{\mu}(\gamma_\alpha)\neq 0$), $\lim_\alpha\bar{\gamma}_\alpha\mu=\sigma$ w^* (in the weak* topology of $M(\bar{\hat{\mathbf{R}}}^n\times G)$), and $\hat{\sigma}=\hat{\nu}$ on $\Omega_1\times\Gamma$. Since $\hat{\nu}(0)=\hat{\sigma}(0)\neq 0$, $\boldsymbol{CLP}\,(\hat{\nu}^{-1}(\mathbf{Z}\setminus\{0\}))$ is non-empty. Now choose $\varrho\in M(\hat{\mathbf{R}}^n)\subset M(\bar{\hat{\mathbf{R}}}^n\times G)$ such that supp $(\hat{\varrho})\subset\Omega_1\times\Gamma$, $\hat{\varrho}(\gamma)=1$ when $\gamma\in\Omega_2\times\Gamma$ for some neighborhood Ω_2 of the origin in $\mathbf{R}^n$, and $\|\varrho\|<1+1/(4\|\mu\|)$. This can always be done (see e.g. Rudin [10, p. 53, Theorem 2.6.8]). Next we prove that

$$\lim_\alpha\bar{\gamma}_\alpha(\varrho*\mu) \;=\; \varrho*\nu\;\; w^*\,.$$

Since $\bar{\gamma}_\alpha\mu\to\sigma$ w^* and consequently $\varrho*(\bar{\gamma}_\alpha\mu)\to\varrho*\sigma$ w^*, and since $\varrho*\sigma=\varrho*\nu$ (look at the FSTs), we must show that

$$\bar{\gamma}_\alpha(\varrho*\mu)-\varrho*(\bar{\gamma}_\alpha\mu) \;=\; (\bar{\gamma}_\alpha\varrho-\varrho)*(\bar{\gamma}_\alpha\mu)\to 0\;\; w^*\,.$$

But, since $\varrho\in M(\hat{\mathbf{R}}^n)$ $(\subset M(\bar{\hat{\mathbf{R}}}^n\times G))$ and since $\lim_\alpha\gamma_\alpha'=0$ in the topology of $\mathbf{R}^n$ and consequently $\lim_\alpha\|\bar{\gamma}_\alpha\varrho-\varrho\|=0$, this is obviously true. We have proved that

$\bar{\gamma}_\alpha(\varrho * \mu) \to \varrho * \nu \ w^*$. Now Theorem 1.1.4 shows that there is a σ-compact subgroup F of $\bar{\hat{\mathsf{R}}}^n \times G$ such that

$$\lim_\alpha \|\bar{\gamma}_\alpha(\varrho * \mu)|_F - \varrho * \nu\| = 0 .$$

Set $\lambda = (\varrho * \mu)|_F$. Choose $\delta_0 > 0$ so that $\hat{\varrho}(\gamma) = 1$ for all $\gamma \in \Omega_{2\delta_0} \times \Gamma$. Then choose an α_0 such that

$$\|\lambda - \gamma_{\alpha_0}(\varrho * \nu)\| < \tfrac{1}{8} \quad \text{and} \quad \hat{\varrho}(\gamma - \gamma_{\alpha_0}) = 1$$

for all $\gamma \in \Omega_{\delta_0} \times \Gamma$. Since $\lim_\alpha \|\lambda - \gamma_\alpha(\varrho * \nu)\| = 0$, since $\lim_\alpha \gamma'_\alpha = 0$ in the topology of R^n, and since $\varrho \in M(\hat{\mathsf{R}}^n)$ $(\subset M(\bar{\hat{\mathsf{R}}}^n \times G))$, this is always possible.

Now observe that

(1) $(\varrho * \mu - \gamma_{\alpha_0}(\varrho * \nu))\hat{}$ is integer-valued on $\Omega_{\delta_0} \times \Gamma$,

(2) $\frac{1}{4} + \|\mu\| > \|\varrho * \mu\| = \|\lambda\| + \|\varrho * \mu - \lambda\| > \|\gamma_{\alpha_0}(\varrho * \nu)\| - \frac{1}{8} + \|\varrho * \mu - \gamma_{\alpha_0}(\varrho * \nu)\| - \frac{1}{8}$
$= \|\varrho * \nu\| + \|\varrho * \mu - \gamma_{\alpha_0}(\varrho * \nu)\| - \frac{1}{4}$
$> |\hat{\varrho}(0)\hat{\nu}(0)| + \|\varrho * \mu - \gamma_{\alpha_0}(\varrho * \nu)\| - \frac{1}{4}$
$\geqq \|\varrho * \mu - \gamma_{\alpha_0}(\varrho * \nu)\| + \frac{3}{4}$,
i.e., $\|\varrho * \mu - \gamma_{\alpha_0}(\varrho * \nu)\| < \|\mu\| - \frac{1}{2}$,

(3) $\hat{\mu} = (\gamma_{\alpha_0}\nu)\hat{} + (\varrho * \mu - \gamma_{\alpha_0}(\varrho * \nu))\hat{}$ on $\Omega_{\delta_0} \times \Gamma$,

(4) $(\gamma_{\alpha_0}\nu)\hat{}$ is integer-valued.

Thus, if the theorem is true for all measures in $M(\bar{\hat{\mathsf{R}}}^n \times G)$ with norm less than $\|\mu\| - \frac{1}{2}$, it is also true for μ. But the theorem is obviously true for all measures in $M(\bar{\hat{\mathsf{R}}}^n \times G)$ with norm less than one (the FST of a measure with norm less than one must be zero whenever it is integer-valued). Hence Theorem C follows by induction in Case ii.

2. Theorem A.

2.1. *A local equality theorem* (*Theorem* 2.1.5).

Theorem 2.1.5 essentially says that if we restrict ourself to narrow vertical strips in $\mathsf{R}^n \times \Gamma$, then $B(\mathsf{R}^n_d \times \Gamma)$ is the same as $B(\mathsf{T}^n_d \times \Gamma)$.

The proof of the following lemma is straightforward and is omitted.

LEMMA 2.1.1. *Let O be a non-empty open subset of a compact abelian group H. Then there exists a finite disjoint subdivision of H into Borel sets, each having a non-empty interior, and each being included in some translate of O.*

LEMMA 2.1.2. *Let $\varphi \in B(\Gamma_d)$. Then there exists a net $\{\varphi_\alpha\}$ in $B(\Gamma)$ such that*

1° $\lim_\alpha \varphi_\alpha(\gamma) = \varphi(\gamma)$ *for each fixed* $\gamma \in \Gamma$,
2° $\|\varphi_\alpha\| \leqq \|\varphi\|$ *for all* α.

PROOF. Let $\{O_\alpha\}_{\alpha\in A}$ be a neighborhood base at the origin in $\bar{G}$. For each α we apply Lemma 2.1.1 with $O=O_\alpha$ and $H=\bar{G}$. For each α this gives finitely many subsets of $\bar{G}$, say $T_{\alpha,1},\ldots,T_{\alpha,N_\alpha}$, having the properties of Lemma 2.1.1. For each $T_{\alpha,i}$ choose an $x_{\alpha,i}\in T_{\alpha,i}\cap G$. (Recall that G is dense in $\bar{G}$.) Suppose $\varphi=\hat{\mu}$, where $\mu\in M(\bar{G})$. For each α set

$$\mu_\alpha = \sum_{i=1}^{N_\alpha} \mu(T_{\alpha,i})\delta_{x_{\alpha,i}},$$

where $\delta_{x_{\alpha,i}}$ denotes the Dirac measure at the point $x_{\alpha,i}$. Then each $\mu_\alpha\in M(G)$ $(\subset M(\bar{G}))$. Direct the index set A by saying that for $\alpha',\alpha''\in A$, $\alpha'\geqq\alpha''$ if $O_{\alpha'}\subset O_{\alpha''}$. It is easy to verify that the net $\{\varphi_\alpha=\hat{\mu}_\alpha,\ \alpha\in A\}$ satisfies the requirements of the lemma.

DWFINITION 2.1.3. For $\delta>0$ let $\mathscr{E}_{n,\delta}$ be the following subset of R^n,

$$\mathscr{E}_{n,\delta} = \{(\gamma_1,\ldots,\gamma_n)\in \mathsf{R}^n \mid |\gamma_i|\leqq\delta,\ i=1,\ldots,n\}.$$

The next lemma is a slight generalization of the non-trivial part of Theorem 2.1.6 in Rudin [10, p. 56].

LEMMA 2.1.4. *Let $\varphi\in B(\mathsf{R}\times\Gamma)$ be carried by $\mathscr{E}_{1,1/2}\times\Gamma$. Define the function ψ on $\mathsf{R}\times\Gamma$ by*

$$\psi(\gamma',\gamma'') = \sum_{k=-\infty}^{\infty} \varphi(\gamma'+k,\gamma''),\ (\gamma',\gamma'')\in\mathsf{R}\times\Gamma.$$

Then $\psi\in B(\mathsf{T}\times\Gamma)$, and there is an absolute constant C such that $\|\psi\|\leqq C\|\varphi\|$.

PROOF. Choose a compactly supported $g\in C^2(\mathsf{R})$ which is one on $\mathscr{E}_{1,1/2}$. Then $\hat{g}$, the Fourier transform of g, satisfies

$$|\hat{g}(y)| < \frac{A}{1+y^2},\quad y\in\mathsf{R}, \tag{1}$$

for some constant $A>0$. For $\gamma''\in\Gamma$ define $a_k(\gamma'')$ by

$$a_k(\gamma'') = \int_{-1/2}^{1/2} \varphi(\gamma',\gamma'')e^{-i2\pi k\gamma'}\,d\gamma'.$$

Suppose $\varphi=\hat{\mu}$, where $\mu\in M(\hat{\mathsf{R}}\times G)$. Since $\varphi(\gamma',\gamma'')=\varphi(\gamma',\gamma'')g(\gamma')$, it is straightforward to verify, using Fubini's theorem, that

$$a_k(\gamma'') = \int_{\substack{x'\in\hat{\mathsf{R}}\\ x''\in G}} \hat{g}(k+x')(-x'',\gamma'')\,d\mu(x',x''). \tag{2}$$

Because of (1), there is a constant $C>0$ such that $\sum_{k=-\infty}^{\infty}|\hat{g}(k+x')|<C$ for all $x' \in \hat{\mathsf{R}}$. Combining this with (2) we see that $\sum_{k=-\infty}^{\infty}|a_k(\gamma'')|<C\|\mu\|$ for all $\gamma'' \in \Gamma$. It follows that

$$\psi(\gamma',\gamma'') = \sum_{k=-\infty}^{\infty} a_k(\gamma'')e^{i2\pi k\gamma'} \quad \text{for all } (\gamma',\gamma'') \in \mathsf{R}\times\Gamma .$$

For each $k \in \mathsf{Z}$ set $d\mu_k(x',x'')=\hat{g}(k+x)\,d\mu(x',x'')$. Since each $\|\hat{g}_k\|_\infty<\infty$, each $\mu_k \in M(\hat{\mathsf{R}}\times G)$. For each k define the set function ν_k on the Borel subsets of G by setting $\nu_k(E)=\mu_k(\hat{\mathsf{R}}\times E)$ if E is a Borel subset of G. Since $\hat{\mathsf{R}}$ is a σ-compact subgroup of $\hat{\mathsf{R}}\times G$, Lemma 1.1.2 shows that each $\nu_k \in M(G)$. For each k let $\delta_k \in M(\mathsf{Z})$ $(\subset M(\hat{\mathsf{R}}))$ be the Dirac measure at the point $-k \in \mathsf{Z}$ $(\subset\hat{\mathsf{R}})$. Set $\sigma=\sum_{k=-\infty}^{\infty}\delta_k\otimes\nu_k$. Then $\sigma \in M(\mathsf{Z}\times G)$ $(\subset M(\hat{\mathsf{R}}\times G))$ and

$$\hat{\sigma}(\gamma',\gamma'') = \sum_{k=-\infty}^{\infty} e^{i2\pi k\gamma'}a_k(\gamma'') = \psi(\gamma',\gamma'') .$$

Hence $\psi \in B(\mathsf{T}\times\Gamma)$ $(\subset B(\mathsf{R}\times\Gamma))$. We also have,

$$\|\psi\| = \|\sigma\| \leqq \sum_{k=-\infty}^{\infty} \|\delta_k\otimes\nu_k\| = \sum_{k=-\infty}^{\infty}\|\nu_k\|$$

$$\leqq \sup_{x'\in\mathsf{R}} \sum_{k=-\infty}^{\infty} |\hat{g}(x'+k)|\,\|\mu\| < C\|\mu\| = C\|\varphi\| .$$

REMARK. By a partition of unity argument Lemma 2.1.4 can be extended to functions $\varphi \in B(\mathsf{R}\times\Gamma)$ having supports on an arbitrary $\mathscr{E}_{1,\delta}\times\Gamma$. The constant C then depends on δ.

We can now prove

THEOREM 2.1.5. *Let $\varphi \in B(\mathsf{R}_d^n\times\Gamma)$ be carried by $\mathscr{E}_{n,1/4}\times\Gamma$. Define the function ψ on $\mathsf{R}^n\times\Gamma$ by*

$$\psi((\gamma_1',\ldots,\gamma_n'),\gamma'') = \sum_{(i_1,\ldots,i_n)\in\mathsf{Z}^n} \varphi((\gamma_1'+i_1,\ldots,\gamma_n'+i_n),\gamma'') ,$$
$$((\gamma_1',\ldots,\gamma_n'),\gamma'') \in \mathsf{R}^n\times\Gamma .$$

Then $\psi \in B(\mathsf{T}_d^n\times\Gamma)$, and there is a constant C_n only depending on n such that $\|\psi\| < C_n\|\varphi\|$.

PROOF. Choose $\omega' \in C^\infty(\mathsf{R}^n)$ so that $\omega'=1$ on $\mathscr{E}_{n,1/4}$ and $\omega'=0$ outside $\mathscr{E}_{n,1/2}$. Then $\omega' \in B(\mathsf{R}^n)$. Define ω on $\mathsf{R}^n\times\Gamma$ by $\omega(\gamma',\gamma'')=\omega'(\gamma')$. Then $\omega \in B(\mathsf{R}^n\times\Gamma)$ (see e.g. Rudin [10, p. 79, Lemma 4.2.1]). Since $B(\mathsf{R}_d^n\times\Gamma)\subset B(\mathsf{R}_d^n\times\Gamma_d)$, Lemma

2.1.2 shows that there is a net $\{\varphi'_\alpha\}$ in $B(\mathsf{R}^n \times \Gamma)$ such that $\lim_\alpha \varphi'_\alpha(\gamma) = \varphi(\gamma)$ for each fixed $\gamma \in \mathsf{R}^n \times \Gamma$, and each $\|\varphi_\alpha\| \leqq \|\varphi\|$. For each α set $\varphi_\alpha = \omega\varphi'_\alpha$. Then each $\varphi_\alpha \in B(\mathsf{R}^n \times \Gamma)$, $\lim_\alpha \varphi_\alpha(\gamma) = \varphi(\gamma)$ for each fixed $\gamma \in \mathsf{R}^n \times \Gamma$, each $\|\varphi_\alpha\| \leqq \|\omega\| \, \|\varphi\|$, and each φ_α is carried by $\mathscr{E}_{n,1/2} \times \Gamma$. For each α set

$$\psi_\alpha((\gamma'_1, \ldots, \gamma'_n), \gamma'') = \sum_{(i_1, \ldots, i_n) \in \mathsf{Z}} \varphi_\alpha((\gamma'_1 + i_1, \ldots, \gamma'_n + i_n), \gamma'') ,$$

$$((\gamma'_1, \ldots, \gamma'_n), \gamma'') \in \mathsf{R}^n \times \Gamma .$$

Then $\lim_\alpha \psi_\alpha(\gamma) = \psi(\gamma)$ for each fixed $\gamma \in \mathsf{R}^n \times \Gamma$, and n applications of Lemma 2.1.4 show that each $\psi_\alpha \in B(\mathsf{T}^n \times \Gamma)$ and each $\|\psi_\alpha\| < C^n \|\varphi\|$. It follows that $\psi \in B(\mathsf{T}^n_d \times \Gamma_d)$ and $\|\psi\| < C^n \|\varphi\|$ (1) (see e.g. Rudin [10, p. 34, Theorem 1.9.2]). But $\psi(\gamma', \gamma'')$ is continuous on Γ for each fixed γ' (2). Combining (1) and (2) we get $\psi \in B(\mathsf{T}^n_d \times \Gamma)$ and $\|\psi\| < C^n \|\varphi\|$.

REMARK. By a partition of unity argument Theorem 2.1.5 can be extended to functions $\varphi \in B(\mathsf{R}^n_d \times \Gamma)$ having supports on an arbitrary $\mathscr{E}_{n,\delta} \times \Gamma$. The constant C_n then depends also on δ.

2.2. PROOF OF THEOREM A. We begin by proving the following theorem.

THEOREM 2.2.1. *Let Ω be a neighborhood of the origin in $\mathsf{R}^m \times \mathsf{T}^n$. Let φ be a function defined on $\Omega \times \Gamma$. Then the following two conditions are equivalent.*

1° *There exists a neighborhood Ω_1 of the origin in $\mathsf{R}^m \times \mathsf{T}^n$ such that φ on $\Omega_1 \times \Gamma$ is integer-valued and coincides with a function in $B(\mathsf{R}^m_d \times \mathsf{T}^n_d \times \Gamma_d)$.*

2° *There exists a neighborhood Ω_2 of the origin in $\mathsf{R}^m \times \mathsf{T}^n$ such that φ on $\Omega_2 \times \Gamma$ coincides with a finite sum of functions, each being canonical on $\Omega_2 \times \Gamma$.*

PROOF. Suppose 1° is true. Then Theorem C combined with Cohen's theorem shows that there exist a neighborhood Ω'_1 of the origin in $\mathsf{R}^m \times \mathsf{R}^n$ and finitely many functions $\psi_1, \ldots, \psi_N$ each being canonical on $\mathsf{R}^m \times \mathsf{R}^n \times \Gamma$ so that $\varphi = \psi_1 + \ldots + \psi_N$ on $\Omega'_1 \times \Gamma$. For each $k \in \{1, \ldots, N\}$, let ψ'_k on $\mathsf{R}^m \times \mathsf{R}^n \times \Gamma$ be equal to ψ_k on $\mathsf{R}^m \times \mathscr{E}_{n,1/4} \times \Gamma$ and zero elsewhere, and define φ_k on $\mathsf{R}^m \times \mathsf{R}^n \times \Gamma$ by

$$\varphi_k(\gamma', (\gamma''_1, \ldots, \gamma''_n), \gamma''') = \sum_{(i_1, \ldots, i_n) \in \mathsf{Z}^n} \psi'_k(\gamma', (\gamma'_1 + i_1, \ldots, \gamma'_n + i_n), \gamma''') ,$$

$$(\gamma', (\gamma''_1, \ldots, \gamma''_n), \gamma''') \in \mathsf{R}^m \times \mathsf{R}^n \times \Gamma .$$

Then, using the trivial fact that if a function is canonical on a set E it is also canonical on every subset of E, it is easy to see that there is a neighborhood Ω_2

of the origin in $\mathsf{R}^m \times \mathsf{T}^n$ such that $\varphi = \varphi_1 + \ldots + \varphi_k$ on $\Omega_2 \times \Gamma$ and each φ_k is canonical on $\Omega_2 \times \Gamma$. Thus $1° \Rightarrow 2°$.

We now prove that $2° \Rightarrow 1°$. Suppose φ is canonical on $\Omega_2 \times \Gamma$ where Ω_2 is a neighborhood of the origin in $\mathsf{R}^m \times \mathsf{T}^n$. By Theorem 1.2.3 there exist a neighborhood Ω_2' of the origin in $\mathsf{R}^m \times \mathsf{R}^n$ and a $\hat{\mu} \in B(\mathsf{R}_d^m \times \mathsf{R}_d^n \times \Gamma_d)$ such that $\varphi = \hat{\mu}$ on Ω_2'. Choose $\omega \in B(\mathsf{R}_d^m \times \mathsf{R}_d^n \times \Gamma_d)$ so that $\omega = 1$ on $\mathsf{R}^m \times \mathscr{E}_{n,1/8} \times \Gamma$ and $\omega = 0$ outside $\mathsf{R}^m \times \mathscr{E}_{n,1/4} \times \Gamma$. Set $\psi' = \hat{\mu}\omega$ and define ψ on $\mathsf{R}^m \times \mathsf{R}^n \times \Gamma$ by

$$\psi(\gamma', (\gamma_1'', \ldots, \gamma''), \gamma''') = \sum_{(i_1, \ldots, i_n) \in \mathsf{Z}^n} \psi'(\gamma', (\gamma_1'' + i_1, \ldots, \gamma_n'' + i_n), \gamma'''),$$

$$(\gamma', (\gamma_1'', \ldots, \gamma_n''), \gamma''') \in \mathsf{R}^m \times \mathsf{R}^n \times \Gamma .$$

Then $\psi \in B(\mathsf{R}_d^m \times \mathsf{T}_d^n \times \Gamma_d)$ by Theorem 2.1.5, and there clearly exists a neighborhood Ω_2'' of the origin in $\mathsf{R}^m \times \mathsf{T}^n$ such that $\varphi = \psi$ on $\Omega_2'' \times \Gamma$. This shows that $2° \Rightarrow 1°$.

We are now ready to prove Theorem A.

THEOREM A. *Let φ be a function defined on a neighborhood of a point γ_0 in Γ. Then the following conditions are equivalent.*

1° *There exists a neighborhood Ω_1 of γ_0 such that φ on Ω_1 is integer-valued and coincides with a function in $B(\Gamma_d)$.*

2° *There exists a neighborhood Ω_2 of γ_0 such that φ on Ω_2 coincides with a finite sum of functions, each being canonical on Ω_2.*

PROOF. Since

(1) The formulation on Theorem A is translation invariant.
(2) Every LCAG has an open subgroup of type $\mathsf{R}^m \times H$, where H is a compact abelian group (see e.g. Rudin [10, p. 40, Theorem 2.4.1]).
(3) If Γ' is a subgroup of Γ'', then $B(\Gamma_d') = B(\Gamma_d'')|_{\Gamma'}$ (see e.g. Rudin, [10, p. 53, Theorem 2.7.2]).

there is no restriction to suppose $\gamma_0 = 0$ and $\Gamma = \mathsf{R}^m \times H$ where H is a compact abelian group. From now on we do that.

Using the well-known fact that every compact abelian group is a closed subgroup of some complete direct sum of circle groups (this follows e.g. from the proof of Theorem 2.5.1 in Rudin [10, p. 44], it is easy to see that we have

(4) Let Ω be an arbitrary neighborhood of the origin in $\mathsf{R}^m \times H$. Then there exist a group $\mathsf{R}^m \times \mathsf{T}^n \times K$ where K is a compact abelian group and a neighborhood Ω' of the origin in $\mathsf{R}^m \times \mathsf{T}^n$ such that $\mathsf{R}^m \times H$ is a closed subgroup of $\mathsf{R}^m \times \mathsf{T}^n \times K$ and $\Omega \supset (\Omega' \times K) \cap (\mathsf{R}^m \times H)$.

It is also easy to see that we have

(5) Let Γ' be a subgroup of Γ''.

(a) If φ'' is canonical on E'' in Γ'', then $\varphi''|_{\Gamma'}$ is canonical on $E'' \cap \Gamma'$ in Γ'.

(b) If E'' is a subset of Γ'', if φ' is canonical on $E'' \cap \Gamma'$ in Γ', and if φ'' is defined on $E''-E''+E''$ in Γ'' by $\varphi''=\varphi'$ on $(E''-E''+E'')\cap\Gamma'$ and $\varphi''=0$ elsewhere, then φ'' is canonical on E'' *in* Γ''.

(4), (3), $1^\circ \Rightarrow 2^\circ$ in Theorem 2.2.1, and (5a (in that order) show that $1^\circ \Rightarrow 2^\circ$ in Theorem A. (4), (5b), $2^\circ \Rightarrow 1^\circ$ in Theorem 2.2.1, and (3) (in that order) show that $2^\circ \Rightarrow 1^\circ$ in Theorem A.

3. Theorem B.

3.1. *A continuation theorem* (*Theorem* 3.1.4).

In this section we shall define the dense part (Definition 3.1.3) of a measure in $M(\bar{\bar{\mathsf{R}}}^n \times G)$ of type $\sum_{i=1}^N P_i m_{H_i}$ and give a continuation result (Theorem 3.1.4) for the dense part. We begin with a lemma which is the key to the result of this section.

LEMMA 3.1.1. *Let* $a_1,\ldots,a_N \in \mathsf{C}$. *Let* $\Delta_1,\ldots,\Delta_N$ *be cosets of subgroups of* $\mathsf{R}^n\times\Gamma$. *Let* Ω *be a non-empty open subset of* R^n. *Suppose*

1° $\boldsymbol{CLP}(\Delta_i)=\mathsf{R}^n$, $i=1,\ldots,N$,

2° $\sum_{i=1}^N a_i\chi_{\Delta_i}(\gamma)=0$ *for all* $\gamma\in\Omega\times\Gamma$ (χ_{Δ_i} *denotes the characteristic function of* Δ_i).

Then $\sum_{i=1}^N a_i\chi_{\Delta_i}(\gamma)=0$ *for all* $\gamma\in\mathsf{R}^n\times\Gamma$.

PROOF. Let $\Lambda_1,\ldots,\Lambda_M$ $(M\leqq N)$ be the different groups appearing in the cosets $\Delta_1,\ldots,\Delta_N$. Set $\Lambda=\bigcap_{i=1}^M\Lambda_i$. We get two cases.

CASE i. $\boldsymbol{CLP}(\Lambda)=\mathsf{R}^n$. Choose an arbitrary $\gamma_0=(\gamma_0',\gamma_0'')\in\mathsf{R}^n\times\Gamma$. Since $\boldsymbol{CLP}(\Lambda)=\mathsf{R}^n$, there is a $\gamma_1=(\gamma_1',\gamma_1'')\in\Lambda$ such that $\gamma_0'-\gamma_1'\in\Omega$. We get

$$\sum_{i=1}^N a_i\chi_{\Delta_i}(\gamma_0) = \sum_{i=1}^N a_1\chi_{\Delta_i}(\gamma_0-\gamma_1) = 0\,.$$

Hence, since $\gamma_0\in\mathsf{R}^n\times\Gamma$ was arbitrary, $\sum_{i=1}^N a_i\chi_{\Delta_i}(\gamma)=0$ for all $\gamma\in\mathsf{R}^n\times\Gamma$. The lemma is proved in Case i.

CASE ii. $\boldsymbol{CLP}(\Lambda)$ is a proper subgroup of R^n. In this case there is an integer M', $1\leqq M'\leqq M-1$, with the following two properties:

(a) If $k>M'$ and if $i_1,\ldots,i_k$ are different integers $\in\{1,\ldots,M\}$, then $\boldsymbol{CLP}\,(\bigcap_{j=1}^{k} \Lambda_{i_j})$ is a proper subgroup of R^n,

(b) There exist M' different integers, say $l_1,\ldots,l_{M'}\in\{1,\ldots,M\}$, such that $\boldsymbol{CLP}\,(\bigcap_{j=1}^{M'} \Lambda_{l_j})=\mathsf{R}^n$.

Set $\Lambda'=\bigcap_{j=1}^{M'}\Lambda_{l_j}$. Write $\mathsf{R}^n\times\Gamma$ as a union of disjoint cosets of Λ', $\mathsf{R}^n\times\Gamma=\bigcup_\alpha(\gamma_\alpha+\Lambda')$ (each $\gamma_\alpha\in\mathsf{R}^n\times\Gamma$). For each α set $\Delta_\alpha=\gamma_\alpha+\Lambda'$. Note that for each α

(1) $\sum_{i=1}^N a_i\chi_{\Delta_i}(\gamma)=0$ for all $\gamma\in\Omega\times\Gamma$ $\Rightarrow$

(2) $\sum_{i=1}^N a_i\chi_{\Delta_i\cap\Delta_\alpha}(\gamma)$ for all $\gamma\in\Omega\times\Gamma$.

Each $\Delta_i\cap\Delta_\alpha$ is either the empty set or a coset of a subgroup of $\mathsf{R}^n\times\Gamma$. Let I be the set of all integers $i\in\{1,\ldots,N\}$ such that Δ_i is a coset of one of $\Lambda_{l_1},\ldots,\Lambda_{l_{M'}}$. It is easy to see that for each $i\in\{1,\ldots,N\}\setminus I$ and each α there is a (not unique) set $E_{i\alpha}$, consisting of denumerably many parallel equidistant hyperplanes in R^n, and covering $\boldsymbol{CLP}\,(\Delta_i\cap\Delta_\alpha)$. For each α set

$$\Omega_\alpha = \Omega\setminus\Bigg(\bigcup_{\substack{1\le i\le N\\ i\notin I}} E_{i\alpha}\Bigg).$$

Then each Ω_α is a non-empty open subset of R^n and it follows from (2) that for each α

$$\sum_{i\in I} a_i\chi_{\Delta_i\cap\Delta_\alpha}(\gamma) = 0 \quad \text{for all } \gamma\in\Omega_\alpha\times\Gamma\,. \tag{3}$$

If $i\in I$, then either $\Delta_i\cap\Delta_\alpha=\varnothing$ or $\boldsymbol{CLP}\,(\Delta_i\cap\Delta_\alpha)=\mathsf{R}^n$. We also have $1\le\operatorname{card} I\le N-1$. Now suppose (*) the lemma has been proved for all positive integers less than N and all non-empty open subsets Ω of R^n. Then it follows from (3) that for each α

$$\sum_{i\in I} a_i\chi_{\Delta_i\cap\Delta_\alpha}(\gamma) = 0 \quad \text{for all } \gamma\in\mathsf{R}^n\times\Gamma\,. \tag{4}$$

(For some α we might have $\Delta_i\cap\Delta_\alpha\neq\varnothing$ for all $i\in I$.) But since $\bigcup_\alpha\Delta_\alpha=\mathsf{R}^n\times\Gamma$, having (4) for each α is equivalent with

$$\sum_{i\in I} a_i\chi_{\Delta_i}(\gamma) = 0 \quad \text{for all } \gamma\in\mathsf{R}^n\times\Gamma\,. \tag{5}$$

From (1) and (5) it follows that

$$\sum_{\substack{1\le i\le N\\ i\notin I}} a_i\chi_{\Delta_i}(\gamma) = 0 \quad \text{for all } \gamma\in\Omega\times\Gamma\,.$$

Under the assumption (*) this implies

(6) $$\sum_{\substack{1 \leqq i \leqq N \\ i \notin I}} a_i \chi_{\Delta_i}(\gamma) = 0 \quad \text{for all } \gamma \in \mathbf{R}^n \times \Gamma .$$

Cobining (5) and (6) we get

(7) $$\sum_{i=1}^{N} a_i \chi_{\Delta_i}(\gamma) = 0 \quad \text{for all } \gamma \in \mathbf{R}^n \times \Gamma .$$

(7) is proved under the assumption (*). But the lemma is obviously true for $N = 1$ and arbitrary non-empty open subsets Ω of $\mathbf{R}^n$. Hence the lemma follows by induction in Case ii.

We can now prove the following result.

THEOREM 3.1.2. *Let Ω be a non-empty open subset of $\mathbf{R}^n$. Let each P_i be a trigonometric polynomial on $\bar{\hat{\mathbf{R}}}^n \times G$, let each H_i be a compact subgroup of $\bar{\hat{\mathbf{R}}}^n \times G$, and let each m_{H_i} be the Haar measure of H_i $(i=1,\ldots,N)$. Set*

$$I = \{i \in \{1,\ldots,N\} \mid \boldsymbol{CLP}\,(H_i^{\perp}) = \mathbf{R}^n\}$$

($H_i^{\perp}$ denotes the annihilator of H_i in $\mathbf{R}_d^n \times \Gamma$). Suppose $(\sum_{i=1}^{N} P_i m_{H_i})\hat{}\,(\gamma) = 0$ for all $\gamma \in \Omega \times \Gamma$. Then $\sum_{i \in I} P_i m_{H_i} \equiv 0$.

PROOF. For $i \in \{1,\ldots,N\}$ we can write $P_i(x) = \sum_{j=1}^{N_i} a_{ij}(x, -\gamma_{ij})$, where $a_{i1},\ldots,a_{iN_i} \in \mathbf{C}$ and $\gamma_{i1},\ldots,\gamma_{iN_i} \in \mathbf{R}^n \times \Gamma$. We have

(1) $$\sum_{i=1}^{N} \sum_{j=1}^{N_i} a_{ij} \chi_{(\gamma_{ij} + H_i^{\perp})}(\gamma) = \left(\sum_{i=1}^{N} P_i m_{H_i}\right)^{\hat{}}(\gamma) = 0 \quad \text{for all } \gamma \in \Omega \times \Gamma .$$

For each $i \in \{1,\ldots,N\} \setminus I$ and each $j \in \{1,\ldots,N_i\}$ there is a (not unique) set E_{ij}, consisting of denumerably many parallel equidistant hyperplanes in $\mathbf{R}^n$, and covering $\boldsymbol{CLP}\,(\gamma_{ij} + H_i^{\perp})$. Set

$$\Omega_1 = \Omega \setminus \left(\bigcup_{\substack{1 \leqq i \leqq N \\ i \notin I}} \bigcup_{j=1}^{N_i} E_{ij} \right).$$

Then Ω_1 is a non-empty open subset of $\mathbf{R}^n$ and it follows from (1) that

$$\sum_{i \in I} \sum_{j=1}^{N_i} a_{ij} \chi_{(\gamma_{ij} + H_i^{\perp})}(\gamma) = 0 \quad \text{for all } \gamma \in \Omega_1 \times \Gamma .$$

Using Lemma 3.1.1 we conclude that

$$\sum_{i\in I}\sum_{j=1}^{N_i} a_{ij}\chi_{(\gamma_{ij}+H_i^\perp)}(\gamma) = 0 \qquad \text{for all } \gamma \in \mathbf{R}^n \times \Gamma .$$

Thus $\sum_{i\in I} P_i m_{H_i} \equiv 0$.

Theorem 3.1.2 with $\Omega = \mathbf{R}^n$ shows that if $\mu \in M(\bar{\hat{\mathbf{R}}}^n \times G)$ can be represented as a sum $\mu = \sum_{i=1}^N P_i m_{H_i}$ where P_i, H_i and m_{H_i} are as in Theorem 3.1.2, then $\sum_{i\in I} P_i m_{H_i}$, where

$$I = \{i \in \{1,\ldots,N\} \mid CLP(H_i^\perp) = \mathbf{R}^n\} ,$$

is independent of the particular representation and only depends on μ. Hence the following definition is not ambiguous.

Definition 3.1.3. Let $\mu \in M(\bar{\hat{\mathbf{R}}}^n \times G)$ be a measure of type $\mu = \sum_{i=1}^N P_i m_{H_i}$ where P_i, H_i and m_{H_i} are as in Theorem 3.1.2. Set $I = \{1,\ldots,N\} \mid CLP(H_i^\perp) = \mathbf{R}^n\}$. Then $\sum_{i\in I} P_i m_{H_i}$ is called the dense part of μ.

The following theorem is now an immediate consequence of Theorem 3.1.2 and this definition.

Theorem 3.1.4. *Let Ω be a non-empty open subset of $\mathbf{R}^n$. Let $\mu \in M(\bar{\hat{\mathbf{R}}}^n \times G)$ be a measure of type $\mu = \sum_{i=1}^N P_i m_{H_i}$ where P_i, H_i and m_{H_i} are as in Theorem* 3.1.2. *Suppose $\hat{\mu}(\gamma) = 0$ for all $\gamma \in \Omega \times \Gamma$. Then the dense part of μ is identically zero.*

3.2. *A consequence* (*Theorem* 3.2.3) *of a result of Davenport.*

It is easy to see that the following result follows directly from Davenport [3].

Lemma 3.2.1. *Let $\mu \in M(G)$. Let r be a positive integer. Let*

$$\gamma_0, \gamma_{11}, \ldots, \gamma_{1r}, \gamma_{21}, \ldots, \gamma_{2r}, \ldots, \gamma_{r^2 1}, \ldots, \gamma_{r^2 r} \in \Gamma .$$

Define the sets A_k inductively by $A_0 = \{\gamma_0\}$ and

$$A_k = A_{k-1} \cup \left(A_{k-1} + \bigcup_{1\leqq i<j\leqq r} \{\gamma_{ki} - \gamma_{kj}\} \right) \cup \{\gamma_{k1}, \ldots, \gamma_{kr}\}, \qquad 1 \leqq k \leqq r^2 - 1 .$$

Suppose

1° $|\mu(\gamma_0)| \geqq 1$ *and* $|\hat{\mu}(\gamma_{ij})| \geqq 1$ $\quad (1 \leqq i \leqq r^2,\ 1 \leqq j \leqq r)$,

2° *for each $k \in \{1,\ldots,r^2\}$ we have $\hat{\mu}(\gamma + \gamma_{ki} - \gamma_{kj}) = 0$ whenever $\gamma \in A_{k-1}$ and $1 \leqq i < j \leqq r$.*

Then $\|\mu\| > \frac{1}{100}\sqrt{r}$.

Before we can prove the main result of this section we need another lemma.

LEMMA 3.2.2. *Let* Ω *be a non-empty bounded open subset of* $\mathbf{R}^n$. *Let* $E \subset \Omega \times \Gamma$ *be such that*

1° *for each* $\gamma' \in \Omega$ *there exist a neighborhood* Ω' *of* γ' *in* $\mathbf{R}^n$ *and finitely many hyperplanes* $H_1, \ldots, H_N$ *in* $\mathbf{R}^n$ *such that* $E \cap (\Omega' \times \Gamma) \subset \bigcup_{i=1}^N H_i \times \Gamma$,

2° *there do not exist finitely many hyperplanes* $H_1, \ldots, H_N$ *in* $\mathbf{R}^n$ *such that* $E \subset \bigcup_{i=1}^N H_i \times \Gamma$.

Let $A \subset \Omega \times \Gamma$ *be a finite set. Let* r *be a positive integer. Then there exist* $\gamma_1, \ldots, \gamma_r \in E$ *such that* $\gamma + \gamma_i - \gamma_j \in (\Omega \times \Gamma) \setminus E$ *whenever* $\gamma \in A$ *and* $1 \leqq i < j \leqq r$.

PROOF. Since E satisfies 1° and 2° a compactness argument shows that there is a $\gamma_0' \in \partial\Omega$ with the property: for each neighborhood Ω' of γ_0' in $\mathbf{R}^n$ it is impossible to find finitely many hyperplanes $H_1, \ldots, H_M$ in $\mathbf{R}^n$ such that $E \cap (\Omega' \times \Gamma) \subset \bigcup_{i=1}^M H_i \times \Gamma$. Write $A = \{\alpha_1, \ldots, \alpha_N\}$. For each α_i' ($\alpha_i = (\alpha_i', \alpha_i'')$) take a neighborhood $\Omega_i \subset \Omega$ of α_i' in $\mathbf{R}^n$ and finitely many hyperplanes $H_{i1}, \ldots, H_{iN_i}$ in $\mathbf{R}^n$ such that $\alpha_i' \in H_{ij}$ ($1 \leqq j \leqq N_i$) and $E \cap (\Omega_i \times \Gamma) \subset \bigcup_{j=1}^{N_i} H_{ij} \times \Gamma$. This is possible because of 1°. For each $i \in \{1, \ldots, N\}$ and each $j \in \{1, \ldots, N_i\}$ let H_{ij}^0 be the unique hyperplane in $\mathbf{R}^n$ which passes through γ_0' and is parallel with H_{ij}. Set

$$E^0 = E \setminus \left(\bigcup_{i=1}^{N} \bigcup_{j=1}^{N_i} H_{ij}^0 \times \Gamma \right).$$

Because of our choice of γ_0' we have $E^0 \cap (\Omega' \times \Gamma) \neq \emptyset$ for each neighborhood Ω' of γ_0' in $\mathbf{R}^n$. Choose a neighborhood Δ_1 of γ_0' in $\mathbf{R}^n$ such that for each $i \in \{1, \ldots, N\}$ we have $\Delta_1 - \Delta_1 + \alpha_i' \subset \Omega_i$. Then choose $\gamma_1 \in (\gamma_1', \gamma_1'') \in E_0 \cap (\Delta_1 \times \Gamma)$. Observe that the definition of E^0 implies that $\gamma_1' \notin H_{ij}^0$ ($1 \leqq i \leqq N$, $1 \leqq j \leqq N_i$). Next choose a neighborhood $\Delta_2 \subset \Delta_1$ of γ_0' in $\mathbf{R}^n$ such that $\gamma_1' - \gamma'$ is not parallel with any H_{ij}^0 ($1 \leqq i \leqq N$, $1 \leqq j \leqq N_i$) for any $\gamma' \in \Delta_2$. Such a neighborhood Δ_2 can always be found. Every sufficiently small neighborhood of γ_0' is an acceptable choice for Δ_2, because $\gamma_1' \notin H_{ij}^0$ ($1 \leqq i \leqq N$, $1 \leqq j \leqq N_i$). Then choose $\gamma_2 = (\gamma_2', \gamma_2'') \in E^0 \cap (\Delta_2 \times \Gamma)$. As above $\gamma_2' \notin H_{ij}^0$ ($1 \leqq i \leqq N$, $1 \leqq j \leqq N_i$). Next choose a neighborhood $\Delta_3 \subset \Delta_2$ of γ_0' in $\mathbf{R}^n$ such that $\gamma_2' - \gamma'$ is not parallel with any H_{ij}^0 ($1 \leqq i \leqq N$, $1 \leqq j \leqq N_i$) for any $\gamma' \in \Delta_3$. Continue this construction of Δ_i:s and γ_i:s r times. Since $E^0 \cap (\Omega' \times \Gamma) \neq \emptyset$ for each neighborhood Ω' of γ_0' in $\mathbf{R}^n$, this is always possible. Note that the construction is done in such a way that

$$A + \gamma_i - \Delta_{i+1} \times \Gamma \subset (\Omega \times \Gamma) \setminus E \qquad (i = 1, \ldots, r-1)$$

(just look at the $\mathbf{R}^n$-components). Also note that $\gamma_j \in \Delta_{i+1} \times \Gamma$ if $j>i$. Thus

$$A+\gamma_i-\gamma_j \subset (\Omega \times \Gamma) \setminus E \quad \text{if } 1 \leqq i<j \leqq r .$$

This proves the lemma.

The main result of this section is

THEOREM 3.2.3. *Let Ω be a bounded open subset of $\mathbf{R}^n$. Let $\hat{\mu} \in B(\mathbf{R}_d^n \times \Gamma)$ be integer-valued on $\Omega \times \Gamma$. Set $E=\{\gamma \in \Omega \times \Gamma \mid \hat{\mu}(\gamma) \neq 0\}$. Suppose E satisfies*

(*) *for each $\gamma' \in \Omega$ there exist a neighborhood Ω' of γ' in $\mathbf{R}^n$ and finitely many hyperplanes $H'_1,\ldots,H'_M$ in $\mathbf{R}^n$ such that $E \cap (\Omega' \times \Gamma) \subset \bigcup_{i=1}^M H'_i \times \Gamma$.*

Then there exist finitely many hyperplanes $H_1,\ldots,H_N$ in $\mathbf{R}^n$ such that $E \subset \bigcup_{i=1}^N H_i \times \Gamma$.

PROOF. Suppose there do not exist finitely many hyperplanes $H_1,\ldots,H_N$ *in* $\mathbf{R}^n$ such that $E \subset \bigcup_{i=1}^N H_i \times \Gamma$. Since (*) is the same as 1° in Lemma 3.2.2, then E satisfies both 1° and 2° of Lemma 3.2.2. Let r be a positive integer such that $\frac{1}{100}\sqrt{r} > \|\mu\|$. Choose $\gamma_0 \in E$. Set $A_0=\{\gamma_0\}$. Then choose $\gamma_{11},\ldots,\gamma_{1r} \in E$ such that $A_0+\gamma_{1i}-\gamma_{1j} \subset (\Omega \times \Gamma) \setminus E$ if $1 \leqq i<j \leqq r$. This is possible according to Lemma 3.2.2. Set

$$A_1=A_0 \cup \left(A_0+\bigcup_{1 \leqq i<j \leqq r}\{\gamma_{1i}-\gamma_{1j}\}\right) \cup \{\gamma_{11},\ldots,\gamma_{1r}\} .$$

Note that $A_1 \subset \Omega \times \Gamma$. Then choose $\gamma_{21},\ldots,\gamma_{2r} \in E$ such that $A_1+\gamma_{2i}-\gamma_{2j} \subset (\Omega \times \Gamma) \setminus E$ if $1 \leqq i<j \leqq r$. Since $A_1 \subset \Omega \times \Gamma$ and A_1 is finite, this is possible according to Lemma 3.2.2. Set

$$A_2 = A_1 \cup \left(A_1+\bigcup_{1 \leqq i<j \leqq r}\{\gamma_{2i}-\gamma_{2j}\}\right) \cup \{\gamma_{21},\ldots,\gamma_{2r}\} .$$

Note that $A_2 \subset \Omega \times \Gamma$. Then choose $\gamma_{31},\ldots,\gamma_{3r} \in E$ such that

$$A_2+\gamma_{3i}-\gamma_{3j} \subset (\Omega \times \Gamma) \setminus E \quad \text{if } 1 \leqq i<j \leqq r .$$

Continue this construction r^2 times. Note that the construction is carried out so that

1° $|\hat{\mu}(\gamma_0)| \geqq 1$ and $|\hat{\mu}(\gamma_{ij})| \geqq 1 \quad (1 \leqq i \leqq r^2,\ 1 \leqq j \leqq r)$,
2° for each $k \in \{1,\ldots,r^2\}$ we have $\hat{\mu}(\gamma+\gamma_{ki}-\gamma_{kj})=0$ whenever $\gamma \in A_{k-1}$ and $1 \leqq i<j \leqq r$.

Also note that the sets $A_0,\ldots,A_{r^2-1}$ are constructed from the characters

$\gamma_0, \gamma_{11}, \ldots, \gamma_{1r}, \gamma_{21}, \ldots, \gamma_{2r}, \ldots, \gamma_{r^21}, \ldots, \gamma_{r^2r}$ in a similar way as the corresponding sets in Lemma 3.2.1 from the corresponding characters there. Hence it follows from Lemma 3.2.1 that $\|\mu\| > \frac{1}{100}\sqrt{r}$. But r is chosen so that $\frac{1}{100}\sqrt{r} > \|\mu\|$. We have reached a contraction. It follows that our assumption in the beginning of the proof was false, and thus there exist finitely many hyperplanes $H_1, \ldots, H_N$ in $\mathbf{R}^n$ such that $E \subset \bigcup_{i=1}^N H_i \times \Gamma$.

3.3. Proof of Theorem B. We first prove

Theorem 3.3.1. *Let Ω be a bounded open convex subset of $\mathbf{R}^n$. Suppose $\hat{\mu} \in B(\mathbf{R}_d^n \times \Gamma)$ is integer-valued on $\Omega \times \Gamma$. Then there exists an integer-valued $\hat{\nu} \in B(\mathbf{R}_d^n \times \Gamma)$ such that $\hat{\mu} = \hat{\nu}$ on $\Omega \times \Gamma$.*

Proof. Using Theorem C combined with Cohen's theorem it is easy to see that there exist denumerably many open subsets $\Omega_1, \Omega_2, \ldots$ of $\mathbf{R}^n$ and denumerably many measures $\varrho_1, \varrho_2, \ldots$ each of which is a finite sum of canonical measures in $M(\bar{\bar{\mathbf{R}}} \times G)$ so that

(1) $\bigcup_{i=1}^\infty \Omega_i = \Omega$,
(2) each compact subset of Ω only intersects finitely many of the Ω_i:s
(3) $\hat{\mu} = \hat{\varrho}_i$ on $\Omega_i \times \Gamma$.

If $\Omega_i \cap \Omega_j \neq \emptyset$ then it follows from Theorem 3.1.4 that ϱ_i and ϱ_j have the same dense part. Since a convex set is connected we conclude that all ϱ_i have the same dense part. Call this common dense part ϱ. Set $\sigma = \mu - \varrho$. Ω and σ satisfy the hypotheses of Theorem 3.2.3. Hence there exist finitely many hyperplanes $H_1, \ldots, H_N$ in $\mathbf{R}^n$ intersecting Ω and satisfying

$$\{\gamma \in \Omega \times \Gamma \mid \hat{\sigma}(\gamma) \neq 0\} \subset \bigcup_{i=1}^N H_i \times \Gamma\,.$$

We now make two observations.

(i) Suppose $n=1$. A hyperplane in $\mathbf{R}$ is a point. Hence in this case there are finitely many distinct points $\gamma_1', \ldots, \gamma_N' \in \Omega$ such that

$$\{\gamma \in \Omega \times \Gamma \mid \hat{\sigma}(\gamma) \neq 0\} \subset \bigcup_{i=1}^N \{\gamma_i'\} \times \Gamma\,.$$

For each $i \in \{1, \ldots, N\}$ let φ_i be the function on $\mathbf{R} \times \Gamma$ which is equal to $\hat{\sigma}$ on $\{\gamma_i'\} \times \Gamma$ and zero elsewhere. Then each $\varphi_i \in B(\mathbf{R}_d \times \Gamma)$. Set $\varphi = \hat{\varrho} + \sum_{i=1}^N \varphi_i$. Then φ is integer-valued, $\varphi \in B(\mathbf{R}_d \times \Gamma)$, and $\hat{\mu} = \varphi$ on $\Omega \times \Gamma$.

(ii) Suppose the theorem has been proved for the dimension $n-1$ ($n \geqq 2$). Let $\Omega_i = \Omega \cap H_i$ ($1 \leqq i \leqq N$). Then Ω_i is a bounded convex relatively open subset of

H_i $(1 \leqq i \leqq N)$. (Each H_i is endowed with the relative topology from $\mathbf{R}^n$.) Since the theorem is supposed true for the dimension $n-1$ it is easy to see that there are N inter-valued functions $\varphi_1, \ldots, \varphi_N \in B(\mathbf{R}_d^n \times \Gamma)$ such that $\varphi_1 = \hat{\sigma}$ on $H_1 \times \Gamma$ and $\varphi_1 = 0$ off $H_1 \times \Gamma$, $\varphi_2 = \hat{\sigma} - \varphi_1$ on $H_2 \times \Gamma$ and $\varphi_2 = 0$ off $H_2 \times \Gamma$, $\varphi_3 = \hat{\sigma} - \varphi_1 - \varphi_2$ on $H_3 \times \Gamma$ and $\varphi_3 = 0$ off $H_3 \times \Gamma, \ldots,$ and $\varphi_N = \hat{\sigma} - \sum_{i=1}^{N-1} \varphi_i$ on $H_N \times \Gamma$ and $\varphi_N = 0$ off $H_N \times \Gamma$. Set $\varphi = \hat{\varrho} + \sum_{i=1}^N \varphi_i$. Then φ is integer-valued, $\varphi \in B(\mathbf{R}_d^n \times \Gamma)$, and $\hat{\mu} = \varphi$ on $\Omega \times \Gamma$. Hence the theorem is true for dimension n if it is true for dimension $n-1$ $(n \geqq 2)$.

Combining (i) and (ii) the theorem follows by induction.

Theorem B is now obtained as an easy corollary of Theorem 3.3.1.

THEOREM B. *Let Ω be an open convex slice of $\mathbf{R}^n$. Suppose $\hat{\mu} \in B(\mathbf{R}_d^n)$ is integer-valued on Ω. Then there exists an integer-valued $\hat{\nu} \in B(\mathbf{R}_d^n)$ such that $\hat{\mu} = \hat{\nu}$ on Ω.*

PROOF. Combine Theorem 3.3.1 with Lemma 4.2.1 in Rudin [10, p. 79].

4. Some examples and remarks.

4.1. *A counterexample for* T (*Example* 4.1.2).

In this section we give Example 4.1.2 referred to in the introduction. But first a lemma.

LEMMA 4.1.1. *Let $a_1, \ldots, a_N \in \mathbf{C}$. Let $\Delta_1, \ldots, \Delta_N$ be cosets of subgroups of* $\mathbf{R}$. *Let*

$$I = \{i \in \{1, \ldots, N\} \mid \text{closure of } \Delta_i = \mathbf{R}\}.$$

Suppose $\sum_{i=1}^N a_i \chi_{\Delta_i}(\gamma)$ is 1-periodic. Then $\sum_{i \in I} a_i \chi_{\Delta_i}(\gamma)$ is 1-periodic.

PROOF. Set

$$\Omega = \mathbf{R} \setminus \left(\left(\bigcup_{\substack{1 \leqq i \leqq N \\ i \notin I}} \right) \cup \left(\bigcup_{\substack{1 \leqq i \leqq N \\ i \notin I}} (\Delta_i - 1) \right) \right).$$

Note that Ω is a non-empty open subset of $\mathbf{R}$. Also note that

$$\sum_{i=1}^N a_i \chi_{\Delta_i}(\gamma) = \sum_{i=1}^N a_i \chi_{\Delta_i}(\gamma + 1) \qquad \text{for all } \gamma \in \mathbf{R}$$

$$\Rightarrow \quad \sum_{i=1}^N a_i \chi_{\Delta_i}(\gamma) = \sum_{i=1}^N a_i \chi_{(\Delta_i - 1)}(\gamma) \qquad \text{for all } \gamma \in \mathbf{R}$$

$$\Rightarrow \quad \sum_{i \in I} a_i \chi_{\Delta_i}(\gamma) = \sum_{i \in I} a_i \chi_{(\Delta_i - 1)}(\gamma) \qquad \text{for all } \gamma \in \Omega .$$

Using Lemma 3.1.1 it follows that $\sum_{i\in I} a_i\chi_{\Delta_i}(\gamma)=\sum_{i\in I} a_i\chi_{(\Delta_i-1)}(\gamma)$ for all $\gamma \in \mathsf{R}$, which means

$$\sum_{i\in I} a_i\chi_{\Delta_i}(\gamma) = \sum_{i\in I} a_i\chi_{\Delta_i}(\gamma+1) \quad \text{for all } \gamma \in \mathsf{R}\,.$$

EXAMPLE 4.1.2. There exists a $\hat{\mu} \in B(\mathsf{T}_d)$ such that

1° $\hat{\mu}$ is canonical on a neighborhood of the origin in T,
2° it is impossible to find a neighborhood Ω of the origin in T and an integer-valued $\hat{\nu} \in B(\mathsf{T}_d)$ such that $\hat{\mu}=\hat{\nu}$ on Ω.

PROOF. Let φ be a 1-periodic function on R which is equal to $\chi_{\sqrt{2}\mathsf{Q}}$ (Q is the rationals and $\chi_{\sqrt{2}\mathsf{Q}}$ is the characteristic function of $\sqrt{2}\mathsf{Q}$) on a neighborhood of the origin in R. It is then obvious that φ considered as a function on T is canonical on some neighborhood of the origin in T. By Theorem A there exist a neighborhood Ω of the origin in T and a $\hat{\mu} \in B(\mathsf{T}_d)$ such that $\varphi=\hat{\mu}$ on Ω. Suppose there exist a neighborhood Ω' of the origin in T and an integer-valued $\hat{\nu} \in B(\mathsf{T}_d)$ such that $\hat{\mu}=\hat{\nu}$ on Ω'. Now consider ν as a measure in $M(\bar{\mathsf{R}})$ ($M(\bar{\mathsf{Z}}) \subset M(\bar{\mathsf{R}})$) and let ϱ be the dense part of ν (ν is of type $\sum P_i m_{H_i}$ by Cohen's theorem). Since $\hat{\nu}=\hat{\mu}=\chi_{\sqrt{2}\mathsf{Q}}$ near the origin in R, it follows using Lemma 3.1.1 that $\hat{\varrho}\equiv\chi_{\sqrt{2}\mathsf{Q}}$. But Lemma 4.1.1 shows that since $\hat{\nu}$ is 1-periodic, $\hat{\varrho}$ must be 1-periodic. Since $\chi_{\sqrt{2}\mathsf{Q}}$ is not 1-periodic we have reached a contradiction, and therefore there does not exist an integer-valued $\hat{\nu} \in B(\mathsf{T}_d)$ such that $\hat{\mu}=\hat{\nu}$ near the origin in T.

Example 4.1.2 when combined with Cohen's theorem shows that "each being canonical on Ω_2" in 2° of Theorem A cannot, in general, be replaced by "each being canonical on Γ". Since a function canonical on the whole of Γ is integer-valued on Γ and belongs to $B(\Gamma_d)$, Example 4.1.2 also shows that a function which is canonical only on a neighborhood of a point in Γ is not, in general, a restriction of a function canonical on the whole of Γ.

4.2. *The case* R^n.

In the section we shall further illustrate what can and what cannot be expected of functions in $B(\mathsf{R}^n_d)$ which are integer-valued on a given open subset of R^n.

EXAMPLE 4.2.1. Let Ω be a non-empty bounded open subset of R^n whose closure $\bar{\Omega}$ in R^n is non-convex. Then there exists a $\hat{\mu} \in B(\mathsf{R}^n_d)$, integer-valued on Ω and such that $\hat{\mu}|_\Omega \neq \hat{\nu}|_\Omega$ for all integer-valued $\hat{\nu} \in B(\mathsf{R}^n_d)$.

PROOF. A simple argument shows that there are three line-segments of the same line L such that the two outer ones, I_1 and I_2 say, lie in Ω, and the middle one, I_3 say, lies in the complement of $\bar{\Omega}$. Give the line L a real line order $>$ so that $\gamma' \in I_1$ and $\gamma'' \in I_2 \Rightarrow \gamma' > \gamma''$ and choose $\gamma_1, \gamma_2 \in I_3$ with $\gamma_1 < \gamma_2$. Then choose $\gamma_3 \in L$ so that $\gamma \in L$ and $\gamma > \gamma_3 \Rightarrow \gamma \notin \Omega$. Let ψ be a compactly supported C^∞ function on L which is 1 if $\gamma \in L$ and $\gamma_2 < \gamma < \gamma_3$ and which is 0 if $\gamma \in L$ and $\gamma < \gamma_1$. Define φ on $\mathbf{R}^n$ by setting $\varphi = \psi$ on L and $\varphi = 0$ elsewhere. Then $\varphi \in B(\mathbf{R}^n_d)$ and φ is integer-valued on Ω. Now suppose there exists an integer-valued $\hat{v} \in B(\mathbf{R}^n_d)$ such that $\varphi = \hat{v}$ on Ω. Let η be the restriction of $\hat{v}$ to L. Since $\eta(\gamma) = \psi(\gamma)$ for all $\gamma \in \Omega$ we have

(1) $\eta(\gamma) = 1$ on I_1,
(2) $\eta(\gamma) = 0$ on I_2.

η is a sum of type $\sum_{i=1}^N a_i \chi_{\Delta_i}$ where each $a_i \in \mathbf{Z}$ and each Δ_i is a coset included in L. Let

$$I = \{i \in \{1, \ldots, N\} \mid \text{closure of } \Delta_i = L\} .$$

Combining (1) with Lemma 3.1.1 we get $\sum_{i \in I} a_i \chi_{\Delta_i} \equiv 1$. But combining (2) with Lemma 3.1.1 we get $\sum_{i \in I} a_i \chi_{\Delta_i} \equiv 0$. This contradiction shows that there does not exist an integer-valued $\hat{v} \in B(\mathbf{R}^n_d)$ such that $\varphi = \hat{v}$ on Ω.

REMARK 4.2.2. If $n \geqq 2$ and if Ω is a non-empty bounded non-convex open subset of $\mathbf{R}^n$ such that $\bar{\Omega}$ is convex, then a result corresponding to Theorem B may or may not be true.

If Ω is an open ball minus its centre, then such a result is true.

If Ω is an open ball minus a closed line-segment contained in the ball, then such a result is not true. A counterexample may be constructed as in Example 4.2.1.

REMARK 4.2.3. If Ω is a non-empty bounded non-convex open subset of $\mathbf{R}$ such that $\bar{\Omega}$ is convex, then a result corresponding to Theorem B is always true. This is an easy consequence of Theorems B, 3.1.4 and 3.2.3.

Example 4.2.1 shows that if Ω is a bounded open subset of $\mathbf{R}^n$ and if a result corresponding to Theorem B should be true, then $\bar{\Omega}$ must be convex. However, for unbounded open subsets of $\mathbf{R}^n$ this is not the case, as the following remark shows.

REMARK 4.2.4. Let

$$\Omega = \{(\gamma_1, \ldots, \gamma_n) \in \mathbf{R}^n \mid |\gamma_1| > 1\} .$$

Then a result corresponding to Theorem B is true. This may be seen by using

(1) the theorem for open half-spaces given in the introduction,
(2) the well-known fact that if Γ is a totally ordered discrete abelian group and $\hat{\varrho} \in B(\Gamma)$, then $\lim_{\gamma\to\infty} \hat{\varrho}(\gamma)=0 \Leftrightarrow \lim_{\gamma\to-\infty} \hat{\varrho}(\gamma)=0$,
(3) Theorem 3.1.4.

In fact, suppose $\hat{\mu} \in B(\mathbf{R}_d^n)$ is integer-valued on Ω. Using (1) we get an integer-valued $\hat{\sigma} \in B(\mathbf{R}_d^n)$ such that $\hat{\mu}(\gamma)=\hat{\sigma}(\gamma)$ if $\gamma_1>1$ $(\gamma=(\gamma_1,\ldots,\gamma_n) \in \mathbf{R}^n)$. Since $\hat{\mu}-\hat{\sigma}$ is integer-valued on Ω we also have $\hat{\mu}(\gamma)=\hat{\sigma}(\gamma)$ if $\gamma_1<\alpha_0$ for some $\alpha_0 \leqq -1$. This follows from (2) if we give $\mathbf{R}^n$ the dictionary order. Using (1) once more we get an integer-valued $\hat{\tau} \in B(\mathbf{R}_d^n)$ such that $\hat{\mu}(\gamma)-\hat{\sigma}(\gamma)=\hat{\tau}(\gamma)$ if $\gamma_1<-1$, and since $\hat{\mu}(\gamma)-\hat{\sigma}(\gamma)=0$ if $\gamma_1<\alpha_0$ it is not hard to see because of (3) that this $\hat{\tau}$ can be chosen so that $\hat{\tau}(\gamma)=0$ if $\gamma_1 \geqq -1$. If we do that and set $\varphi=\hat{\sigma}+\hat{\tau}$, then $\varphi \in B(\mathbf{R}_d^n)$, φ is integer-valued, and $\hat{\mu}=\varphi$ on Ω.

REFERENCES

1. I. Amemiya and T. Ito, *A simple proof of a theorem of P. J. Cohen*, Bull. Amer. Math. Soc. 70 (1964), 774–776.
2. P. J. Cohen, *On a conjecture of Littlewood and idempotent measures*, Amer. J. Math. 82 (1960), 191–212.
3. H. Davenport, *On a theorem of P. J. Cohen*, Mathematika 7 (1960), 93–97.
4. I. Glicksberg *Fourier–Stieltjes transforms with an isolated value*, in *Conference on Harmonic Analysis* (University of Maryland, College Park, 1971), Lecture Notes in Math. 266, Springer-Verlag, Berlin - Heidelberg - New York, 1972.
5. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, II (Grundlehren Math. Wissensch. 152), Springer-Verlag, Berlin - Heidelberg - New York, 1972.
6. J. L. Kelley, *General topology*, Van Nostrand, New York, 1961.
7. I. Kessler, *Semi-idempotent measures on abelian groups*, Bull. Amer. Math. Soc. 73 (1967), 258–260.
8. I. Kessler, *Semi-idempotent measures on abelian groups*, Southern Illinois University at Edwardsville, Papers in Mathematics and the Mathematical Sciences, no. 9, 1972.
9. Y. Meyer, *Algebraic numbers and harmonic analysis*, North Holland, Amsterdam, 1972.
10. W. Rudin, *Fourier analysis on groups* (Interscience Tracts in Pure and Applied Mathematics 12), Interscience Publ., New York, 1967.

DEPARTMENT OF MATHEMATICS
UPPSALA UNIVERSITY
THUNBERGSVÄGEN 3
752 38 UPPSALA
SWEDEN