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NON-HYPERBOLIC CLOSED GEODESICS

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On negatively curved compact Riemannian manifolds and more generally on compact Riemannian manifolds of Anosov type the geodesic flow is ergodic and every periodic orbit is hyperbolic. On the other hand it is known that there exist stable periodic orbits in the geodesic flow of surfaces which are not too dissimilar from a three-axial ellipsoid. Hence the geodesic flow is not ergodic.

The purpose of this paper is to find conditions which imply that the geodesic flow behaves similar to the ellipsoid case. More precisely we will search for conditions on a Riemannian manifold which ensure the existence of a non-hyperbolic closed geodesic.

The main result will be that a Riemannian manifold for which the sectional curvature satisfies

$$\tfrac{4}{9}\max K < \min K$$

has a closed geodesic of length $<3\pi(\max K)^{-1/2}$ which is not hyperbolic. In the two dimensional case it will be shown that

$$\tfrac{1}{4}\max K \leqq \min K$$

ensures the existence of an infinitesimally stable closed geodesic of length $\leqq 4\pi(\max K)^{-1/2}$. This implies that the existence of a stable periodic orbit and hence of a non-ergodic geodesic flow is an open and dense property in the set of convex surfaces whose Gaussian curvature satisfies

$$\tfrac{1}{4}\max K < \min K\,.$$

(See [11] for a discussion of stability problems and [7] or [6, p. 114] for a perturbation theorem.)

In the literature the following results on non-hyperbolic closed geodesics are known to us.

Poincaré [13], referring to his continuation method, asserted that an infinitesimally stable closed geodesic without self-intersections exists on any analytic convex surface. But the sketch of a proof he gave is rather

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unconvincing and a satisfactory proof has never been given. (Morse discusses some of the difficulties in connection with Poincaré's continuation method in [10, p. 305].)

In [6, pp. 165–166] it is shown that the shortest closed geodesic in a free homotopy class cannot be hyperbolic when the sectional curvature is positive.

In [15] Ziller proves that every closed geodesic on a naturally reductive homogeneous space is infinitesimally stable. He also gives an example of a homogeneous space where this is not the case.

In section 1 we review some definitions and theorems which will be used in the sequel.

In section 2 we extend a result of Klingenberg on the existence of short closed geodesics on pinched spheres.

In section 3 we prove the theorem on (4/9)-pinched spheres mentioned above. Furthermore we prove that short closed geodesics of odd index k cannot be hyperbolic if the manifold satisfies a certain pinching condition which depends on k.

In section 4 we study the influence of the isometry group on the existence of non-hyperbolic closed geodesics. The main theorem in this section states that the isometry group of a compact manifold is finite if all closed geodesics are hyperbolic.

1. Preliminaries.

Let M be a Riemannian manifold. Let $\varphi_t X$, $t \in \mathbf{R}$, be a periodic orbit in the geodesic flow in T_1M. $\varphi_t X$ is called *stable*, if for every tubular neighbourhood $\mathscr{U}$ of $\varphi_t X$ there is a neighbourhood $\mathscr{V}$ of X such that $\varphi_t Y \in \mathscr{U}$ for every $Y \in \mathscr{V}$ and every $t \in \mathbf{R}$.

Let Σ be a hypersurface in T_1M which is transversal to $\varphi_t X$ in X. Then there is a neighbourhood Σ^* of X in Σ and maps $\mathscr{P}: \Sigma^* \to \Sigma$ and $T: \Sigma^* \to \mathbf{R}$ such that $\mathscr{P}(Y) = \varphi_{T(Y)} Y$ for every $Y \in \Sigma^*$. The map $\mathscr{P}$ is called a *Poincaré map* of $\varphi_t X$. X is a fixed point of $\mathscr{P}$. Two Poincaré maps of $\varphi_t X$ are locally conjugate. The eigenvalues of $d\mathscr{P}_X$ are therefore invariants of the periodic orbit $\varphi_t X$.

If $d\mathscr{P}_X$ has an eigenvalue λ with $|\lambda| \neq 1$, then $\varphi_t X$ is not stable.

The periodic orbit $\varphi_t X$ and the closed geodesic $\pi(\varphi_t X)$ are called *hyperbolic* if no eigenvalue of $d\mathscr{P}_X$ has absolute value one; *elliptic* if all eigenvalues have absolute value one and are $\neq 1$; *degenerate* if 1 is an eigenvalue; and *infinitesimally stable* if all eigenvalues are of absolute value one.

If λ is an eigenvalue of $d\mathscr{P}_X$, then also $\bar{\lambda}$, λ^{-1} and $\bar{\lambda}^{-1}$. This shows that the eigenvalues of $d\mathscr{P}_X$ are either real or of absolute value one if $\dim M = 2$. Hence a non-hyperbolic closed geodesic on a surface is infinitesimally stable.

The index of a closed geodesic $c: [0, \omega] \to M$ is equal to the index of c as a geodesic segment plus concavity [8]. The concavity is $\geqq 0$ and $\leqq (n-1)$, where $n = \dim M$.

We will make a repetitive use of the *Index Theorem for Hyperbolic Closed Geodesics* which we now shortly describe. The proof of this theorem which is much easier than the general case in [6] can be found in [5].

Let $c\colon [0,\omega] \to M$ be a closed geodesic with arc-length parameter and assume that $\dim M = n+1$. TTM has a canonical splitting into the so-called horizontal and vertical bundles both of which are isomorphic to TM. This splitting induces a Riemannian metric on TM.

Let $V^{2n}(t)$ be the subspace in $T_{\dot c(t)}T_1M$ which consists of all vectors which are orthogonal to the tangent vector of the geodesic flow in $\dot c(t)$.

The splitting of TTM determines a splitting

$$V^{2n}(t) = V_h^n(t) \oplus V_v^n(t).$$

$V^{2n}(t)$ also splits in a canonical way into $d\varphi_t$-invariant bundles

$$V^{2n}(t) = V_s^n(t) \oplus V_u^n(t)$$

where

$$V_s^n(t) := d\varphi_t\Big(\bigoplus_{|\lambda|<1} V(\lambda)\Big) \quad \text{and} \quad V_u^n(t) := d\varphi_t\Big(\bigoplus_{|\lambda|>1} V(\lambda)\Big).$$

$V_s^n(t)$ and $V_u^n(t)$ are called the stable and the unstable bundles, respectively. $V(\lambda)$ is the generalized eigenspace of the eigenvalue λ of $d\mathscr{P}_{\dot c(t)}$; $\mathscr{P}$ is defined with respect to a hypersurface Σ orthogonal to the orbit $\dot c(t)$.

Then the Index Theorem for Hyperbolic Closed Geodesics says that

$$\operatorname{index} c = \sum_{0 \leqq t < \omega} \dim\,(V_v^n(t) \cap V_s^n(t)).$$

An immediate consequence is the property of hyperbolic closed geodesics that

$$\operatorname{index} c^n = n \operatorname{index} c.$$

The connection with Jacobi fields is as follows:

Let $(v,w) \in V^{2n}(0)$ (where v and w are horizontal and vertical components). Then the horizontal component of $d\varphi_t(v,w) = (Y(t), \nabla Y(t)) \in V^{2n}(t)$ is a Jacobi field. A Jacobi field with the property that $(Y(t), \nabla Y(t)) \in V_s^n(t)$ for every $t \in [0,\omega]$ is called *stable*. Hence

$$\dim\,(V_v^n(t') \cap V_s^n(t')) > 0$$

if and only if a stable Jacobi field vanishes in t'.

In section 2 we construct cycles in $\Pi M \operatorname{mod} \Pi^0 M$. ΠM is the set of unparameterized piecewise differentiable closed curves on M with the topology induced by the compact-open topology. Here an unparameterized closed curve is an equivalence class of closed curves with respect to the equivalence relation "$c \sim d$ if and only if $c = d \circ \alpha$ for an $\alpha \in O(2)$". For $\varkappa \geqq 0$ we denote by $\Pi^{\varkappa} M$ the set of all curves in ΠM which have energy $\leqq \varkappa$.

2. Short closed geodesics on pinched spheres.

In this and the next section M will be a simply connected n-dimensional Riemannian manifold for which the sectional curvature satisfies

$$\tfrac{1}{4} \max K < \min K .$$

It is the content of the Sphere Theorem that these curvature conditions imply that M is homeomorphic to S^n.

By multiplication by a constant it is possible to normalize the Riemannian metric such that $\max K = 1$. Since this change of scale has no influence on the properties of the geodesic flow which we are going to investigate, we will assume $\max K = 1$ or

$$\tfrac{1}{4} < K \leqq 1 .$$

For convenience we set $\delta := \min K$ and $\sigma := \delta^{-1/2}$.

We will frequently use the fact that $d(p, C(p)) \geqq \pi$ for every $p \in M$. ($C(p)$ denotes the cut locus of p.) This shows that the length of a geodesic loop on M is $\geqq 2\pi$. The closed geodesics which we are going to study will be of length $< 2\sigma\pi < 4\pi$. They are therefore obviously without selfintersections. For the rest of this section we will assume that *every closed geodesic on M of length $< 2\sigma\pi$ is non-degenerate.* (M is therefore in particular not a sphere of constant curvature.)

The following theorem is due to Klingenberg.

2.1. THEOREM (Klingenberg). *There is at least one closed geodesic of index k on M for every natural number k in the interval $[(n-1), 2(n-1)]$ which is of length $< 2\sigma\pi$ and therefore without self-intersections.*

PROOF. See [6]. The existence of n closed geodesics of length $< 2\sigma\pi$ is proven in [4] and on p. 54 in [6] (without the non-degeneracy assumption). For the claim about the indices see the proof of Theorem (2.5) below.

Our next aim is the construction of cycles in $\Pi M \bmod \Pi^0 M$ which will be used in the proof of Theorem (2.5) below. A necessary condition on the sectional curvature will be

$$\tfrac{4}{9} < \delta .$$

We begin with some preliminaries.

Choose a real number η in the open interval $((2\sigma-2)\pi, \pi)$. This interval is non-empty since $2\sigma<3$. Fix a point $r \in M$. The closed normal coordinate ball $B_{\eta/2}(r)$ is convex in the sense that for any two points $s, s' \in B_{\eta/2}(r)$ there is a unique minimal geodesic γ connecting s and s' and the image of γ is contained in $B_{\eta/2}(r)$ [2, p. 160]. Let N be the boundary of $B_{\eta/2}(r)$. N is a differentiable submanifold. Define a differentiable involution $A\colon N \to N$ as follows: p is mapped into the first point in which the geodesic from p through r intersects N again. Obviously $d(p, Ap)=\eta$.

2.2. LEMMA. *Let $s \in M$ have the property that*

$$d(s,p) = d(s,Ap)$$

for some $p \in N$. Then $d(s,p)=d(s,Ap)<\pi$.

PROOF. Suppose that $d(s,p)=d(s,Ap)\geqq\pi$. The circumference of a triangle on M is $\leqq 2\sigma\pi$ (Toponogov's Theorem). Hence $d(p,Ap)\leqq 2\sigma\pi-2\pi<\eta=d(p,Ap)$ —a contradiction.

We now define with help of the involution $A\colon N \to N$ the following sets for $p \in N$.

$$M_+(p) := \{s \in M \mid d(p,s)\leqq d(Ap,s)\}$$

$$M_-(p) := \{s \in M \mid d(p,s)\geqq d(Ap,s)\}$$

$$E(p) := M_+(p) \cap M_-(p)\,.$$

2.3. LEMMA. *A geodesic c of length π emanating from $p \in N$ meets $E(p)$ in exactly one point.*

PROOF. Let $c\colon [0,\pi] \to M$ be a geodesic with $c(0)=p$ and $\|\dot{c}(t)\|=1$. c must meet $E(p)$ at least once because the function

$$f(t) := d(p,c(t))-d(Ap,c(t))$$

takes on positive and negative values as can be seen with the methods of the proof of Lemma (2.2).

If c meets $E(p)$ twice, then there are $0<t<t'\leqq\pi$ such that

$$d(Ap,c(t))-d(p,c(t)) = d(Ap,c(t'))-d(p,c(t')) = 0\,.$$

Since the geodesic c is minimal between each of its points, we have

$$d(p,c(t')) = d(p,c(t))+d(c(t),c(t'))\,.$$

Hence $d(Ap,c(t')) = d(Ap,c(t))+d(c(t),c(t'))$. This implies that the minimal geodesic joining Ap and $c(t')$ contains the geodesic segment $c\,|\,[0,t']$. Hence

$$d(Ap,c(t')) = d(Ap,p)+d(p,c(t')) > d(p,c(t'))$$

—a contradiction.

We can now begin with the construction of the cycles in $\Pi M \bmod \Pi^0 M$ which we will use in the proof of Theorem (2.5) below. In $\Pi S^n \bmod \Pi^0 S^n$ we have for $0 \leqq a \leqq b \leqq n-1$ the cycle $\{a,b\}$ defined as follows: $\{a,b\}$ consists of all circles parallel to a great circle in the subsphere $S^{b+1} = S^n \cap \{x_{b+2} = \ldots = x_n = 0\}$ which meets the subsphere $S^a = S^n \cap \{x_{a+1} = \ldots = x_n = 0\}$. $\dim\{a,b\} = a+b+n-1$ (see [6, p. 51]).

The cycle $\{0,a\}$ is homologous to the cycle $z(0,a)$ which is defined as follows: A non-constant circle in $\{0,a\}$ is replaced by the curve which consists of the two meridians through the two points of intersection of the circle with the subsphere $S^n \cap \{x_0 = 0\}$. A point-curve in $\{0,a\}$ is replaced by the curve whose image is the meridian through the point. Finally we add the curves whose images are the segments of meridians which have equally long parts in each hemisphere. (These curves include constants.) See [4] and [6, p. 54].

Let R_1 be the set of rotations which move points in the (x_0,x_1)-plane, but keep its orthogonal complement fixed. Define

$$z(1,n-1) = \{r \circ c \mid r \in R_1,\ c \in z(0,n-1)\}\ .$$

$z(1,n-1)$ is homologous to $\{1,n-1\}$; see [6, pp. 168–170]. We also define the projection

$$\pi_1 \colon z(1,n-1) \to P_1(\mathbf{R}) \colon r \circ c \mapsto (r(\text{north pole}), r(\text{south pole}))\ .$$

Let R_2 be the set of rotations which move points in the (x_0,x_2)-plane, but keep its orthogonal complement fixed. Define

$$z(2,n-1) = \{r \circ c \mid r \in R_2,\ c \in z(1,n-1)\}\ .$$

$z(2,n-1)$ is homologous to $\{2,n-1\}$. As above we define

$$\pi_2 \colon z(2,n-1) \to P_2(\mathbf{R})$$

by mapping $r_2 \circ r_1 \circ c \in z(2,n-1)$, where $r_2 \in R_2$, $r_1 \in R_1$, $c \in z(0,n-1)$, onto $(r_2 \circ r_1(\text{north pole}), r_2 \circ r_1(\text{south pole}))$.

Recursively we thus define the cycle $z(k,n-1)$ for $1 \leqq k \leqq n-1$ which is homologous to $\{k,n-1\}$ and the projection $\pi_k \colon z(k,n-1) \to P_k(\mathbf{R})$.

Let $h \colon S^n \to M$ be a homeomorphism which is everywhere differentiable with the possible exception of the south pole and which maps $S^{n-1} = S^n \cap \{x_n = 0\}$ onto N such that $(h\,|\,S^{n-1})^{-1} \circ A \circ (h\,|\,S^{n-1})$ is the antipodal map of S^{n-1}.

Such a map can be constructed similar to the map in the Sphere Theorem with respect to r and its diametral point except that it must be smoothed out on the equator.

Let $h_*: \Pi S^n \to \Pi M$ be the map induced by $h: S^n \to M$.

2.4. LEMMA. *The cycles* $h_*(z(0,k))$ *and* $h_*(z(k,n-1))$, $1 \leqq k \leqq n-1$, *can be deformed such that the length of every curve in them becomes* $<4\pi$.

PROOF. Let $c \in h_*(z(k,n-1))$. (The case $c \in h_*(z(0,k))$ can be treated in a similar way.) Then $\pi_k(h_*^{-1}(c))$ is a pair of antipodal points $(q, -q)$ on S^k. Set $p := h(q)$. Then $Ap = h(-q)$. c either passes through the points p and Ap or is a part of such a curve in $h_*(z(k,n-1))$. Having defined the deformation on the first kind of curves it can be extended to the rest in an obvious way.

Note that c has possibly a corner in p and Ap.

Let c_1 (respectively c_2) be the geodesic hinge (i.e. a once broken geodesic) with corner in p (respectively Ap) which begins in $E(p)$, is tangential to c in p (respectively Ap), and ends in $E(p)$.

It follows from (2.2) and (2.3) that $M_+(p) - \{p\}$ (respectively $M_-(p) - \{Ap\}$) can be projected from p (respectively Ap) along geodesics into $E(p)$. c minus p and Ap consists of two curves c_3' and c_4'. Project the parts of c_3' and c_4' which lie in $M_+(p)$ (respectively $M_-(p)$) from p (respectively Ap) into $E(p)$. Call the images of c_3' and c_4' under these projections c_3 and c_4. Now move the endpoints of the hinges c_1 and c_2 along the curves c_3 and c_4 into the midpoints of c_3 and c_4.

The image of c under this deformation—which depends continuously on c—is a closed broken geodesic which consists of four geodesic segments each of length $<\pi$.

2.5. THEOREM. *If* $\delta > (4/9)$, *then there is at least one closed geodesic of index* k *for every* $(n-1) \leqq k \leqq 3(n-1)$ *which has length* $<2\sigma\pi$ *and therefore no self-intersections. (In particular we have at least* $2n-1$ *geometrically different closed geodesics on* M *of length* $<2\sigma\pi$.)

PROOF. Deforming the cycles

$$h_*(z(0,0)), \ldots, h_*(z(0,n-1)), h_*(z(1,n-1)), \ldots, h_*(z(n-1,n-1))$$

first as in Lemma (2.4) and then either as in part (1) of the proof of (3.2) in [14] or with the Hilbert manifold theory in [6] will give us closed geodesics of length $<4\pi$. The index of the shortest closed geodesic at which the cycle $h_*(z(0,k))$ (respectively $h_*(z(k,n-1))$) "remains hanging" is equal to $\dim h_*(z(0,k)) = k+n-1$ (respectively $\dim h_*(z(k,n-1)) = k+2(n-1)$. This conclusion is independent of the curvature assumption and can be used in the proof of (2.1).

It is left to prove that the closed geodesics have length $<2\sigma\pi$. This will follow from Lemma (2.6) below.

REMARK. Using a technique of Klingenberg [6, pp. 174–175] it is possible to use the above theorem to prove the existence of $2n-1$ closed geodesics of length $<2\sigma\pi$ on Riemannian manifolds satisfying the same pinching condition but without any non-degeneracy assumption. But then nothing can be said about the indices of the closed geodesics. This is independent of the most difficult part of Klingenberg's method to prove the existence of many closed geodesics which is his statement about the behaviour of multiplicities of certain closed geodesics. We can avoid this since their multiplicity is equal to one in our case.

2.6. LEMMA. *Let M be a simply connected n-dimensional Riemannian manifold with*

$$\tfrac{4}{9} < \delta \leqq K \leqq 1 .$$

Then there are no closed geodesics on M with length in the interval $[2\sigma\pi, 4\pi)$, if M is not a sphere of constant curvature δ.

PROOF. Suppose that $c\colon [0,\omega] \to M$ is a closed geodesic on M with arc-length parameter and $\omega = L(c) \geqq 2\sigma\pi$. Set $p := c(0)$. We consider the function

$$f(t) := d(c(t),p) .$$

The injectivity radius of the exponential mapping is $\geqq \pi$, so we have $f(t) = t$ for $t \in [0,\pi]$ and $f(t) = \omega - t$ for $t \in [\omega-\pi,\omega]$. Note that $\frac{2}{3}\sigma\pi < \pi$. If $f(t)$ would not have a local minimum, then we could construct a geodesic triangle with circumference $\geqq 2\sigma\pi$ which is in contradiction with Toponogov's Theorem. (A geodesic triangle with circumference $\geqq 2\sigma\pi$ does only exist on spheres of constant curvature δ.) The local minimum must therefore be in the open interval $(\pi, \omega-\pi)$. If a geodesic $\tau\colon [0,1] \to M$ with $d(p,\tau(0)) \geqq \sigma\pi/2$ and $d(p,\tau(l)) > \sigma\pi/2$ is such that the function $d(p,\tau(t))$ has a local minimum, then $L(\tau) > \sigma\pi$ [1, p. 114]. This shows that the local minimum of $f(t)$ must be smaller than $\sigma\pi/2$ which implies that it is attained in the interval $(2\pi - (\sigma\pi/2), \omega - 2\pi + (\sigma\pi/2))$ and that the length of this interval is $\geqq \sigma\pi$. Hence

$$(\omega - 2\pi + (\sigma\pi/2)) - (2\pi - (\sigma\pi/2)) \geqq \sigma\pi .$$

So $L(c) = \omega \geqq 4\pi$ which proves the lemma.

3. Non-hyperbolic closed geodesics on pinched spheres.

In this section we show that some of the closed geodesics whose existence was proven in section 2 cannot be hyperbolic. We use the same notations as there (see the beginning of section 2) and continue to assume that *all closed geodesics of length* $<2\sigma\pi$ *are non-degenerate.*

Note that the theorems in this section imply the existence of non-hyperbolic closed geodesics on manifolds which are not simply connected because a covering of a hyperbolic closed geodesic is hyperbolic.

3.1. THEOREM. *If* $\delta > (4/9)$, *then a closed geodesic of length* $<2\sigma\pi$ *and index* $3(n-1)$ *cannot be hyperbolic. (The existence of at least one such closed geodesic was proven in* (2.5).)

PROOF. Suppose that c is a hyperbolic closed geodesic of length $<2\sigma\pi$ and index $3(n-1)$. Choose a rational number p/q such that

$$\sigma < \frac{p}{q} < \frac{3}{2}.$$

$L(c^q) < 2p\pi$ so we can use the Comparison Theorem of Morse–Schoenberg [2, p. 176] which says that the index of c^q as a geodesic segment is $\leqq (2p-1)(n-1)$. The index theorem in [8] says that the index of c^q as a closed geodesic is equal to its index as a geodesic segment plus concavity which is $\leqq (n-1)$. Hence we have

$$\operatorname{index} c^q \leqq 2p(n-1).$$

Since c is hyperbolic we have that $\operatorname{index} c^q = q \operatorname{index} c$. This implies

$$\operatorname{index} c \leqq \frac{2p}{q}(n-1) < 3(n-1)$$

—a contradiction.

3.2. THEOREM. *If* $(k/k+1)^2 < \delta \leqq K \leqq 1$, *where* k *is an odd number in the interval* $[(n-1), 3(n-1)]$, *then a closed geodesic of length* $<2\sigma\pi$ *and index* k *cannot be hyperbolic. (For the existence of such closed geodesics see* (2.1) *and* (2.5).)

REMARK. (1) Note that a short hyperbolic closed geodesic can have an odd index as an example of Poincaré shows [13, p. 260]. His example is a disturbed ellipsoid of revolution with a pinching constant $\cong 1/16$. The index is equal to 1.

(2) The shortest closed geodesic c on a surface satisfying $1/4 \leqq K \leqq 1$ has index 1 if the closed geodesics of length $\leqq 4\pi$ are non-degenerate. Using the fact that $L(c) > 2\pi$ we can show with the methods of the following proof that c is non-hyperbolic and hence infinitesimally stable.

PROOF. Suppose that $c\colon [0,\omega] \to M$ is a closed geodesic of length $\omega < 2\sigma\pi$ and index k. The Index Theorem for Hyberbolic Closed Geodesics says that

$$\operatorname{index} c = \sum_{0 \leqq t < \omega} \dim (V_s(t) \cap V_v(t)) .$$

Let $t_1 \in [0,\omega)$ be such that $\dim (V_s(t_1) \cap V_v(t_1)) > 0$. Then there is a non-trivial Jacobi field Y along c with $(Y(t), \nabla Y(t)) \in V_s(t)$ and $Y(t_1) = 0$. Since $K > 0$ there is an infinite sequence of distinct positive numbers $\{t'_m\}$ such that $Y(t'_m) = 0$ and $\{t'_m\}$ contains all positive zeros of Y. Let $t_m \in [0,\omega)$ be equal to t'_m modulo ω. Obviously

$$\dim (V_s(t_m) \cap V_v(t_m)) > 0 \qquad \text{for every } m .$$

The sequence $\{t_m\}$ has only finitely many values because the index of c is finite. So there is a smallest $l \in \mathrm{N}$ such that $Y(t_j + l\omega) = Y(t_j) = 0$ for some j and every positive zero of Y is equal modulo ω to some of its zeros in $[t_j, t_j + l\omega)$. Without loss of generality we can assume that $t_j = 0$.

Denote the number of zeros of Y in the interval $[0, l\omega)$ by i. We want to show that i is even which means that the contribution of a single stable Jacobi field to the index is even. So assume that i is odd.

It follows from our considerations above that $i \leqq \operatorname{index} c = k$. The Rauch Comparison Theorem implies that the ith positive zero of Y is contained in the interval $[i\pi, i\sigma\pi]$. Obviously

$$\sigma < \frac{k+1}{k} \leqq \frac{i+1}{i} < \frac{i}{i-1} .$$

Hence

$$\sigma i < i+1 \qquad \text{and} \qquad \sigma(i-1) < i$$

and therefore

$$\left(\frac{i+1}{2} - 1\right)\omega < i\pi < i\sigma\pi < \left(\frac{i+1}{2}\right)\omega .$$

Here we have used that $2\pi \leqq \omega \leqq 2\sigma\pi$. $p := (i+1)/2$ is a natural number since i is odd. The above calculations show that the ith zero of Y—which is $l\omega$—lies in the open interval $((p-1)\omega, p\omega)$—a contradiction. So we have shown that i is even.

The proof of the theorem is finished, when we have proved the following: Suppose that

$$\dim (V_s(t') \cap V_v(t')) := m \geqq 2 .$$

Let $Y_1,\ldots,Y_m$ be linearly independent stable Jacobi fields which satisfy

$$Y_1(t') = \ldots = Y_m(t') = 0 .$$

Let Y be some stable Jacobi field which vanishes in t'. Then Y does not contribute anything new to the index of c. I.e. if Y vanishes in a point $t^* > t'$, then some of the Y_i vanishes there too and

$$\nabla Y(t^*) = \sum \nabla Y_{i_j}(t^*), \qquad Y_{i_j}(t^*) = 0 .$$

Let $Y_{m+1},\ldots,Y_{n-1}$ be Jacobi fields which vanish in t' and form a linearly independent set together with $Y_1,\ldots,Y_m$.

The next zero of Y or of one of the Y_i, $1 \leqq i \leqq n-1$, after t' lies in the interval $[t'+\pi, t'+\sigma\pi]$ and neither Y nor the Y_i have more than one zero in this interval because $\sigma < 2$. The index of the segment $c\,|\,[t', t'+2\pi]$ is $=(n-1)$, see [2, p. 176].

The first zero t'' of Y after t' must be equal to the zero of some of the Jacobi fields $Y_1,\ldots,Y_m$ since the index of $c\,|\,[t', t'+2\pi]$ would otherwise be too big. Furthermore

$$\nabla Y(t'') = \sum \nabla Y_{i_j}(t''), \qquad Y_{i_j}(t'') = 0, \qquad i_j \leqq m ,$$

for the same reason.

4. Some remarks on isometries and closed geodesics.

The purpose of this section is to investigate the influence of the isometry group on the existence of non-hyperbolic closed geodesics.

It is well known that all closed geodesics on a negatively curved compact manifold are hyperbolic and that its isometry group is finite. The same is true for Anosov manifolds [5]. The next theorem generalizes these results.

4.1. THEOREM. (a) *Let X be a non-trivial Killing vector field on a Riemannian manifold. Then all hyperbolic closed geodesics on which X does not vanish have index equal to* 0.

(b) *Let M be a compact Riemannian manifold all of whose closed geodesics are hyperbolic. Then the isometry group $I(M)$ of M is finite.*

PROOF. (a) Let c be a hyperbolic closed geodesic on M and let X be a non-trivial Killing vector field on M which does not vanish on c. If there is some t such that $X(c(t))$ and $\dot{c}(t)$ are linearly independent, then X restricted to c is a periodic non-trivial Jacobi field along c and c would be degenerate. Hence the 1-parameter group φ_s of isometries which X generates translates c.

$$\operatorname{index} c = \sum_{0 \leqq t < \omega} \dim (V_s(t) \cap V_v(t)) ,$$

where ω is the length of c. φ_s induces a bundle automorphism of $V^{2n-2} \to S^1$ which leaves V_s and V_v invariant. Moreover there is an $s \in \mathbf{R}$ for any $t_1, t_2 \in S^1$ such that the induced automorphism maps the fiber $V^{2n-2}(t_1)$ into $V^{2n-2}(t_2)$. This implies that $\dim(V_s(t) \cap V_v(t)) = \text{const}$. This constant must be equal to zero because the index is finite.

(b) Suppose that $I(M)$ is not finite. The isometry group of a compact Riemannian manifold is compact, so it must have an 1-parameter subgroup when it is not finite. The closure of this 1-parameter subgroup in $I(M)$ is a compact abelian Lie group, i.e. a torus. In the torus there is a subgroup which is homomorph to S^1. Let X be the Killing vector field which generates this group. The integral curve through the maximum of $\|X\|$ is a closed geodesic [9]. It has index >0 because it is non-degenerate. It follows from (a) that this closed geodesic cannot be hyperbolic.

In the next theorem we investigate isometry invariant geodesics. Grove et al. have made an intensive study of the subject in recent years, see e.g. [3]. We will use an existence result from [12].

4.2. Theorem. *Let M be a compact n-dimensional Riemannian manifold with a non-trivial isometry $A\colon M \to M$ which satisfies $Ap \notin C(p)$ for every $p \in M$. Let q be a maximum of the displacement function of A.*

Then the unique minimal geodesic $c_{qAq}\colon [0,1] \to M$ connecting q and Aq is invariant under A (i.e. A_* maps $\dot{c}(0)$ onto $\dot{c}(1)$).

Let $c\colon \mathbf{R} \to M$ be the extension of c_{qAq} to $\mathbf{R}$.

If A is of finite order, then c is a closed geodesic which is not hyperbolic when $n = 2$. It is not hyperbolic in the case $n > 2$ when $2d(p, Ap) < d(p, C(p))$ for every $p \in M$.

Proof. A proof of the invariance of c_{qAq} under the isometry A can be found in [12].

So it is left to prove that the closed geodesic c cannot be hyperbolic.

(1) We first treat the case $n=2$. The index of c as a closed geodesic is greater than zero or q is a degenerate maximum of the displacement function. In the second case c is degenerate. So we can assume that the index of c is greater than zero. Suppose that c is hyperbolic. Let Y be the stable Jacobi field along c. Since the index of c is positive, Y must be zero in some point, say in t_0. $A_* Y$ is a stable Jacobi field along $Ac(t) = c(t+1)$ and since the space of stable Jacobi fields is one-dimensional on surfaces, it follows that $X(t) := A_* Y(t-1)$ and $Y(t)$ must be linearly dependent. X is zero in $t_0 + 1$ and Y therefore too. This implies that $c(t_0)$ and $c(t_0 + 1) = Ac(t_0)$ are conjugate along c which is a contradiction.

(2) We now assume that $2d(p, Ap) < d(p, C(p))$. The index of c_{qAq} and c_{qA^2q} in

the variational problem of A- (respectively A^2-) invariant geodesics is obviously in both cases $\geqq (n-1)$ (assuming non-degeneracy). From [8] we know that the index of an isometry invariant geodesic is equal to the number of conjugate points plus concavity. The concavity is $\leqq (n-1)$ and the number of conjugate points of c_{qAq} and c_{qA^2q} is $=0$ because of the assumption $2d(p, Ap) < d(p, C(p))$. Hence the index of c_{qAq} and c_{qA^2q} as isometry invariant geodesics is $=(n-1)$ in both cases.

We can identify points which lie in the same orbit of A in an A-invariant tubular neighbourhood of c. c_{qAq} becomes a closed geodesic of index $(n-1)$ and c_{qA^2q} its double covering which also has index $(n-1)$. Since the index of a hyperbolic closed geodesic grows linearly it follows that the closed geodesic c cannot be hyperbolic.

REMARK. The following fact is related to the last part of the above proof (but independent of the cut locus assumption in the theorem): Let M be a Riemannian manifold, $A: M \to M$ an isometry of finite order, $c: \mathbf{R} \to M$ a hyperbolic closed geodesic which is invariant but not fixed under A. Let n be the order of A on c. Then the index of c divides by n.

On an ellipsoid which is not too dissimilar from a round sphere, the closed geodesics in the intersection with the coordinate planes have indices 1, 2 and 3. Now it follows that the two of index 1 and 3 cannot be hyperbolic since they are invariant under an isometry of order 2.

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