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L^q-ESTIMATES FOR GREEN POTENTIALS IN LIPSCHITZ DOMAINS

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1. Introduction.

Let $D \subset \mathbf{R}^n$ be a bounded domain. Let G be the Green function of D and for $f \in L^1(D)$ let $Gf(P) = \int_D f(Q)G(P,Q)\,dQ$ be the Green potential of f. It is a classical fact that if ∂D is sufficiently smooth and if $f \in L^p(D)$ then

$$\left(\int_D |\nabla Gf|^q \,dP\right)^{1/q} \leqq C_p\left(\int_D |f|^p \,dP\right)^{1/p} \tag{1.1}$$

where ∇ denotes the gradient and q is given by $1/q = 1/p - 1/n$, $1<p<n$. Also if $1<p<\infty$ then

$$\left(\int_D |\nabla_2 Gf|^p \,dP\right)^{1/p} \leqq C_p\left(\int_D |f|^p \,dP\right)^{1/p}, \tag{1.2}$$

where $\nabla_2 u$ denotes the second order derivatives with respect to a basis $x_1,\ldots,x_n$ arranged in some order.

The purpose of this paper is to study the possibility of extending the properties (1.1) and (1.2) to Lipschitz domains. Regarding (1.1) we shall establish the following result.

THEOREM 1. *Let $D \subset \mathbf{R}^n$ be a Lipschitz domain and put $p_2 = 4/3$, $p_n = 3n(n+3)^{-1}$ for $n \geqq 3$. Then there is a number $\varepsilon = \varepsilon(D) > 0$ such that if $1 < p < p_n + \varepsilon$ and q is given by $1/q = 1/p - 1/n$ then*

$$\left(\int_D |\nabla Gf|^q \,dP\right)^{1/q} \leqq C\left(\int_D |f|^p \,dP\right)^{1/p}, \tag{1.3}$$

where C only depends on p and D. Also there is a constant C only depending on D such that

$$|\{P \in D : |\nabla Gf(P)| > \lambda\}| \leqq C\left(\lambda^{-1} \int_D |f|\,dP\right)^{n/n-1}, \tag{1.4}$$

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where $|E|$ denotes the Lebesgue measure of a set E.

We remark that if q_n is defined by $1/q_n = 1/p_n - 1/n$ then $q_2 = 4$ and $q_n = 3$ for $n \geqq 3$. Also the exponent p_n is optimal; we shall show in Section 4 that given any $q > q_n$ there is a Lipschitz domain $D \subset \mathbf{R}^n$ and a bounded function f such that $\nabla Gf \notin L^q(D)$.

We shall in Section 4 construct a Lipschitz domain in $\mathbf{R}^2$ such that for some bounded function f in D we have that $\nabla_2 Gf \notin L^p$ for all $p > 1$, which shows that no analogue of (1.2) is possible for Lipschitz domains.

2. Technical preliminaries.

In this section we have collected some results on which the proof of Theorem 1 is based.

We start by recalling that a bounded domain $\Omega \subset \mathbf{R}^n$ is called a Lipschitz domain if $\partial\Omega$ can be covered by finitely many open right circular cylinders whose bases have positive distance from $\partial\Omega$ and corresponding to each cylinder L there is a coordinate system (x, y) with $x \in \mathbf{R}^{n-1}$, $y \in \mathbf{R}$, with the y-axis parallel to the axis of L and a function $\varphi \colon \mathbf{R}^{n-1} \to \mathbf{R}$ satisfying a Lipschitz condition (i.e., $|\varphi(x) - \varphi(z)| \leqq M|x - z|$) such that

$$L \cap \Omega = \{(x,y) : \varphi(x) < y\} \cap L \quad \text{and}$$

$$L \cap \partial\Omega = \{(x,y) : y = \varphi(x)\} \cap L .$$

Let $L(M)$ denote the class of functions $\varphi \colon \mathbf{R}^{n-1} \to \mathbf{R}$ such that $\varphi(0) = 0$ and $|\varphi(x) - \varphi(z)| \leqq M|x - z|$. As is easily seen, there are now positive constants a and b, which can be taken to depend only on M, such for all $\varphi \in L(M+2)$ we have that the domain

$$D(\varphi) = \{(x,y) : |x| < 10,\ \varphi(x) < y < a\}$$

is star shaped with respect to $P_M = (0, b) \in D(\varphi)$. Moreover, there is a $\theta > 0$ only depending on M such that for any $P \in \partial D(\varphi)$ the circular cone $\Gamma(P)$ with vertex at P, opening angle θ height $|P - P_M|$ and axis along the line segment between P and P_M is contained in $D(\varphi)$. Also if $\Gamma_-(P)$ denotes the unbounded cone which has the same vertex, opening angle and axis as $\Gamma(P)$ but has its opening opposite that of $\Gamma(P)$ then $\Gamma_-(P)$ is contained in the complement of $D(\varphi)$. We also have that if $P \in \{(x, \varphi(x)) : |x| \leqq 5\}$ and $0 < t < 5$ then $P + (0, t) \in \Gamma(P)$.

We next pick a $\delta > 0$ such that $B(P_M, 2\delta)$ is contained in $D(\varphi)$ for all $\varphi \in L(M+2)$, where $B(Q, r)$ means the ball with center at Q and radius r. We put $D^*(\varphi) - B(P_M, \delta)$. We denote by σ the surface measure of $\partial D(\varphi)$, $d(P)$ the distance from P to $\partial D(\varphi)$ and $\omega(E)$ the harmonic measure of a set $E \subset \partial D(\varphi)$ evaluated at P_M.

We shall next give some estimates for function harmonic in $D(\varphi)$ $\varphi \in L(M)$. Unless otherwise is specified, the constants C that will appear only depend on M and n. In the next lemma we have collected some results from Dahlberg [2], [3], [4].

LEMMA 1. *Let $\varphi \in L(M+1)$ and denote by g the Green function of $D(\varphi)$ with pole at P_M. There are constants C such that the following holds:*

$$\int_{\partial D(\varphi)} \sup\{|\nabla g(Q)|^2 \ : \ Q \in \Gamma(P) \cap D^*(\varphi)\}\, d\sigma \leqq C . \tag{2.1}$$

If $P \in D^*(\varphi)$ and $\{P^*\} = \partial D(\varphi) \cap \{r(P-P_M)+P_M \ : r>0\}$ then

$$C^{-1}\omega(A(P)) \leqq d(P)^{n-2} g(P) \leqq C\omega(A(P)) , \tag{2.2}$$

where $A(P) = \{Q \in \partial D(\varphi) : |Q-P^*| \leqq d(P)\}$. If $f \in L^2(\partial D(\varphi), \sigma)$ and Hf denotes the Poisson integral of f then

$$\int_{\partial D(\varphi)} \sup\{|Hf(Q)|^2 \ : \ Q \in \Gamma(P)\} \leqq C \int_{\partial D(\varphi)} |f|^2\, d\varphi . \tag{2.3}$$

We shall need to compare positive harmonic functions which simultaneously vanish on the boundary.

LEMMA 2. *Let u and v be two positive harmonic functions in $D(\varphi)$ which both vanish on $\{(x,\varphi(x)) : |x| \leqq 10\}$. If $u(P_M) = v(P_M) = 1$ then*

$$\sup\{u(P) \ : \ P \in R\} \leqq C \tag{2.4}$$

and

$$C^{-1} \leqq u/v \leqq C \quad \text{in } R , \tag{2.5}$$

where $R = \{(x,y) : |x| < 4,\ \varphi(x) < y < \varphi(x)+4\}$.

PROOF. First inequality (2.4) follows from Hunt–Wheeden [6, p. 512].

Let g denote the Green function of $D(\varphi)$. Then

$$\sup\{g(P,P_M) \ : \ |P-P_M| = \delta\} \leqq C ,$$

where δ is as in the definition of $D^*(\varphi)$. Since δ was chosen such that $B(P_M, 2\delta) \subset D(\varphi)$ it follows from Harnack's inequality that if $P \in B(P_M, \delta)$ then $u(P) \geqq C > 0$. Hence it follows from the maximum principle that if $P \in D^*(\varphi)$ for some $C>0$

$$u(P) \geqq Cg(P,P_M) . \tag{2.6}$$

For a number t let t^+ denote t if $t>0$ and zero otherwise. We now put $\varphi^*(x) = \varphi(x)-(|x|-6)^+$ and observe that $\varphi^* \in L(M+1)$. Let

$$D' = \{(x,y) : |x|<9, \varphi(x)<y<10+\varphi(x)\} ,$$
$$E = \{(x,\varphi(x)) : |x|\leqq 9\}$$

and put $F=\partial D'-E$. From Harnack's inequality follows that $\inf_{P\in F} G(P)\geqq c >0$, where G is the Green function of $D(\varphi^*)$ with pole at P_M. A repeated application of the aforementioned result in Hunt–Wheeden [6] yields that $\sup_{P\in F} v(P)\leqq C$. Since G is superharmonic it follows now from the minimum principle that $v\leqq CG$ in D' and hence in R. In view of (2.6) we see that the right hand side inequality of (2.5) follows if we can show that

$$G(P) \leqq Cg(P,P_M), \quad P\in R . \tag{2.7}$$

From the computation in Naim [8, p. 223] follows that if $Q\in D(\varphi)$ and $P_j\in D(\varphi)$ and $P_j \to P \in \{(x,\varphi(x)) : |x|\leqq 5\}$ then

$$\frac{g(P_j,Q)}{G(P_j)} \to K(P,Q)-h(P,Q) \tag{2.8}$$

where $K(P,\cdot)$ is the kernel function of $D(\varphi^*)$ with pole at P normalized by $K(P,P_M)=1$ and $h(P,\cdot)$ the harmonic function in $D(\varphi)$ with the same boundary values as $K(P,\cdot)$ on $D(\varphi^*)\cap\partial D(\varphi)$ and zero otherwise.

Let $T(\varphi)=\{(x,\varphi(x)) : |x|\leqq 5\}$ and put

$$e = \inf_{\varphi\in L(M)} \inf_{P\in T(\varphi)} \{K(P,P_M)-h(P,P_M\} .$$

A compactness argument shows that $e=K(P,P_M)-h(P,P_M)$ for some $\varphi\in L(M)$ and some $P\in T(\varphi)$. It is easily seen that $K(P,P_M)-h(P,P_M)>0$ (see Dahlberg [2, p. 281]) which shows that $e>0$. Denote by ω_1 and ω_2 the harmonic measure evaluated at P_M of $D(\varphi)$ and $D(\varphi^*)$ respectively. If k_i denotes the density of ω_i with respect to the surface measure then k_i is a.e. given as the normal derivative of the Green function with pole at P_M. Since $e>0$ it follows from (2.8) that there is a constant C such that $k_2\leqq Ck_1$ a.e. on $T(\varphi)$. If $P\in\{(x,\varphi(x)) : |x|\leqq 4\}$ and $0<t<1$ it follows from (2.2) that if $P_t=P_+(0,t)$ then

$$\begin{aligned} G(P_t) &\leqq Ct^{2-n}\int_{A(P_t)} k_2\,d\sigma \\ &\leqq Ct^{2-n}\int_{A(P_t)} k_1\,d\sigma = Cg(P_t,P_M) . \end{aligned} \tag{2.9}$$

Observing that from Harnack's inequality it follows that

$$G(P) \leqq Cg(P_1P_M) \quad \text{if } P \in \{(x,y) : \varphi(x)+1<y<\varphi(x)+4, |x|<4\}$$

we see that (2.9) yields (2.6) which establishes the right hand side of inequality (2.5). Reversing the roles of u and v establishes the left hand side of (2.5) which yields the lemma.

We shall use the following notations. For $x=(x_1,\ldots,x_{n-1}) \in \mathbf{R}^{n-1}$ put $\|x\| = \max|x_i|$ and put

$$S(\varphi,r) = \{(x,y) : \|x\|<r, \varphi(x)<y<\varphi(x)+2r\} ,$$

$$Z_r = (0,r) \qquad T(\varphi,r) = \{(x,\varphi(x)) : \|x\|<r\} .$$

We have the following straight-forward consequences of Lemma 2.

Lemma 3. *If u and v are positive and harmonic in $S(\varphi,2r)$ with vanishing boundary values on $T(\varphi,2r)$ then there is a constant $C>0$ such that $u \leqq Cu(Z_r)$ in $S(\varphi,r)$ and*

$$C^{-1}\frac{u(Z_r)}{v(Z_r)} \leqq \frac{u}{v} \leqq C\frac{u(Z_r)}{v(Z_r)} \quad \text{in } S(\varphi,r) .$$

Proof. Since the Lipschitz constant is invariant under changes of scale its sufficient to prove the lemma for $r=1$ in which case a repeated application of Lemma 2 yields the result.

For $0<\alpha<n$ let

$$I_\alpha f(P) = \int_{\mathbf{R}^n} f(Q)|P-Q|^{\alpha-n}\,dQ .$$

We shall need estimates for the operator I_α.

Lemma 4. *If $1<p<n/\alpha$ then*

$$\left(\int_{\mathbf{R}^n} |I_\alpha f|^q\,d\theta\right)^{1/q} \leqq C\left(\int_{\mathbf{R}^n} |f|^p\,dQ\right)^{1/p} ,$$

where q is given by $1/q=1/p-\alpha/n$ and C depends only on p and n. If $f \in L^1(\mathbf{R}^n)$ then

$$|\{P : |I_\alpha f(P)|>S\}| \leqq C\left(S^{-1}\int_{\mathbf{R}^n} |f|\,dP\right)^{n/n-\alpha}$$

where C only depends on n.

For a proof of these results we refer to Stein [9, Chapter V].

3. The main result.

We shall continue working with a fixed function $\varphi \in L(M)$ and make our estimates uniformly with respect to M.

We put

$$V(\varphi) = \{(x,y) : x \in \mathrm{R}^{n-1}, \varphi(x)<y\},$$

$\Lambda=\Lambda(\varphi)$ is defined as the class of domains of the form

$$S = \{(x,y) : \|x-x_1\|<r, \varphi(x)<y-a<\varphi(x)+2r\}$$

for some $x_1 \in \mathrm{R}^{n-1}$ and some $a,r>0$. We call the point $Z=(x_1,\varphi(x_1)+r+a)$ the center of S. If $N>0$ we put

$$NS = \{(x,y) : \|x-x_1\|<Nr, |y-a-\varphi(x)-r|<Nr\}.$$

We shall call $\{(x,\varphi(x)+a) : \|x-x_1\|<r\}$ for the bottom part of ∂S. We let $d(P)$ denote the distance from P to $\partial V(\varphi)$.

Lemma 5. *Suppose U is positive and harmonic in $D(\varphi)$ and is vanishing on $\{(x,\varphi(x)) : |x|<10\}$. Then there is a constant C such that for all $S \in \Lambda$, contained $S(\varphi,1)$ we have*

$$C^{-1}\left(\frac{1}{|S|}\int_S (u(P)/d(P))^q\,dP\right)^{1/q} \leqq u(Z)/d(Z)$$

$$\leqq C\frac{1}{|S|}\int_S (u(P)/d(P))\,dP,$$

when Z is the center of S and $q=4$ when $n=2$ and $q=3$ when $n\geqq 3$.

Proof. If ϱ denotes the diameter of S then $B(Z,2C_1\delta)\subset S\subset B(Z,C_2\delta)$ for suitable constants $C_i>0$. Hence it follows from Harnack's inequality that $u \geqq cu(Z)$ in $B(Z,C_1\delta)$ which yields the right hand side of the inequality. If say $5S\cap\partial V(\varphi)=\varphi$ then it follows from Harnack's inequality and elementary geometry that $u(P)/d(P)\leqq Cu(Z)/d(Z)$, for all P in S, which gives the left hand side inequality in this case. If $5S\cap\partial V(\varphi)\neq\varphi$ we see that there is no loss in generality in assuming that the botoom part of ∂S touches $\partial V(\varphi)$, for otherwise we pick an $S' \in \Lambda$ such that $S\subset S'\subset 5S$ and the bottom part of the boundary of S' touches ∂V. Since the conclusion of the lemma is invariant under changes of scale it's therefore sufficient to show the lemma for the case $S=S(\varphi,1)$, $Z=(0,1)$ and $u(Z)=1$.

Let

$$U = \{P=(x,y) : y>-2M|x|, |P|\leqq r_0\}$$

where $r_0 = r_0(M)$ has been chosen so small that for all $P \in \{(x, \varphi(x)) : \|x\| \leqq 1\} = T$ we have that

$$P + U \subset S(\varphi, 3/2) \cup \{(x, y) : \|x\| < 3/2, y \leqq \varphi(x)\} = E .$$

Put $F = \partial U \cap \partial B(0, r_0)$ and denote by h the harmonic measure of F with respect to U. Then it's well known that $h(P) \leqq C|P|^\delta$, where $\delta > 1/2$ if $n = 2$ and $\delta > 0$ if $n \geqq 3$. If we extend u to E by putting $u = 0$ outside $D(\varphi)$ then u is subharmonic in E. It follows from Lemma 3 that $u \leqq C = C(M)$ in E, which means that if $P_0 \in T$ and $P \in P_0 + U$ then

$$u(P) \leqq Ch(P - P_0) \leqq C|P - P_0|^\delta .$$

Therefore, we have that

$$u(P) \leqq Cd(P)^\delta \quad \text{in } S . \tag{3.1}$$

If g denotes the Green function of $D(\varphi)$ with pole at P_M we have from Lemma 3 that $u(P) \leqq Cg(P)$ in S. For $P \in T$ let

$$f(P) = \sup\{|\nabla g(Q)| : Q \in \Gamma(P) \cap D^*(\varphi)\} .$$

Then it follows from (2.1) that $\int_{\delta D(\varphi)} f^2 \, d\sigma \leqq C$ and we have that if $P = (x, y) \in S(\varphi, 1)$ then

$$\begin{aligned} u(P)/d(P) &\leqq C(g(P)/d(P)) \\ &\leqq Cf(P^*) \end{aligned}$$

where $P^* = (x, \varphi(x))$. From out estimates of u we have that

$$\int_S (u(P)/d(P))^q \, dP \leqq C \int_0^1 y^{(\delta - 1)(q - 2)} \int_T f^2 \, d\sigma \leqq C$$

if q is as in the formulation of the lemma, which completes the proof.

We shall next study the integrability properties of Green functions.

LEMMA 6. *There is a constant $C > 0$ such that if g denotes the Green function of $V = V(\varphi)$ with pole at $Z = (0, 1)$ then*

$$\int_V (g(P)/d(P))^{n/n-1} \, dP \leqq C .$$

PROOF. We shall give the proof only for the case $n \geqq 3$. The case $n = 2$ is left to the reader.

From Lemma 5 and the fact that $g(P) \leqq |P - Z|^{2-n}$ follows that

$$\int_{V'} (g(P)/d(P))^{n/n-1}\,dP \leqq C\,,$$

where $V' = V \cap B(0, 10)$. If we extend g to all of $\mathbf{R}^n$ by putting $g = 0$ outside V we find that g is subharmonic in $\Gamma = \{(x, y) : y > 2 - 2M(x)\}$ and has vanishing boundary values on $\partial\Gamma \cap (\mathbf{R}^n - B(0, r_0))$, where $r_0 = r_0(M)$. It is well known that this implies

$$g(P) \leqq C|P|^{2-n-\delta} \quad \text{for } P \in \Gamma,\ \delta = \delta(M) > 0\,. \tag{3.2}$$

It is easily seen that there is a constant $C > 0$ such that if $P \in K = \{(x, y) : y > 2M|x|, |x| > 2\}$ then $d(P) \geqq C|P|$. Hence

$$\int_K (g(P)/d(P))^{n/n-1}\,dP \leqq C \int_K |P|^{-(n+\delta_1)}\,dP \leqq C\,.$$

To complete the proof of the lemma it now remains to estimate $\int_{V''} (g(P)/d(P))^{n/n-1}\,dP$, where $V'' = V - (V' \cup K)$. To this end we observe that there is a number N, only depending on n, such that for $j \geqq 0$ an integer the set $\{x \in \mathbf{R}^{n-1} : 2^j \leqq \|x\| \leqq 2^{j+1}\}$ can be written as a union of N cubes $K_1, \ldots, K_N$ with sides 2^j. We denote the center of K_i by x_i and put

$$\Omega_i = \{(x, y) : x \in K_i,\ \varphi(x) < y < \varphi(x) + 2^{j+3}M\}$$

and $P_i = (x_i, \varphi(x_i) + 2^j M)$. From Lemma 5 and (3.2) follows

$$\int_{\Omega_i} (g(P)/d(P))^{n/n-1}\,dP \leqq C 2^{jn} (g(P_i)/d(P_i))^{n/n-1} \leqq C 2^{-\delta_1 j}$$

where $\delta_1 > 0$. Since $V'' \cap \{x : 2^j \leqq \|x\| \leqq 2^{j+1}\}$ is contained in $\bigcup_1^N \Omega_i$ it follows that $\int_{V''} (g(P)/d(P))^{n/n-1}\,dP \leqq C$ which yields the lemma.

We have the following consequence of Lemma 6.

LEMMA 7. *There is a constant C such that for all $S \in \Lambda$ we have*

$$\int_{V-10S} \sup\{(G(P, Q)/d(P))^{n/n-1} : Q \in S\}\,dP \leqq C\,,$$

where $V = V(\varphi)$ and G is the Green function of V.

PROOF. Since the assertion is invariant under translation and changes of scale we see that there is no loss in generality in assuming that $Z = (0, 1)$, where Z is the center of S. If $5S \cap \partial V = \varphi$ it follows from Harnack's inequality that

$$\sup\{G(P, Q) : Q \in S\} \leqq CG(P, Z)\,.$$

If $5S \cap \partial V \neq \varphi$ we may without loss of generality assume $S = S(\varphi, 1)$. If $P \in V - 10S$ then $G(P, \cdot)$ fulfils the conditions of Lemma 3 so that $\sup_{Q \in S} G(P,Q) \leqq CG(P,Z)$. Therefore the lemma follows from Lemma 6.

We shall next compare the boundary behaviour of a potential with a harmonic function which vanishes on a piece of the boundary.

LEMMA 8. *Let u be postive and harmonic in $D(\varphi)$ and suppose u vanishes on $\{(x,\varphi(x)) : |x| \leqq 10\}$. Also assume $u(Z)=1$, where $Z=(0,1)$. Then there are positive constants C and α such that if $f \in L(S)$, where $S=S(\varphi,1)$ then*

$$\int_E (u(P)/d(P))^{n/n-1}\, dP \leqq C\left(\int_S |f|\, dP\right)^{\alpha},$$

where $E = \{P \in S : |Gf(P)| \geqq u(P)\}$ and G is the Green function of $V = V(\varphi)$.

PROOF. It is no loss in generality in assuming $f \geqq 0$, otherwise we replace f with $|f|$. We put $\int_S f\, dP = \varepsilon$ and $F(x,y) = f(x, y + \varphi(x))$. We shall assume that $\varepsilon \leqq \varepsilon(M)$, where $\varepsilon(M)$ has been chosen so small that $\int_K |F|\, dP \leqq C\varepsilon \leqq C\varepsilon(M) = 1$, where

$$K = \{(x,y) : \|x\| < 1,\, 0 < y < 2\}.$$

Observing that $f(x,y) = F(x, y - \varphi(x))$ for $P = (x,y) \in S$ we see that if we make a Calderon–Zygmund decomposition of F, see Stein [9, Chapter I], then we can write $f = f_1 + f_2$ where $\int_S |f_1|^p\, dP \leqq C\varepsilon$ for all $p \geqq 1$ and $f_2 = \sum b_j$, where each b_i is supported on an $S_j \in \Lambda$, $S_j \subset S$, the S_j's are pairwise disjoint with $|\bigcup S_j| \leqq C\varepsilon$ and $\sum \int |b_j|\, dP \leqq C\varepsilon$. Put $k(P) = (u(P)/d(P))^{n/n-1}$ and define $\lambda(E) = \int_E k\, dP$. By putting

$$E_i = \{P \in S : Gf_i > \tfrac{1}{2}u(P)\}$$

we see that

$$E \subset E_1 \cup E_2 . \tag{3.3}$$

Letting $U = S - (\bigcup 10S_j)$ we see from Lemma 7 that

$$\left(\int_U |Gb_j(P)/d(P)|^{n/n-1}\, dP\right)^{(n-1)/n} \leqq C \int_S |b_j|\, dP,$$

which yields that

$$\begin{aligned} \lambda(E_2 \cap U)^{(n-1)/n} &\leqq \left(C \int_U |Gf_2(P)/d(P)|^{n/n-1}\, dP\right)^{(n-1)/n} \\ &\leqq C \sum \int |b_j|\, dP \leqq C\varepsilon . \end{aligned} \tag{3.4}$$

From Lemma 5 and Harnack's inequality follows that $\lambda(10S_j) \leqq C\lambda(S_j)$. Hence $\lambda(E_2 - U) \leqq C\sum\lambda(S_j)$. We now observe that if $p = 3(n-1)/n > 1$, $n \geqq 2$, then it follows from Lemma 5 that

$$\left(\int_E k^p\, dP\right)^{1/p} \leqq C\,.$$

Hence for any set $E \subset S$ we have

$$\lambda(E) = \int_E k\, dP \leqq \left(\int_S k^p\, dP\right)^{1/p} |E|^{1-1/p},$$

which implies that

$$\lambda(E_2 - U) \leqq C\sum\lambda(S_j) \leqq C\sum\lambda(\bigcup S_j) \leqq C|\bigcup S_j|^{1-1/p} \leqq C\varepsilon^{\alpha},$$

which taken together with (3.4) shows that

$$\lambda(E_2) \leqq C\varepsilon^{\alpha}\,. \tag{3.5}$$

It now remains to show $\lambda(E_1) \leqq C\varepsilon^{\alpha}$ and we now observe that it's sufficient to treat the case when φ is smooth and f is smooth with compact support and show that the constants only depends on M and n.

Let g be the Green function of $D = D(\varphi)$. We have $Gf_1 = gf_1 + h$, where $h \geqq 0$ is harmonic in D with vanishing boundary values on $\{(x, \varphi(x)) : |x| < 10\}$. Since

$$h(P_M) \leqq \int_S \sup\{G(P, P_M) : P \in S\} f_1(Q)\, dQ \leqq C_{\varepsilon}$$

it follows from Lemma 5 that

$$\int_S (h(P)/d(P))^{n/n-1}\, dP \leqq C\varepsilon^{n/n-1}$$

which yields that

$$\lambda\{P \in S : h(P) > \tfrac{1}{4}u(P)\} \leqq C\varepsilon^{n/n-1}\,.$$

Since $E_1 \subset \{P \in S : h(P) > u(P)/4\} \cup E_3$, where $E_3 = \{P \in S : gf_1(P) > u(P)/4\}$ and $\int_S |f_1|^{2n}\, dP \leqq C\varepsilon$ we see that the lemma follows if we can show that for any smooth $f \geqq 0$ supported on S we have

$$\|gf/d\|_2 \leqq C\|f\|_{2n}\,, \tag{3.6}$$

where $\|f\|_p = (\int_S |f|^p\, dP)^{1/p}$ and C only depends on M an n.

We shall now prove (3.6). Let $v = \partial gf/\partial n$, where $\partial/\partial n$ denotes differentiation with respect to the unit inward normal of ∂D. For $k \in L(\partial D, \sigma)$ let Hk denote the Poisson integral of k. It follows from Green's formula that

$$\int_{\partial D} kv\,d\sigma = \int_D Hk\,f\,dP\,. \tag{3.7}$$

It follows from (2.3) that $\int_D |Hk|^2\,dP \leqq C\int_{\partial D} k^2\,d\sigma$ which taken together with (3.7) gives that $\int_{\partial D} v^2\,d\sigma \leqq C\int_D f^2\,dP$. Let $\gamma=\gamma(n)$ be such that if

$$If(P) = \gamma \int f(Q)|P-Q|^{2-n}\,dQ$$

then $\Delta If = f$. Letting $z = gf - If$ we see that z is harmonic in D and since $gf=0$ on ∂D we have that

$$\left(\int_{\partial D} |\nabla z|^2\,d\sigma\right)^{1/2} \leqq \left(\int_{\partial D} v^2\,d\sigma\right)^{1/2} + \left(\int_{\partial D} |\nabla If|\,d\sigma\right)^{1/2}.$$

Furthermore, since

$$|\nabla If| \leqq C\|f\|_{2n} \tag{3.8}$$

we have that

$$\int_{\partial D} |\nabla z|^2\,d\sigma \leqq C\|f\|_{2n}^2\,.$$

If we for $P=(x,y)\in S$ put $P^*=(x,\varphi(x))$ and $F(P)=\sup_{\Gamma(P^*)}|\nabla z|$, then it follows from (2.3) that

$$\|F\|_2 \leqq C\left(\int_{\partial D} |\nabla z|^2\,d\sigma\right)^{1/2} \leqq C\|f\|_{2n}\,.$$

We now have from (3.8) that

$$|gf(P)/d(P)| \leqq C(F(P)+\|f\|_{2n})$$

which yields (3.6). Lemma 8 is proved.

We have the following consequence from Lemma 8.

LEMMA 9. *Let u be as in Lemma 8 and put $k(P)=(u(P)/d(P))^{n/n-1}$. There are constants C and α, only depending on M and n such that for any $S\in\Lambda$ such that $S\subset S(\varphi,1)$ and any integrable f supported on S satisfying*

$$\int_S |f|\,dP \leqq \varepsilon\left(\int_S k\,dP\right)^{(n-1)/n}$$

we have that

$$\int_E k\,dP \leqq C\varepsilon^\alpha \int_S k\,dP\,,$$

where G is the Green function of $V = V(\varphi)$ and $E = \{P \in S : |Gf(P)| > u(P)\}$.

PROOF. It is no loss in generality in assuming $f \geqq 0$. It is also no loss in generality in assuming that the center Z of S is $(0, r)$, $0 < r < 1$. We start by observing that

$$C^{-1} r \leqq d(Z) \leqq Cr$$

for some constant $C = C(M)$. We now make a change of scale and put $S^* = \{P : rP \in S\}$, $v(P) = Gf(rP)$. Let $d^*(P)$ denote the distance from P to ∂V^* (notice $rd^*(P) = d(rP)$), G^* denotes the Green function of V^*. Putting $q(P) = u(rP)/u(Z)$ and $k^*(P) = (q(P)/d^*(P))^{n/n-1}$ we see that

$$\int_E k\,dP = r^n (u(Z)/r)^{n/n-1} \int_{E^*} k^*\,dP\,. \tag{3.9}$$

Hence the lemma follows if we establish that

$$\int_{E^*} k^*\,dP \leqq C\varepsilon^\alpha \int_{S^*} k^*\,dP\,.$$

We notice that

$$E^* = \{P \in S^* : v(P) > u(Z)q(P)\} \tag{3.10}$$

and if we define F by $v = G^*F$ then $F(P) = r^2 f(rP)$ and it follows from (3.9) that

$$\begin{aligned}\int_{S^*} F(P)\,dP &= r^{2-n} \int_S f\,dP \\ &\leqq \varepsilon r^{2-n} \left(\int_S k\,dP\right)^{(n-1)/n} \\ &= \varepsilon u(Z) \left(\int_{S^*} k^*\,dP\right)^{(n-1)/n}.\end{aligned}$$

Putting $h(P) = F(P)/u(Z)$ we therefore have from (3.10) that

$$E^* = \{P \in S^* : G^*h(P) > q(P)\} \quad \text{and}$$

$$\int_{S^*} h\,dP \leqq \varepsilon \left(\int_{S^*} k^*\,dP\right)^{(n-1)/n}.$$

From Lemma 5 follows the existence of a constant $C = C(M)$ such that

$$C^{-1}|S^*| \leqq \int_{S^*} k^* \leqq C|S^*|\,. \tag{3.11}$$

There are now two cases to consider.

If $5S \cap V = \emptyset$ it follows from Harnack's inequality that $k(P) \leqq C$ in S^* and $q(P) \geqq C > 0$ in S^*. Hence

$$E^* \subset \{P \in S^* : G^*h(P) \geqq C\}$$

and since $G(P,Q) \leqq C|P-Q|^{1-n}$ in S^* it follows from Lemma 4 that

$$\int_{E^*} k^* \, dP \leqq C|E^*| \leqq C|\{P : I_1 h(P) \geqq C\}|$$

$$\leqq C\varepsilon^{n/n-1} \int_{S^*} k^* \, dP .$$

If $5S \cap V \neq \emptyset$ we may without loss of generality assume that the bottom boundary touches ∂V. Since in this case $C^{-1} \leqq |S^*| \leqq C$ for some constant $C = C(M)$ it follows from (3.11) and Lemma 8 that

$$\int_{E^*} k^* \, dP \leqq C\varepsilon^{\alpha} \int_{S^*} k^* \, dP ,$$

which proves the lemma.

We shall now fix a positive harmonic function u in $D(\varphi)$ with vanishing boundary values on $\{(x, \varphi(x)) : |x| < 10\}$ taking the value 1 at $(0,1)$. For f an integrable function supported on $S_0 = S(\varphi, 1)$ we define for $P \in S_0$

$$Tf(P) = Gf(P)/u(P) ,$$

where G is the Green function of $V(\varphi)$, and

$$Kf(P) = \sup \int_S |f| \, dQ/(\lambda(S))^{(n-1)/n} ,$$

where the sup is taken over all $S \in \Lambda$ such that $P \in S \subset 5S_0$ and $\lambda(S) = \int_S k \, dP$, $k(P) = (u(P)/d(P))^{n/n-1}$. The maximal function Kf is related to the maximal function

$$\sup \left\{ \int_\Omega |f| \, dQ/|\Omega|^{(n-\alpha)/n} : P \in \Omega, \ \Omega \text{ a cube} \right\}$$

introduced in Muckenhoupt–Wheeden [7] for studying the operator I_α.

LEMMA 10. *There are constants A, B and α, which can be taken to depend only on M and n with the following property. If $S \in \Lambda$ is contained in S_0 and $f \geqq 0$ is integrable on S_0 and if $Tf(P) \leqq 1$ for some $P \in S$ then*

$$\lambda\{P \in S : Tf(P) \geqq B, Kf(P) \leqq \varepsilon\} \leqq C\varepsilon^{\alpha}\lambda(S) .$$

PROOF. We start by observing that if $Q \notin 10S$ then

$$\sup_{P \in S} G(P,Q)/u(P) \leqq C \inf_{P \in S} G(P,Q)/u(P) . \tag{3.12}$$

For if $5S \cap \partial V = \emptyset$ this is a consequence of Harnack's inequality. If $5S \cap \partial V \neq \emptyset$ it is a consequence of Lemma 3.

Let $g=f$ in $10S$ and zero otherwise and put $h=f-g$. If $P_0 \in S$ is chosen so that $Tf(P_0) \leqq 1$ we have from (3.12) that

$$\sup_{P \in S} Th(P) \leqq CTh(P_0) \leqq CTf(P_0) \leqq C .$$

Hence if B is chosen large enough then

$$\{P \in S : Tf > B, Kf \leqq \varepsilon\} \subset \{P \in S : Tg > 1, Kf \leqq \varepsilon\} = E .$$

If $E \neq \emptyset$ and $P_1 \in E$ we have that

$$\int g\,dP \leqq CKf(P_1)(\lambda(S))^{(n-1)/n} \leqq C\varepsilon(\lambda(S))^{(n-1)/n} ,$$

which together with Lemma 9 yield Lemma 10.

It is well known that from Lemma 10 follows that for all $q>1$ such the reversed Hölder inequality in Lemma 5 holds we have the estimate

$$\|hTf\|_q \leqq C\|hKf\|_q, \quad h = u/d , \tag{3.13}$$

where C is independent of f and $\|f\|_q = (\int_{S_0} |f|^q\,dP)^{1/q}$ see e.g. Coifman–Fefferman [1].

LEMMA 11. *Let $q > n/n-1$ and put $h(P)=u(P)/d(P)$. Suppose that there is a constant N such that*

$$\left(|S|^{-1} \int_S h^q\,dP\right)^{1/q} \leqq N|S|^{-1} \int_S h\,dP$$

for all $S \in \Lambda$ contained in $5S_0$. If p is determined by $q^{-1}=p^{-1}-n^{-1}$ then we have for all functions f supported in S_0 that

$$\|Gf/d\|_q \leqq C\|f\|_p ,$$

where C only depends on N, M, q and n.

PROOF. It is well known that it is possible to find positive constants α and β, only depending on N, q and n such that if $-\alpha < r < q+\beta$ and $r \neq 0$ then

$$C^{-1}A \leqq \left(|S|^{-1} \int_S h^r\,dP\right)^{1/r} \leqq CA , \tag{3.14}$$

where $A=|S|^{-1}\int_S h\,dP$ and C is independent of S_1 (see Coifman–Fefferman [1] where the case $-\alpha<r<0$ and $1\leqq r<q+\beta$ is treated. The remaining case follows easily from this and Hölder's inequality). Define $Hf=K(fh^{1+q/n})$. Let $t>0$ and put $E=\{P\in S_0 : Hf(P)>t\}$. If $E\neq\varnothing$ then E can be written as a union of at most countably many pairwise disjoint sets $S_j\in\Lambda$.

Let q' be close to q and determine p' by $(q')^{-1}=(p')^{-1}-n^{-1}$. We shall use the following notation:

$$\gamma = (q/q')-1, \qquad \delta = 1+(q/n),$$

$$\mu(F) = \int_F h^q\,dP \quad\text{and}\quad \lambda(F) = \int_F h^{n/n-1}\,dP .$$

With this notation we have that $p'/q'<1$ and hence

$$(\mu(E))^{p'/q'} \leqq \sum (\mu(S_j))^{p'/q'} .$$

Since $1\leqq t^{-1}\lambda(S_j)^{-(n-1)/n}\int_{S_j} fh^\delta\,dP$ we have that

$$(\mu(E))^{p'/q'} \leqq t^{-p'} \sum (\mu(S_j))^{p'/q'}(\lambda(S_j))^{-p'(n-1)/n}\left(\int_{S_j} fh^\delta\,dP\right)^{p'} .$$

Hölder's inequality gives that

$$\int_S fh^\delta\,dP = \int_S fh^{\delta+\gamma}h^{-\gamma}\,dP$$

$$\leqq \left(\int_S (fh^{\delta+\gamma})^{p'}\,dP\right)^{1/p'}\left(\int_S h^{-\gamma r}\,dP\right)^{1/r},$$

where r is determined by $r^{-1}+(p')^{-1}=1$. Since $p'(\delta+\gamma)=q$ we find that

$$(\mu(E))^{p'/q'} \leqq t^{-p'} \sum B_j \int_{S_j} f^{p'}h^q\,dP' ,$$

where

$$B_j = (\mu(S_j))^{p'/q'}(\lambda(S_j))^{-p'(n-1)/n}\left(\int_{S_j} h^{-\gamma r}\,dP\right)^{p'/r} .$$

From (3.14) follows that if $-\alpha<-\gamma r<q+\beta$ then

$$B_j \leqq C|S_j|^a A_j^b ,$$

where $A_j=|S_j|^{-1}\int_{S_j} h\,dP$. Here we have $a=p'((q')^{-1}-r^{-1}-(n-1)/n)=0$ and $b=p'((q/q')-1-\gamma)=0$, which implies

$$\mu(E) \leqq C\left(t^{-q'}\int_{S_0} f^{p'}\,d\mu\right)^{q'/p'} .$$

Hence the operator H is of the weak type (p', q') with respect to the measure μ, whenever p' is sufficiently close to p. Hence it follows from Marcinkiewicz interpolation theorem (Zygmund [10, Vol. II, p. 112]) that

$$\left(\int_{S_0} (Hf)^q h^q \, dP\right)^{1/q} \leqq C\left(\int_{S_0} f^p h^q \, dP\right)^{1/q}.$$

Since $Kf = H(fh^{-\delta})$ and $p\delta = q$ it follows that

$$\|hKf\|_q \leqq C\|f\|_p .$$

Recalling (3.13) and observing that $hTf = Gf/d$ yields the lemma.

We shall next obtain the weak type estimate for $p=1$.

LEMMA 12. *If f is supported on S_0 and $t>0$ then*

$$|\{P \in S_0 : |Gf(P)| > td(P)\}| \leqq C(t^{-1}\|f\|_1)^{n/n-1},$$

where C only depends on M and n.

PROOF. Using a Calderon–Zygmund decomposition, we can to any given $z > \|f\|_1$ find $S_j \in S$ contained in S_0 such that the S_j's are paurwise disjoint, $|\bigcup S_j| \leqq C\|f\|_1/z$, and $|f| \leqq z$ outside $\bigcup S_j$. Put $g=f$ in $S_0 - \bigcup S_j$ and zero otherwise and let $h = f - g$. Let $p = 3n/(n+3)$ and $q=3$. From Lemmas 5 and 11 follows

$$\|Gg/d\|_q \leqq C\|g\|_p \leqq Cz^{1-1/p}\|f\|_1^{1/p} .$$

If $E_1 = \{P : |Gg(P)| > td(P)/2\}$ we therefore have

$$|E_1| \leqq C\|f\|_1^{q/p} z^{q-q/p} t^{-q} .$$

Let $U = S_0 - \bigcup (10S_j)$. It follows from Lemma 7 that

$$\left(\int_U |Gh_j|^{n/n-1} \, dP\right)^{(n-1)/n} \leqq C \int_{S_0} |h_j| \, dP$$

where $h_j = h$ in S_j and zero otherwise. Hence

$$\left(\int_U |Gh|^{n/n-1} \, dP\right)^{(n-1)/n} \leqq C\|f\|_1$$

which implies that

$$|E_2 \cap U| \leqq C(\|f\|_1 t^{-1})^{n/n-1} ,$$

where $E_2 = \{P \in S_0 : |Gh(P)| > td(P)/2\}$. Since

$$|E_2 \cap (U - S_0)| \leqq C \sum |S_j| \leqq C\|f\|_1 z^{-1}$$

and

$$E = \{P : |Gf(P)| > td(P)\} \subset E_1 \cup E_2$$

we have that

$$|E| \leqq C(\|f\|_1^{q/p} z^{q-q/p} t^{-q} + (\|f\|_1 t^{-1})^{n/n-1} + \|f\|_1 z^{-1}) .$$

Choosing $z = t^{n/n-1} \|f\|_1^{-1/n-1}$ yields the lemma. We can now give a preliminary version of Theorem 1:

LEMMA 13. *Let p_n be as in Theorem 1 and let $D \subset R^n$ be a Lipschitz domain. Then there is a number $p_0 = p_0(D)$ such that $p_n < p_0 < n$ and if $1 < p \leqq p_0$ then*

$$\|Gf/d\|_q \leqq C\|f\|_p ,$$

where $d(P)$ denotes the distance from P to ∂D, $\|f\|_p = (\int_D |f|^p \, dP)^{1/p}$ and $q = pn/n - p$. If $p = 1$ then

$$|\{P \in D : |Gf(P)| > td(P)\}| \leqq C(t^{-1}\|f\|_1)^{n/n-1} .$$

Here the constants C only depend on p and D.

PROOF. We can find finitely many open sets $\Omega_0, \Omega_1, \ldots, \Omega_N$ such that $D = \bigcup \Omega_j$, $\bar{\Omega}_0 \cap \partial D = \varnothing$ and if $1 \leqq i \leqq N$ then $\Omega_i \subset B(P_i, \varepsilon)$ for some $P_i \in \partial D$. Also we may assume that each $i \geqq 1$ there is a coordinate system (x, y) and a Lipschitz function φ such that $V(\varphi_i) \cap L_i = D \cap L_i$ for some right circular cylinder L_i with its axis along the y-axis and P_i corresponds to the origin in the coordinate system. Since $G(P, Q) \leqq C|P - Q|^{1-n}$ in D it follows from Lemma 4 that we only have to estimate Gf/d in $\bigcup_{i \geqq 1} \Omega_i$. We may without loss of generality assume that there are positive number r_i and R_i such that

$$\Omega_i \subset S(\varphi_i, r_i) \subset r_i D(\varphi_i) \subset S(\varphi_i, R_i)$$

$$2R_i D(\varphi_i) \subset L_i \subset D - \bar{\Omega}_0$$

where $RD(\varphi) = \{(x, y) : |x| < 10R, \ \varphi(x) < y < aR\}$ and a is in the definition of $D(\varphi)$ and that $f \geqq 0$.

If f has its support outside $D_i = r_i D(\varphi_i)$ then Gf is a positive, harmonic function in $r_i D(\varphi_i)$, vanishing on the bottom boundary of D_i such that

$$Gf(Z_i) \leqq \sup\{G(P, Z_i) : P \in D - D_i\} \|f\|_1 ,$$

where Z_i is the center of $S(\varphi_i, r_i)$. Hence it follows from Lemma 5 and (3.13) that

$$\left(\int_{\Omega_i} (Gf(P)/d(P))^{q_0}\right)^{1/q_0} \leqq C\|f\|_1 \quad \text{for some } q_0=q_0(D)>q_n\,,$$

where q_n is given by $q_n^{-1}=p_n^{-1}-n^{-1}$. If f is supported in $r_iD(\varphi_i)$ we observe that $Gf=gf+h$ in $S(\varphi_i, R_i)$ where g is the Green function of $S(\varphi_i, R_i)$. We may without loss of generality assume that the center of $S(\varphi_i, R_i)$ is outside $r_i\dot{S}(\varphi_i, r_i)$ arguing as above we see that

$$\left(\int_{\Omega_i} (h(P)/d(P))^{q_0}\right)^{1/q_0} \leqq C\|f\|_1\,.$$

An application of (3.13), Lemmas 5, 11 and 12 yields the lemma.

We can now give the proof of Theorem 1.

Proof of Theorem 1. It is no loss on generality in assuming f is nonnegative and smooth. For $P \in D$ let $B=B(P, d(P)/2)$, where $d(P)$ denotes the distance to ∂D, and denote by h the characteristic function of B. Let $f_1=fh$ and $f_2=f(1-h)$. Since Gf_2 is harmonic and positive, we have that

$$|\nabla Gf_2(P)| \leqq CGf_2(P)/d(P) \leqq CGf(P)/d(P)\,.$$

It is easily seen that

$$|\nabla_p G(P,Q)| \leqq C|P-Q|^{1-n} \quad \text{for } Q \in B\,.$$

This implies that $|\nabla Gf_2(P)| \leqq CI_1f_2(P) \leqq CI_1f(P)$. Combining these estimates we find that

$$|\nabla Gf(P)| \leqq C(I_1f(P)+Gf(P)/d(P))\,.$$

The theorem now follows from Lemmas 4 and 13.

In conclusion we remark that for the case when D is assumed to be C^1, then (1.3) holds for the range $1<p<n$. This is so because given $\varepsilon>0$, we can represent ∂D locally by domains $V(\varphi)$, where $|\varphi(x)-\varphi(z)| \leqq \varepsilon|x-z|$. As is easily seen, the best exponent q for which Lemma 5 holds, tends to ∞ as $\varepsilon \to 0$, which in view of Lemma 11 justifies the remark.

4. Concluding remarks.

We shall first give examples to show that Theorem 1 is sharp. We begin with the case $n=2$. We shall write $z=x+iy=re^{i\theta}$, $-\pi \leqq \theta < \pi$, $r=|z|$. Fix α, $\pi<\alpha<2\pi$, and put

$$D_\alpha = \{re^{i\theta} : 0<r<1, |\theta|<\alpha/2\}, \quad u_\alpha(z) = r^{\pi/\alpha}\cos(\theta\pi/\alpha)\,.$$

Now u_α is harmonic in D_α and vanishes on $\partial D_\alpha \cap \{z : |z|<1\}$. Pick a smooth function φ such that $\varphi(z)=1$ for $|z|<1/2$ and $\varphi(z)=0$ for $|z|>3/4$ and define $v_\alpha = \varphi u_\alpha$. It is easily seen that v_α vanishes on ∂D_α and $\Delta v_\alpha \in L^\infty(D_\alpha)$. Since

$$|\nabla v_\alpha(z)| \geqq C|z|^{-1+\pi/\alpha}$$

we have that $\nabla v_\alpha \notin L^\beta(D_\alpha)$ where $\beta = 2(1-\pi/\alpha)^{-1}$. Since $\beta \to 4$ as $\alpha \to 2\pi$ we see that Theorem 1 is sharp for $n=2$.

We shall next consider the case $n=3$. Fix λ, $0<\lambda<1$, and let P_λ be the generalized Legendre function of degree λ, i.e. P_λ is the solution of the differential equation

$$(1-x^2)u''(x) - 2xu'(x) + \lambda(\lambda+1)u(x) = 0, \qquad -1<x<1,$$

normalized by $P_\lambda(1)=1$. It is well known that P_λ has at least one zero in $-1<x<1$, see Hobson [5, p. 386] and if

$$a_\lambda = \sup\{t : -1<t<1, P_\lambda(t)=0\}$$

then $-1<a_\lambda<1$. Put

$$D_\lambda = \{x=(x_2, x_2, x_3) : |x|<1, x_1>a_\lambda|x|\}$$

and define $v_\lambda(x) = \varphi(x)|x|^\lambda P_\lambda(x_1/|x|)$, where φ is smooth, vanishing for $|x|>3/4$ and identically 1 for $|x|<1/2$. Arguing as above shows that $\Delta u_\lambda \in L^\infty(D_\lambda)$, $v_\lambda = 0$ on ∂D_λ and $\nabla v_\lambda \notin L^\beta(D_\lambda)$, where $\beta = 3/(1-\lambda)$. Since $\beta \to 3$ as $\lambda \to 0$ we see that Theorem 1 is sharp also when $n=3$. Finally, considering the function $u(x_1, \ldots, x_n) = v_\lambda(x_1, x_2, x_3)$ in $D_\lambda \times \mathrm{R}^{n-3}$ yields that Theorem 1 is sharp for $n \geqq 4$.

We shall next construct an example showing that there is no analogue of (1.2) for Lipschitz domain.

Let

$$K = \{z=x+iy : -|x|<y<2, |x|<1\}$$

and put $T=\{x-i|x| : |x|<1\}$. We have the following estimate

LEMMA 14. *There is a constant $C>0$ such that if u is a positive, harmonic function in K vanishing on T then*

$$Cu(i) \leqq \int_K |\nabla_2 u|\, dP. \tag{4.1}$$

PROOF. Let V be the class of positive harmonic functions in K which vanish on T and assume the value 1 at i.

Suppose (4.1) is false. Then there is a sequence $u_j \in V$ such that $\int_K |\nabla_2 u_j| \, dP \to 0$. Since V is a normal family we may assume $u_j \to u \in V$ uniformly on compact subsets of K. Since the derivatives of u_j also converge uniformly on compact subsets to the corresponding derivatives of u, we have that $\int_K |\nabla_2 u| \, dP = 0$, which means that $u = A + Bx + Cy$. Since u vanishes on T it follows that $u \in V$ is identically vanishing. This contradiction yields Lemma 14.

We can now construct the example mentioned in the introduction.

THEOREM 2. *There is a Lipschitz domain $D \subset \mathbf{R}^2$ such that for some $f \in L^\infty(D)$ we have that*

$$\int_D |\nabla_2 Gf|^p \, dP = \infty \quad \textit{for all } p > 1 .$$

PROOF. Let

$$r_n = n^{-1}(\log(n+1))^{-4} \quad \text{and} \quad \delta_n = Nn^{-1}(\log(n+1))^{-2} ,$$

where N is a constant to be chosen later. Put

$$\varrho_n = \sum_{i=1}^{n} (\delta_i + 2r_i) \quad \text{for } n \geqq 1, \ \varrho_0 = 0 .$$

We define $\varphi(x)$ by $\varphi(x) = 0$ for $x \leqq 0$, $\varphi'(x) = 1$ for $\varrho_n < x < \varrho_n + r_{n+1}$, $\varphi'(x) = -1$ for $\varrho_n + r_{n+1} < x < \varrho_n + 2r_{n+1}$, $\varphi'(x) = 0$ for $\varrho_n + 2r_{n+1} < x < \varrho_n + 2r_{n+1} + \delta_{n+1}$, $n = 0, 1, 2, \ldots$ We also put $\varphi'(x) = 0$ for $x > \sum_1^\infty (\delta_i + 2r_i)$. Let $x_n = \varrho_{n-1} + r_n$ and put $z_n = x_n + ir_n$, (Notice $\varphi(x_n) = r_n$.) We now put

$$\Omega = \{z : \varphi(x) < y\}, \quad U = \{z : x^2 < y\}$$

and $U_n = U + z_n$. We claim that if N is chosen large enough then

$$U_n \subset \Omega \quad \text{for } n \geqq 1 \tag{4.2}$$

We shall prove (4.2) by induction. We start by observing $\max \varphi(x) = r_1$ which implies that $U_1 \subset \Omega$. Since $\varphi(x) \leqq r_n$ for $x \geqq x_n$ it follows that

$$\{z \in U_n : x \geqq x_n\} \subset \Omega$$

and therefore it remains to show that the left "half" of U_n is contained in Ω. Assume now that $U_n \subset \Omega$. Put $t_n = x_n + r_n$ and notice that $\varphi(x) \leqq r_{n+1}$ for $t_n \leqq x \leqq r_{n+1}$. Hence

$$\{z \in U_{n+1} : t_n \leqq x\} \subset \Omega .$$

Putting $\varphi_n(x) = (x - x_n)^2 + r_n$ we see that ∂U_n is the graph of φ_n. We now have that

$$\begin{aligned}\varphi_{n+1}(t_n)-\varphi_n(t_n) &= (r_{n+1}+\delta_n)^2-(r_n-r_{n+1}+r_n^2)\\ &> \delta_n^2-(r_n-r_{n+1}+r_n^2)\,.\end{aligned}$$

Since

$$r_n-r_{n+1}+r_n^2 \leqq Cn^{-2}(\log(n+1))^{-4}$$

we see that if N has been sufficiently large then $\varphi_{n+1}(t_n)>\varphi_n(t_n)$, which implies that

$$\{z \in U_{n+1} : x \leqq t_n\} \subset U_n \subset \Omega\,,$$

which yields (4.2).

We now define

$$D = \{z : \varphi(x)<y<A,\ |x|<A\}\,,$$

where A is a large constant. If A has been chosen large enough then $\bigcup K_n \subset D$ and there is a square S' such that $S' \subset D$ and $S^1 \cap (\bigcup K_n)=\emptyset$, where $K_n=r_nK+z_n$ and K is as in Lemma 14. Let

$$V = \{z : x^2<y<10\}$$

and put

$$V_n = V+z_n, \qquad S_n = S+z_n\,,$$

where $S=\{z \in \partial v : y=10\}$. Fix a bounded function f supported on S' such that $\int f\,dP>0$. If A has been chosen large enough we have that $\bar{V}_n \subset D$ and

$$\inf\{Gf(P) : P \in \bigcup S_n\} = L > 0\,.$$

If ω denotes the harmonic measure of S with respect to V then since ∂V is smooth near 0 we have

$$\omega(iy) \geqq Cy, \qquad 0<y<10\,. \tag{4.3}$$

Since Gf is superharmonic we have that $Gf(z) \geqq L\omega(z-z_n)$ for $z \in V_n$. In particular, we have from (4.3) that $Gf(z_n+ir_n) \geqq Cr_n$. If $1<p<\infty$ it follows from Lemma 14 that

$$\begin{aligned}CGf(z_n+ir_n) &\leqq \int_{K_n} |\nabla_2 Gf|\,dP\\ &\leqq Cr_n^{2(1-1/p)}\left(\int_{K_n} |\nabla_2 Gf|^p\,dP\right)^{1/p}\,,\end{aligned}$$

which gives $\int_{K_n} |\nabla_2 Gf|^p\,dP \geqq Cr_n^{2-p}$. Since the K_n's are pairwise disjoint it follows that

$$\int_D |\nabla_2 Gf|^p \, dP \geqq C \sum r_n^{2-p} = \infty ,$$

which proves the theorem.

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