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POSITIVE SELF-ADJOINT EXTENSIONS OF OPERATORS AFFILIATED WITH A VON NEUMANN ALGEBRA

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Let N be a von Neumann algebra on the Hilbert space H with a cyclic and separating vector ξ_0. We study positive self-adjoint extensions affiliated with N of operators h that are affiliated with N and such that $\xi_0 \in \mathscr{D}(h)$. We show that the operators h have unique extensions if and only if N is finite. This in turn we show is equivalent to the validity of the so-called T-theorem for N with respect to ξ_0. Finally we show how all this is intimately related to properties of the cone $P^\# = N_+\xi_0^-$.

1. Introduction.

We are going to consider the following problem: Let h be a (densely defined) positive operator affiliated with a von Neumann algebra N (see Definition 2.1). Characterize all positive self-adjoint extensions $\tilde{h}$ of h such that $\tilde{h}$ is again affiliated with N and give criteria for when there is a unique such extension. It is a well known fact that there exists one extension $\tilde{h}$ of the desired type, namely the Friedrichs extension of h (cf. [14, 9. Appendix]).

The above posed problem arises quite naturally. In fact, if ξ_0 is a cyclic and separating vector for N acting on the Hilbert space H with inner product $\langle , \rangle$, we consider the cone $P^\# = N_+\xi_0^-$, i.e. the closure of the set $N_+\xi_0 = \{x\xi_0 \mid x \in N_+\}$. As a generalization of Sakai's Radon–Nikodym theorem it is shown in [20, Theorem 15.1] that to any normal positive linear functional φ on N there exists a unique $\xi \in P^\#$ such that $\varphi = \omega_\xi$, where $\omega_\xi(x) = \langle x\xi, \xi\rangle$, $x \in N$. (Incidentally, this vector is characterized by the property of being the unique vector ξ in H such that

$$\varphi = \omega_\xi \quad \text{and} \quad \|\xi - \xi_0\| = \inf\{\|\eta - \xi_0\| \mid \varphi = \omega_\eta\},$$

cf. [6, Theorem 5.4; p. 83]). Denote by $\pi(\xi)$ the closure of the linear map $x'\xi_0 \to x'\xi$, $x' \in N'$. Then $\pi(\xi)$ is a (densely defined) positive operator affiliated with N. As pointed out in [20], if $\pi(\xi)$ did have a unique positive self-adjoint

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extension k affiliated with N it would be natural to set $k=(d\omega_{\xi}/d\omega_{\xi_0})^{\frac{1}{2}}$, where $d\omega_{\xi}/d\omega_{\xi_0}$ denotes the Radon–Nikodym derivative od ω_{ξ} with respect to ω_{ξ_0}. However, in [12] Perdrizet gave an example with N an infinite type I factor to show that $\pi(\xi)$ may have several different positive self-adjoint extensions affiliated with N. We are going to explore these matters in depth. For example, we prove that $\pi(\xi)$ has a unique positive self-adjoint extension affiliated with N for every $\xi \in P^{\#}$ if and only if N is finite (Theorem 3.1).

To classify all positive self-adjoint extensions of $\pi(\xi)$ affiliated with N we are going to rely heavily on a truly fundamental paper by Krein [7]. To our knowledge this paper has not been translated from Russian and this fact, together with the extensive use we will make of some of the results contained therein, has convinced us that it is necessary to give a brief outline, tailored for our purposes, of some features of this paper. This is done in section 4. It turns out that there are two "extreme" positive self-adjoint extensions of $\pi(\xi)$ which both are affiliated with N. One of these is the above mentioned Friedrichs extension, the other we will call the Krein extension, and all other extensions will lie between these two extremes in a sense that we will make precise. As both the Friedrichs and Krein extensions can be characterized explicitly we will be able to give criteria for when there is a unique positive self-adjoint extension of $\pi(\xi)$ affiliated with N. In this connection let us introduce some relevant subsets of the cone $P^{\#}=N_+\xi_0^-$. We set

$$P^{\#}_{\mathrm{sa}} = \{\xi \in P^{\#} \mid \pi(\xi) \text{ is self-adjoint}\}, \quad P^{\#}_{\mathrm{un}} = \{\xi \in P^{\#} \mid \pi(\xi)$$

has unique positive self-adjoint extension affiliated with $N\}$. Clearly, $N_+\xi_0 \subset P^{\#}_{\mathrm{sa}} \subset P^{\#}_{\mathrm{un}} \subset P^{\#}$. In [12, Corollaire 4.9] it is shown that $C_+\xi_0^- \subset P^{\#}_{\mathrm{sa}}$, where C is the centralizer to ω_{ξ_0}, that is,

$$C = \{x \in N \mid \omega_{\xi_0}(xy)=\omega_{\xi_0}(yx),\ \forall\, y \in N\}\ .$$

It is a remarkable fact that we will have $P^{\#}_{\mathrm{sa}} \neq P^{\#}_{\mathrm{un}}$ for some cyclic and separating vector ξ_0 when N is not finite (Theorem 3.2).

Whether $P^{\#}_{\mathrm{un}}$ (or $P^{\#}_{\mathrm{sa}}$) is equal to $P^{\#}$ is related to a seemingly different question. Specifically, let us consider the map Φ: $P^{\#} \to (N'_*)_+$ of $P^{\#}$ into $(N'_*)_+$, the positive normal linear functionals on N', defined by $\Phi(\xi)=\omega'_{\xi}$ for $\xi \in P^{\#}$. Here ω'_{ξ} denotes the vector functional on N' defined by $\omega'_{\xi}(x')=\langle x'\xi,\xi\rangle$, $x' \in N'$. In [6, Theorem 4.1; p. 79] Araki showed that if $\psi' \in (N'_*)_+$ is dominated by a positive multiple of ω'_{ξ_0} then $\psi'=\omega'_{\xi}$ for some $\xi \in P^{\#}$. We show first that $P^{\#}_{\mathrm{un}}$ (or $P^{\#}_{\mathrm{sa}}$) is equal to $P^{\#}$ if and only if Φ is an injective map (Lemma 5.1). We also show (Lemma 5.1) that Φ is *onto* $(N'_*)_+$ if and only if the T-theorem is valid for N with respect to ξ_0 (see discussion of this in the next paragraph). Finally we show that each one of the above stated conditions are equivalent to N being finite (Theorem 3.1).

The ontoness of the above map Φ is intimately connected with the following question: If ξ is any vector in H, does there exist a *closed* (densely defined) operator t affiliated with N such that $\xi_0 \in \mathscr{D}(t)$ and $\xi = t\xi_0$? For descriptive purposes let us call this result, when valid, the "T-theorem" for N with respect to ξ_0. Murray and von Neumann [8] showed that if N is finite the (unrestricted) T-theorem holds for N, i.e. for any $\xi_1 \in H$ and $\xi_2 \in [N\xi_1]$ there exists a closed (densely defined) operator t affiliated with N such that $\xi_1 \in \mathscr{D}(t)$ and $\xi_2 = t\xi_1$. They deduced this result from their so-called BT-theorem [8, Lemma 9.2.1], using the fact, which they established, that unbounded operators affiliated with a finite von Neumann algebra can be manipulated with in much the same way as one can with bounded operators (cf. [18]). The converse result was established by Dye [5, Theorem 2]. In the setting above, i.e. N has a cyclic and separating vector ξ_0, he proved that if the (unrestricted) T-theorem holds for N, then N is finite (cf. also [3; III, § 8, Exercice 3]). We shall prove a stronger result, namely we show that if the T-theorem holds for N with respect to the (fixed) cyclic and separating vector ξ_0, then N is finite (Proposition 5.4). As a simple corollary (Corollary 2 to Theorem 3.1) we get a strengthening of [5, Theorem 2].

Finally, let us say a word about a method we apply throughout this paper to prove our results. If N, acting on H, is properly infinite we have that N is spatially isomorphic to $\mathscr{B}(K)\otimes N$ acting on $K\otimes H$, where K is a Hilbert space of countable infinite dimension and $\mathscr{B}(K)$ denotes the bounded operators on K (Proposition 2.2). In this situation the method we use is to transport suitable operators defined in K to operators defined in $K\otimes H$ and there being affiliated with $\mathscr{B}(K)\otimes N$. By choosing appropriately "bad" operators in K we get "bad" operators affiliated with $\mathscr{B}(K)\otimes N$.

2. Notation and preliminaries.

For the purpose of completeness, and also to establish our notation, we will include here a number of basic definitions and elementary results which we shall need in the sequel. We refer to [3] and [18] for proofs and more detailed exposition.

By the term "operator" we will mean a linear mapping t defined on a linear manifold $\mathscr{D}(t)$ of the Hilbert space H and with range $\mathscr{R}(t)$ in H. If t_1 and t_2 are two operators, we write $t_1 \subset t_2$ if $\mathscr{D}(t_1) \subset \mathscr{D}(t_2)$ and t_1 and t_2 agree on $\mathscr{D}(t_1)$. We write $t_1 = t_2$ if $\mathscr{D}(t_1) = \mathscr{D}(t_2)$ and t_1 and t_2 agree on their common domain of definition. We shall occasionally study operators that are not densely defined, but when we use the term "operator" without any specification we will assume that the operator in question has a dense domain of definition.

DEFINITION 2.1. Let N be a von Neumann algebra acting on the Hilbert space H. Let t be an operator in H. We say that t is *affiliated* with N; in symbols $t \eta N$, if $x't \subset tx'$ for every x' in N'. This is equivalent to $t = u'tu'^*$ for every unitary operator u' in N'.

We notice that if t is bounded with $\mathscr{D}(t) = H$ then $t \eta N$ is equivalent to $t \in N$.

We assume the reader is familiar with the concepts of closed and closable operators and also with the concepts of symmetric, positive and self-adjoint operators, respectively (cf. [18]). The polar decomposition of a closed operator t in H yields a unique decomposition $t = v|t|$, where $|t|$ is a positive self-adjoint operator and v is a partial isometry with initial space equal to $\mathscr{R}(|t|)^-$, i.e. initial (v) is equal to the closure of $\mathscr{R}(|t|)$. We also have that the final space of v is equal to $\mathscr{R}(t)^-$. Notice that $\mathscr{D}(\mathrm{t}) = \mathscr{D}(|\mathrm{t}|)$. Let the spectral resolution of $|t|$ be

$$|t| = \int_0^\infty \lambda E(d\lambda) .$$

Then it is an elementary fact that $|t| \eta N$ if and only if the spectral projections $\{E(\omega) \mid \omega \text{ Borel set in } \mathbf{R}_+\}$ lies in N. Also $t \eta N$ if and only if $v \in N$ and $|t| \eta N$.

Recall that by a *core* for a closed operator t we mean a linear subset $\mathscr{C}$ of $\mathscr{D}(t)$ such that t is the closure of $t|_{\mathscr{C}}$, the restriction of t to $\mathscr{C}$. We observe that $\mathscr{C}$ is a core for t if and only if $\mathscr{C}$ is a core for $|t|$.

An operator h in H is called (lower) semibounded if there exists a real number α such that $\langle h\xi, \xi \rangle \geqq \alpha \langle \xi, \xi \rangle$ for all $\xi \in \mathscr{D}(h)$. The supremum $m(h)$ of all such α is called the (lower) bound for h. Obviously,

$$m(h) = \inf_{\xi \in \mathscr{D}(h)} \frac{\langle h\xi, \xi \rangle}{\langle \xi, \xi \rangle} .$$

Clearly h is positive if and only if $m(h) \geqq 0$.

Now let N be a von Neumann algebra acting on H and let ξ_0 be a (fixed) vector in H. Let $\xi \in [N\xi_0]$, where $[N\xi_0]$ denotes the closed linear subspace $N\xi_0^-$. (By a common abuse of notation we will also let $[N\xi_0]$ denote the orthogonal projection onto the subspace $[N\xi_0]$. It will be clear from the context what we mean in each case). Define an operator $\pi_0(\xi)$ in H which is 0 on the orthogonal complement of $[N'\xi_0]$ and which is equal to $x'\xi_0 \to x'\xi$, $x' \in N'$, in $[N'\xi_0]$. Observe that $\xi_0 \in \mathscr{D}(\pi_0(\xi))$. It is easily verified that $\pi_0(\xi)$ is well-defined with a dense domain of definition and $\pi_0(\xi) \eta N$. If $\pi_0(\xi)$ is closable we denote the closure with $\pi(\xi)$, and it is easily verified that $\pi(\xi) \eta N$. Now assume that ξ_0 is a cyclic and separating vector for N and let $\xi \in H = [N\xi_0]$. Then $\pi_0(\xi)$ is equal to the map $x'\xi_0 \to x'\xi$, $x' \in N'$, and $\pi_0(\xi)$ is the smallest operator (in the partial ordering $\subset$ introduced above) among operators affiliated with N which has ξ_0 in their domains of definition. Also, if $\pi_0(\xi)$ is closable, then $\pi(\xi)$ is the smallest *closed* operator with these properties.

If N is a von Neumann algebra there exists a unique central projection e in N such that Ne is finite and $N(I-e)$ is properly infinite. For later reference we write down the following proposition whose proof can be found in [3; I, § 2, Proposition 5 and III, § 8, Corollaire 2].

PROPOSITION 2.2. *Let N be a properly infinite von Neumann algebra acting on H. Then N is spatially isomorphic to the von Neumann algebra $\mathscr{B}(K)\otimes N$ acting on $K\otimes H$, where K is a Hilbert space of countable infinite dimension.*

We also write down for later reference the following theorem.

THEOREM 2.3. *Let N be a finite von Neumann algebra acting on H and let $\xi_1 \in H$. If $\xi_2 \in [N\xi_1]$ there exists a closed (densely defined) operator $t \,\eta\, N$ such that $\xi_1 \in \mathscr{D}(t)$ and $\xi_2 = t\xi_1$.*

Let h be a closed symmetric operator with $h \,\eta\, N$. Then h is self-adjoint.

PROOF. The first part is the T-theorem for finite von Neumann algebras, cf. [18, Corollary 2 to Lemma 3.4].

To prove the second part of the above theorem, let $v=(h-iI)(h+iI)^{-1}$ be the Cayley transform of v [16, Chapter VIII, § 123]. Set $e=\text{initial}\,(v)=\mathscr{R}(h+iI)$. Then e is a projection in N and ve is a partial isometry in N. Now $h=i(I+v)(I-v)^{-1}$ and so $\mathscr{R}(e-ve)=\mathscr{D}(h)$ is dense in H. Hence we have

$$\begin{aligned} I &= \mathscr{R}(e-ve)^- = \mathscr{R}((e-ve)e)^- \\ &\sim \mathscr{R}(\{(e-ve)e\}^*)^- = \mathscr{R}(e(e-ve)^*)^- \leqq e\,. \end{aligned}$$

where $\sim$ denotes equivalence of projections in N [3: III, § 1, Proposition 2]. Since N is finite we must have $e=I$ and also $v=ve$ must be a unitary operator in N. Hence h is self-adjoint.

3. Main results.

Let N be a von Neumann algebra acting on the Hilbert space H with a cyclic and separating vector ξ_0. Recall from section 1 that we defined (relative to ξ_0) the following sets:

$$P^\# = N_+\xi_0^-, \quad P^\#_{\text{sa}} = \{\xi \in P^\# \mid \pi(\xi) \text{ is self-adjoint}\}\,,$$

$$P^\#_{\text{un}} = \{\xi \in P^\# \mid \pi(\xi) \text{ has unique positive self-adjoint extension affiliated with } N\}\,.$$

Recall also that if $\xi \in H$, then ω'_ξ denotes the normal positive linear functional on N' (that is, $\omega'_\xi \in (N'_*)_+$) defined by

$$\omega'_\xi(x') = \langle x'\xi, \xi \rangle, \qquad x' \in N' .$$

We can now state our main theorem.

THEOREM 3.1. *Let N be a von Neumann algebra on H with ξ_0 a cyclic and separating vector for N. Then the following conditions are equivalent*:

(1) *N is finite.*
(2) *For every $\xi \in H$ there exists a closed (densely defined) operator t affiliated with N such that $\xi_0 \in \mathscr{D}(t)$ and $\xi = t\xi_0$.*
(3) *The map Φ: $\xi \to \omega'_\xi$ from $P^\#$ into $(N'_*)_+$ is onto.*
(4) *The map Φ in (3) is injective.*
(5) *$P^\#_{\mathrm{sa}} = P^\#$.*
(6) *$P^\#_{\mathrm{un}} = P^\#$.*

COROLLARY 1. *If either one of (2), (3), (4), (5) or (6) holds in the above theorem for one particular cyclic and separating vector ξ_0, then the same properties hold for any cyclic and separating vector.*

The next corollary should be compared with [5, Theorem 2].

COROLLARY 2. *Let M be a von Neumann algebra on K and let $\xi_0 \in K$. Then the following conditions are equivalent*:

(i) *If ξ is any vector in $[M\xi_0]$, there exists a closed (densely defined) operator t affiliated with M such that $\xi_0 \in \mathscr{D}(t)$ and $\xi = t\xi_0$; i.e. the T-theorem holds for M with respect to ξ_0.*
(ii) *The projection $e = [M'\xi_0]$ in M is finite (equivalently, $e' = [M\xi_0]$ is a finite projection in M').*

PROOF. We show first that (i) implies (ii). We have $(eMe)' = M'e$ and so $ee' \in (eMe)'$. Now $N = ee'Mee'$ is a von Neumann algebra acting on the Hilbert space $H = ee'(K) = [M'\xi_0] \cap [M\xi_0]$, and $\xi_0 \in H$ is a cyclic and separating vector for N. Let $\xi \in H$. Then $\xi \in [M\xi_0]$ and so by hypothesis there exists a closed operator $t \;\eta\; M$ such that $\xi_0 \in \mathscr{D}(t)$ and $\xi = t\xi_0$. We may assume that $t = \pi(\xi)$, i.e. t is equal to 0 on $(I-e)(K)$ and in $e(K)$ t is equal to the closure of the map $x'\xi_0 \to x'\xi$, $x' \in M'$ (cf. section 2). Since $[M'\xi] \subset [M'\xi_0] = e(K)$ we get $t = ete$, and since $(ete) \;\eta\; M$ it is easily verified that t restricted to $H = ee'(K)$ is a closed operator in H equal to $ee'tee'$. Set $t_1 = ee'tee'$ and observe that $t_1 \;\eta\; N$. We also have $\xi_0 \in \mathscr{D}(t_1)$ and $\xi = t_1\xi_0$. By the theorem $N = ee'Mee'$ is finite. Now the central support to the projection ee' in eMe is equal to e since $[(M'e)\xi_0] = e(K)$.

So eMe is isomorphic to N [3; I, § 2, Proposition 2]. Hence e is finite.

To prove that (ii) implies (i) we proceed more or less as in [5], and we present the proof here for the sake of completeness. We first recall that if ξ_1 and ξ_2 are vectors in K, then $[M\xi_1] \prec [M\xi_2]$ in M' if and only if $[M'\xi_1] \prec [M'\xi_2]$ in M [3; III, § 1, Théorème 2]. This yields, in particular, that $e = [M'\xi_0]$ is finite if and only if $e' = [M\xi_0]$ is finite. Now let $\xi \in [M\xi_0]$. Then $[M\xi] \leqq [M\xi_0]$ and hence $[M\xi] \prec [M\xi_0]$ in M'. So $[M'\xi] \prec [M'\xi_0] = e$ in M. Let v be a partial isometry in M with initial space $[M'\xi]$ and final space contained in $[M'\xi_0]$. Now eMe is finite by hypothesis, and so by Theorem 2.3 there exists a closed operator t_1 in $e(K)$ affiliated with eMe such that $\xi_0 \in \mathscr{D}(t_1)$ and $v\xi = t_1\xi_0$. From this we get by a simple observation that the closed operator $\pi(v\xi)$, which is affiliated with M, exists (cf. section 2). Since $\mathscr{R}(\pi(v\xi)) \subset [M'(v\xi)] = v([M'\xi])$, we get that the operator $t = v^*\pi(v\xi)$ is a closed operator affiliated with M. Also $\xi_0 \in \mathscr{D}(t)$ and $\xi = t\xi_0$. So (i) is true.

As mentioned in the Introduction the next theorem has the remarkable consequence that we may actually have $P_{\mathrm{sa}}^{\#} \neq P_{\mathrm{un}}^{\#}$.

THEOREM 3.2. *Let N be a von Neumann algebra on H with a cyclic and separating vector. The following conditions are equivalent:*

(i) *N is finite.*

(ii) *For every cyclic and separating vector ξ_0 we have $P_{\mathrm{sa}}^{\#} = P_{\mathrm{un}}^{\#}$.*

REMARK. We conjecture that it is enough to know that $P_{\mathrm{sa}}^{\#} = P_{\mathrm{un}}^{\#}$ for just one cyclic and separating vector ξ_0 to prove that N is finite, by using some more refined technique than we apply in our proof.

The next proposition gives a characterization of $P_{\mathrm{sa}}^{\#}$ and $P_{\mathrm{un}}^{\#}$.

PROPOSITION 3.3. *Let N be a von Neumann algebra on H with a cyclic and separating vector ξ_0 and let $\xi \in P^{\#}$. Then*

(i) *ξ is in $P_{\mathrm{sa}}^{\#}$ if and only if $\xi + \xi_0$ is separating for N.*

(ii) *ξ is in $P_{\mathrm{un}}^{\#}$ if and only if*

$$\inf_{x' \in N'} \frac{\langle x'\xi, x'\xi_0 \rangle}{|\langle \gamma, x'\xi_0 \rangle|^2} = 0$$

for every γ orthogonal to $[N'(\xi + \xi_0)]$.

REMARK. We can show that if the numerator in the fraction in (ii) is 0 for some $x' \in N'$, then the denominator is also 0, cf. remark after Theorem 4.4. When this occurs we make the convention to set the fraction in (ii) equal to ∞.

4. Krein's method.

To prove our theorems we are faced with the following problem: Let h be a (densely defined) positive operator affiliated with the von Neumann algebra N; when does h have a unique positive self-adjoint extension $\tilde{h}$ affiliated with N?

To solve this problem we present below an outline, suitably tailored for our purposes, of those parts of Krein's paper [7] which has a direct bearing on our investigation.

Let $S \geqq 0$ be a positive (densely defined) operator in the Hilbert space H. Define the operator B by the transformation $B=(S-I)(S+I)^{-1}$, where $\mathscr{D}(B) = \mathscr{R}(S+I)$. It is easily verified that B is well-defined, but we emphasize that B is not in general densely defined. We call B the Krein transform of S. The Krein transform is analogous to the Cayley transform $V=(T-iI)(T+iI)^{-1}$ of a symmetric (densely defined) operator T. But whereas $\lambda \to (\lambda-i)(\lambda+i)^{-1}$ maps the real line onto the unit circle (except 1) of the complex plane, the map $\lambda \to (\lambda-1)(\lambda+1)^{-1}$ maps the positive real line onto the interval $[-1,1\rangle$. This is reflected in the fact that whereas the Cayley transform of a symmetric operator T gives rise to a partial isometry V such that $\mathscr{R}(I-V)$ is dense in H, the Krein transform of a positive operator S gives rise to an operator B which has the following properties:

(i) $\langle B\xi_1, \xi_2\rangle = \langle \xi_1, B\xi_2\rangle$ for all $\xi_1, \xi_2 \in \mathscr{D}(B)$.
(ii) $\|B\xi\| \leqq \|\xi\|$ for all $\xi \in \mathscr{D}(B)$.
(iii) $\mathscr{R}(I-B)$ is dense in H.

Let $\mathscr{F}$ denote the family of all operators in H satisfying the properties (i), (ii) and (iii). Let $\mathscr{P}$ denote the family of all densely defined positive operators in H. Then there is a 1–1 correspondence between $\mathscr{F}$ and $\mathscr{P}$; specifically, if $B \in \mathscr{F}$ and $S \in \mathscr{P}$ correspond we have $B=(S-I)(S+I)^{-1}$ and $S=(I+B)(I-B)^{-1}$, where $\mathscr{D}(B)=\mathscr{R}(S+I)$ and $\mathscr{D}(S)=\mathscr{R}(I-B)$. (It is simple to verify that properties (i) and (iii) implies that $(I-B)^{-1}$ exists.) Furthermore, S is closed if and only if $\mathscr{D}(B)$ is closed. Also, if $S_1 \leftrightarrow B_1$, $S_2 \leftrightarrow B_2$, then $S_1 \subset S_2$ if and only if $B_1 \subset B_2$. Finally, $S \in \mathscr{P}$ is self-adjoint if and only if the corresponding $B \in \mathscr{F}$ is in $\mathscr{B}(H)$, i.e. $\mathscr{D}(B)=H$. We refer to [16, Chapter VIII, Section 125] for verification of these facts.

So the problem of finding all positive self-adjoint extensions of a given positive densely defined operator S is transferred to the problem of finding all self-adjoint extensions $\tilde{B}$ in $\mathscr{B}(H)$ of $B=(S-I)(S+I)^{-1}$ such that $\|\tilde{B}\| \leqq 1$. We call such $\tilde{B}$'s for *admissible* extensions of B and denote the set of all these by $\mathscr{E}(B)$. The various positive self-adjoint extensions $\tilde{S}$ of S will then be rendered by the transformation $\tilde{S}=(I+\tilde{B})(I-\tilde{B})^{-1}$, where $\tilde{B}\in\mathscr{E}(B)$. Krein's proof of the existence of at least one element $\tilde{B}$ in $\mathscr{E}(B)$ is presented in [16, Chapter VIII, Section 125], thus giving an alternative proof to Friedrichs' of the existence of a positive self-adjoint extension of a positive operator.

To characterize the set $\mathscr{E}(B)$ we let $\tilde{B}$ be a specific element of $\mathscr{E}(B)$. Let $\tilde{\tilde{B}}\in\mathscr{B}(H)$ and write $\tilde{\tilde{B}}=\tilde{B}+C$. Then it is a simple observation that $\tilde{\tilde{B}}$ is in $\mathscr{E}(B)$ if and only if C is self-adjoint and

(1) $\mathscr{R}(C)\subset H\ominus\mathscr{D}(B)$.
(2) $-(I+\tilde{B})\leqq C\leqq I-\tilde{B}$.

FUNDAMENTAL LEMMA. *Let K be a closed subspace of H and let A be a positive operator in $\mathscr{B}(H)$. The set $\mathscr{I}$ of self-adjoint operators D in $\mathscr{B}(H)$ such that*

(a) $\mathscr{R}(D)\subset K$
(b) $D\leqq A$

contains a largest operator A_K, i.e. $D\in\mathscr{I}$ implies $D\leqq A_K$. Specifically, $A_K=A^{\frac{1}{2}}P_LA^{\frac{1}{2}}$, where P_L is the orthogonal projection onto the closed subspace $L=\{\xi\in H \mid A^{\frac{1}{2}}\xi\in K\}$.

PROOF. Set $K_1=H\ominus K$ and observe that L is equal to the orthogonal complement of $A^{\frac{1}{2}}(K_1)$. Hence we have, with $\xi\in H$:

$$(*)\qquad \begin{aligned}\langle A_K\xi,\xi\rangle &= \langle A^{\frac{1}{2}}P_LA^{\frac{1}{2}}\xi,\xi\rangle = \|P_LA^{\frac{1}{2}}\xi\|^2\\ &= \inf_{\xi_1\in K_1}\|A^{\frac{1}{2}}\xi-A^{\frac{1}{2}}\xi_1\|^2 = \inf_{\xi_1\in K_1}\langle A(\xi-\xi_1),\xi-\xi_1\rangle .\end{aligned}$$

If $D\in\mathscr{I}$, we have $\langle D\xi_1,\xi\rangle=\langle\xi_1,D\xi\rangle=0$ for $\xi\in H$, $\xi_1\in K_1$. Hence $D\xi_1=0$. Thus we have

$$\begin{aligned}\langle D\xi,\xi\rangle &= \inf_{\xi_1\in K_1}\langle D(\xi-\xi_1),\xi-\xi_1\rangle\\ &\leqq \inf_{\xi_1\in K_1}\langle A(\xi-\xi_1),\xi-\xi_1\rangle = \langle A_K\xi,\xi\rangle; \qquad \xi\in H .\end{aligned}$$

Hence $D\leqq A_K$. As clearly A_K is in $\mathscr{I}$ we are done.

REMARK. Observe that $A_K=0$ if and only if $K\cap\mathscr{R}(A^{\frac{1}{2}})=\{0\}$. In fact, $\mathscr{R}(A_K)\subset K\cap\mathscr{R}(A^{\frac{1}{2}})$ and so $K\cap\mathscr{R}(A^{\frac{1}{2}})=\{0\}$ implies $A_K=0$. Conversely, $A_K=0$ implies $A^{\frac{1}{2}}P_L=0$, which in turn implies that $K\cap\mathscr{R}(A^{\frac{1}{2}})=\{0\}$ by the definition of L.

Returning to the arguments preceding the lemma we get as a consequence of the Fundamental Lemma that a necessary condition for C to satisfy (1) and (2) is that

$$-(I+\tilde{B})_K \leqq C \leqq (I-\tilde{B})_K, \quad \text{where } K = H\ominus\mathscr{D}(B).$$

We define

$$B_m = \tilde{B}-(I+\tilde{B})_K, \quad B_M = \tilde{B}+(I-\tilde{B})_K.$$

Then B_m and B_M are in $\mathscr{E}(B)$, and it is easily verified that a necessary and sufficient condition for $\tilde{\tilde{B}}$ to be in $\mathscr{E}(B)$ is that $B_m \leqq \tilde{\tilde{B}} \leqq B_M$. So B has a unique admissible extension if and only if $B_m = B_M$.

From the remark to the Fundamental Lemma we conclude that $\tilde{\tilde{B}} = B_m$ if and only if

$$\{H\ominus\mathscr{D}(B)\} \cap \mathscr{R}((I+\tilde{B})^{\frac{1}{2}}) = \{0\}.$$

Correspondingly, $\tilde{\tilde{B}} = B_M$ if and only if $\{H\ominus\mathscr{D}(B)\} \cap \mathscr{R}((I-\tilde{B})^{\frac{1}{2}}) = \{0\}$.

Now let S_m and S_M be the positive self-adjoint extensions of S corresponding to B_m and B_M, respectively, i.e.

$$S_m = (I+B_m)(I-B_m)^{-1} \quad \text{and} \quad S_M = (I+B_M)(I-B_M)^{-1}.$$

Recall that $S=(I+\tilde{B})(I-\tilde{B})^{-1}$ and $\tilde{B}=(\tilde{S}-I)(\tilde{S}+I)^{-1}$. We get easily that

$$I-\tilde{B} = 2(\tilde{S}+I)^{-1} \quad \text{and} \quad I+\tilde{B} = 2\tilde{S}(\tilde{S}+I)^{-1}.$$

Hence $\mathscr{R}((I-\tilde{B})^{\frac{1}{2}}) = \mathscr{D}(\tilde{S}^{\frac{1}{2}})$ and $\mathscr{R}((I+\tilde{B})^{\frac{1}{2}}) = \mathscr{R}(\tilde{S}^{\frac{1}{2}})$. Then if the spectral resolution of $\tilde{S}$ is $\tilde{S}=\int_0^\infty \lambda E(d\lambda)$, a direct translation of the above relations yields the following theorem. (Recall that $\mathscr{D}(B)=\mathscr{R}(S+I)$).

THEOREM 4.1. *$\tilde{S}$ coincides with S_m if and only if*

$$\int_0^\infty \frac{1}{\lambda}\langle E(d\lambda)\xi,\xi\rangle = \infty \quad \textit{for all } \xi \neq 0 \textit{ orthogonal to } \mathscr{R}(S+I).$$

Likewise, $\tilde{S}$ coincides with S_M if and only if

$$\int_0^\infty \lambda\langle E(d\lambda)\xi,\xi\rangle = \infty \quad \textit{for all } \xi \neq 0 \textit{ orthogonal to } \mathscr{R}(S+I).$$

S has a unique positive self-adjoint extension $\tilde{S}$ if and only if

$$\int_0^\infty \lambda < E(d\lambda)\xi,\xi\rangle = \int_0^\infty \frac{1}{\lambda} E(d\lambda)\xi,\xi\rangle = \infty$$

$$\textit{for all } \xi \neq 0 \textit{ orthogonal to } \mathscr{R}(S+I).$$

We now want to show that S_M is the Friedrichs extension of S. Recall that $B_M = \tilde{B} + (I - \tilde{B})_K$, where $\tilde{B}$ is a specific admissible extension of $B = (S - I)(S + I)^{-1}$ and $K = H \ominus \mathscr{D}(B)$.

LEMMA 4.2. *$\tilde{B} = B_M$ if and only if $\mathscr{D}(B)$ $(= \mathscr{R}(S + I))$ is dense in H, where H is endowed with the seminorm $\|\xi\|_{\tilde{B}} = (\|\xi\|^2 - \langle \tilde{B}\xi, \xi\rangle)^{\frac{1}{2}}$, which we get from the inner-product $\langle \xi, \gamma\rangle_{\tilde{B}} = \langle \xi, \gamma\rangle - \langle \tilde{B}\xi, \gamma\rangle$; $\xi, \gamma \in H$.*

PROOF. We have that $\tilde{B} = B_M$ if and only if $(I - \tilde{B})_K = 0$. From (*) in the proof of the Fundamental Lemma we get, with ξ in H:

$$\langle (I - \tilde{B})_K \xi, \xi\rangle = \inf_{\xi_1 \in \mathscr{D}(B)} \langle (I - \tilde{B})(\xi - \xi_1), \xi - \xi_1\rangle = \inf_{\xi_1 \in \mathscr{D}(B)} \|\xi - \xi_1\|_{\tilde{B}}^2 .$$

Hence $(I - \tilde{B})_K = 0$ if and only if $\mathscr{D}(B)$ is dense in H with respect to $\|\cdot\|_{\tilde{B}}$.

PROPOSITION 4.3. *S_M is equal to the Friedrichs extension of S.*

PROOF. Recall that the Friedrichs extension S_F of S is characterized by being S^* restricted to $\mathscr{D}(S^*) \cap \mathscr{D}_0$, where

$$\mathscr{D}_0 = \{\xi \in H \mid \exists \{\xi_n\} \subset \mathscr{D}(S),\ \xi_n \to \xi \text{ and } \langle S(\xi_n - \xi_m), \xi_n - \xi_m\rangle \to 0 \text{ when } n, m \to \infty\}$$

(cf. [21, Chapter XII, Section 5]). From [16, Chapter VIII, Section 124] we also have that if S_1 is any symmetric extension of S such that $\mathscr{D}(S_1) \subset \mathscr{D}_0$, then $S_1 \subset S_F$. So to show that $S_M = S_F$ it is sufficient to establish that $\mathscr{D}(S_M) \subset \mathscr{D}_0$.

Now $\mathscr{D}(S_M) = \mathscr{R}(I - B_M)$ and so $\xi \in \mathscr{D}(S_M)$ is of the form $\xi = \gamma - B_M\gamma$ for $\gamma \in H$. Also, $S_M\xi = \gamma + B_M\gamma$. Consequently,

$$[[\xi]]^2 = \langle \xi, \xi\rangle + \langle S_M\xi, \xi\rangle = 2(\langle \gamma, \gamma\rangle - \langle B_M\gamma, \gamma\rangle) .$$

Notice that $\xi \in \mathscr{D}(S)$ if and only if $\gamma \in \mathscr{D}(B)$. By Lemma 4.2 we get that $\mathscr{D}(B) = \mathscr{R}(S + I)$ is dense in $\mathscr{D}(S_M)$, where $\mathscr{D}(S_M)$ is endowed with the seminorm $[[\cdot]]$. But this has the following consequence: Let $\xi \in \mathscr{D}(S_M)$, then $\exists \{\xi_n\} \subset \mathscr{D}(S)$ such that $[[\xi_n - \xi]] \to 0$ as $n \to \infty$, that is, $\xi_n \to \xi$ and

$$\langle S(\xi_n - \xi_m), \xi_n - \xi_m\rangle \to 0 \quad \text{as } n, m \to \infty .$$

So $\xi \in \mathscr{D}_0$ and hence $\mathscr{D}(S_M) \subset \mathscr{D}_0$. This concludes the proof.

REMARK. We can show that if S_1 is a positive self-adjoint operator, then S_1 is an extension of S if and only if $S_m \leqq S_1 \leqq S_M$, where we write $h \leqq k$ for two positive self-adjoint operators h and k if $\mathscr{D}(h^{\frac{1}{2}}) \supset \mathscr{D}(k^{\frac{1}{2}})$ and $\|h^{\frac{1}{2}}\xi\| \leqq \|k^{\frac{1}{2}}\xi\|$ for each ξ in $\mathscr{D}(k^{\frac{1}{2}})$ (cf. [10, p. 62]). As mentioned in the Introduction we call S_m

the Krein extension of S. Whereas $m(S_M)=m(S)$, one can show that $m(S_m)=0$, cf. (B) below. (See also [22, § 14 & § 15] and [23, pp. 330–333]).

If in the Fundamental Lemma we assume $K=\text{lin.span}\{\xi_0\}$, where ξ_0 is a vector in H, then a direct argument shows that

$$A_K\xi = \alpha\langle\xi,\xi_0\rangle\xi_0 \quad \text{for } \xi \in H, \quad \text{where } \alpha = \inf_{\xi\in H} \frac{\langle A\xi,\xi\rangle}{|\langle\xi,\xi_0\rangle|^2}.$$

On the other hand, by the characterization of A_K given in the Fundamental Lemma, we deduce that

$$\frac{1}{\alpha} = \int_0^a \frac{\|E(d\lambda)\xi_0\|^2}{\lambda},$$

where $A=\int_0^a \lambda E\,(d\lambda)$ is the spectral resolution of A. Using this and the previous characterization of S_m and S_M, Krein [7] proves by a lengthy argument the following criterion for when S has a unique positive self-adjoint extension. We state it as a theorem, giving no proof.

THEOREM 4.4. *The positive operator S has a unique positive self-adjoint extension if and only if*

$$\inf_{\xi\in\mathscr{D}(S)} \frac{\langle S\xi,\xi\rangle}{|\langle\xi,\gamma\rangle|^2} = 0$$

for every γ orthogonal to $\mathscr{R}(S+I)$.

REMARK. Assume $\langle S\xi,\xi\rangle=0$ for some $\xi\in\mathscr{D}(S)$. Then we get that $S\xi=0$. (This is seen by considering any positive self-adjoint extension $\tilde{S}$ of S. Then we get that $\tilde{S}^{\frac{1}{2}}\xi=0$ and so $(S\xi=)\,\tilde{S}\xi=0$). Now the orthogonal complement to $\mathscr{R}(S+I)$ is the null space of S^*+I. Hence we get that $\langle\xi,\gamma\rangle=0$ for γ orthogonal to $\mathscr{R}(S+I)$. So if the numerator of the fraction in the above theorem is 0, then so is the denominator. By convention we set the fraction equal to ∞ when this occurs.

We will now draw some consequences of the preceding discussion of Krein's method that we shall need to prove the theorems in Section 3. As before S is a positive (densely defined) operator in H.

(A) S is essentially self-adjoint, i.e. the closure $\bar{S}$ of S is self-adjoint, if and only if $\mathscr{R}(S+I)$ is dense in H. So if S is closed, then $S=S^*$ if and only if $\mathscr{R}(S+I)=H$.

(B) Assume $\bar{S}\neq S^*$. For S to have a unique positive self-adjoint extension it is necessary that $m(S)=0$, where $m(S)$ is the (lower) bound for S (cf. Section 2). Indeed, this follows easily from Theorem 4.4: If $\gamma \neq 0$ is orthogonal to $\mathscr{R}(S+I)$, then we get by the Schwartz inequality,

$$\inf_{\xi \in \mathscr{D}(S)} \frac{\langle S\xi, \xi\rangle}{|\langle \xi, \gamma\rangle|^2} \geqq \inf_{\xi \in \mathscr{D}(S)} \frac{\langle S\xi, \xi\rangle}{\langle \xi, \xi\rangle\langle \gamma, \gamma\rangle} = \frac{1}{\langle \gamma, \gamma\rangle} m(S) \;;$$

so by Theorem 4.4 we must have $m(S)=0$ for S to have a unique positive self-adjoint extension.

We remark that this particular fact is also a consequence of [1, Chapter VII, Section 83, Satz 3] by noting that if S has infinite deficiency indices, we first extend S to a positive operator $\tilde{S}$ such that $m(S)=m(\tilde{S})$ and $\tilde{S}$ has (non-zero) finite deficiency indices.

We stress that $m(S)=0$ is not a sufficient condition for S to have a unique positive self-adjoint extension. A simple example to that effect is given in [13, X. 3, p. 178], where even S has deficiency indices $(1,1)$.

(C) We are now going to use Theorem 4.1 to construct a closed positive operator S in H, where H is an infinite dimensional Hilbert space, which is not self-adjoint, but has a unique positive self-adjoint extension.

In fact, let T be an unbounded positive self-adjoint operator in H with spectral resolution $T=\int_0^\infty \lambda E\,(d\lambda)$, such that $0 \in \sigma(T)$, where $\sigma(T)$ denotes the spectrum of T. Then we can find a vector $\gamma \neq 0$ in H such that

$$(1) \qquad \int_0^\infty \lambda \|E(d\lambda)\gamma\|^2 = \infty \qquad \text{and} \qquad (2) \quad \int_0^\infty \frac{1}{\lambda} \|E(d\lambda)\gamma\|^2 = \infty \,.$$

Clearly (1) implies that $\int_0^\infty \lambda^2 \|E(d\lambda)\gamma\|^2=\infty$, and so $\gamma \notin \mathscr{D}(T)$. Now let

$$\mathscr{D}(S) = \{\xi \in \mathscr{D}(T) \mid \langle \xi + T\xi, \gamma\rangle = 0\}$$

and define $S\xi=T\xi$ for $\xi \in \mathscr{D}(S)$. It is easily verified that S is closed. We prove that $\mathscr{D}(S)$ is a *dense* linear manifold in H.

Indeed, assume to the contrary that $\xi_1 \in H$, $\xi_1 \neq 0$, such that $\langle \xi, \xi_1\rangle=0$ for all $\xi \in \mathscr{D}(S)$. Now $\xi_1=(T+I)\xi_2$ for some $\xi_2 \neq 0$ in $\mathscr{D}(T)$ (cf. (A)). So

$$0 = \langle \xi, (T+I)\xi_2\rangle = \langle \xi + T\xi, \xi_2\rangle \quad \text{for all } \xi \in \mathscr{D}(S)\,.$$

Consider the two linear functionals φ and ψ defined on $\mathscr{D}(T)$ by

$$\varphi(\xi) = \langle \xi + T\xi, \gamma\rangle, \qquad \psi(\xi) = \langle \xi + T\xi, \xi_2\rangle; \qquad \xi \in \mathscr{D}(T)\,.$$

Now obviously $\ker(\varphi)=\mathscr{D}(S) \subset \ker(\psi)$, and so $\psi=\alpha\varphi$ for some $\alpha \in \mathbf{C}$. This is easily seen to imply that $\xi_2=\bar{\alpha}\gamma$. Since $\xi_2 \neq 0$ we must have $\alpha \neq 0$, and so we get $\gamma=(1/\bar{\alpha})\xi_2 \in \mathscr{D}(T)$, which is a contradiction. So $\mathscr{D}(S)$ is dense in H.

We show next that the orthogonal complement of $\mathscr{R}(S+I)$ is the one-dimensional space spanned by γ. In fact, by the definition of $\mathscr{D}(S)$ we get first that γ is orthogonal to $\mathscr{R}(S+I)$, and so, in particular, S is not self-adjoint by (A). On the other hand, assume $\xi_3 \in H$ is orthogonal to $\mathscr{R}(S+I)$. Then

$$0 = \langle \xi + S\xi, \xi_3 \rangle = \langle \xi + T\xi, \xi_3 \rangle \quad \text{for } \xi \in \mathscr{D}(S) .$$

By the same argument as in the preceding paragraph we get that $\xi_3 = \beta\gamma$ for some $\beta \in \mathbf{C}$.

Thus we conclude by (1) and (2) that

$$\int_0^\infty \lambda \|E(d\lambda)\xi\|^2 = \int_0^\infty \frac{1}{\lambda} \|E(d\lambda)\xi\|^2 = \infty$$

for every $\xi \neq 0$ which is orthogonal to $\mathscr{R}(S+I)$. By Theorem 4.1 we get that S has a unique positive self-adjoint extension.

(D) We conclude this section by relating some of the preceding material to von Neumann algebras. Let S be a positive operator affiliated with a von Neumann algebra N acting on H. We have shown in Proposition 4.3 that S_M coincides with the Friedrichs extension of S, and thus S_M is affiliated with N (cf. [14, 9. Appendix]). Then $B_M = (S_M - I)(S_M + I)^{-1}$ is an operator in N, a fact which is readily verified. We have shown in our description of Krein's method that we have $B_m = B_M - (I + B_M)_K$, where K is the orthogonal complement of $\mathscr{R}(S+I)$. Since

$$(I + B_M)_K = (I + B_M)^{\frac{1}{2}} P_L (I + B_M)^{\frac{1}{2}} ,$$

where P_L is the orthogonal projection onto the subspace $L = \{\xi \in H \mid (I + B_M)^{\frac{1}{2}}\xi \in K\}$, it is easily verified that $B_m \in N$. As a consequence we get the following result which we state as a theorem.

THEOREM 4.5. *Let S be a positive operator affiliated with the von Neumann algebra N acting on the Hilbert space H. Then S_M and S_m are both affiliated with N. Thus S has a unique positive self-adjoint extension affiliated with N if and only if S has a unique positive self-adjoint extension $\tilde{S}$ in H (with no requirement that $\tilde{S}\ \eta\ N$).*

In particular, for S to have a unique positive self-adjoint extension affiliated with N it is necessary that $m(S) = 0$ (assuming $\bar{S} \neq S^*$).

5. Proofs.

PROOF OF PROPOSITION 3.3. Recall that (with ξ in $P^\sharp$) $\pi(\xi)$ denotes the closure of the map $\pi_0(\xi)$: $x'\xi_0 \to x'\xi$, $x' \in N'$. We know that $\pi(\xi)$ is positive and affiliated with N. Now

$$\mathscr{R}(\pi(\xi) + I) = [N'(\xi + \xi_0)] ,$$

and so by (A) in section 4 we get that $\xi \in P^{\#}_{sa}$ if and only if $[N'(\xi+\xi_0)]=H$, that is, $\xi+\xi_0$ is a separating vector for N. So we have proved (i).

(ii) is an immediate consequence of Theorem 4.4 and Theorem 4.5, by observing that $\pi(\xi)$ has a unique positive self-adjoint extension if and only if the same is the case for $\pi_0(\xi)$.

The proof of Theorem 3.1 will emerge after establishing some lemmas. The first lemma is Proposition 2.8 of [12], tailored for our purpose. However, one should also note the parallell with Lemma 3.4 of [5].

We retain the notation from section 3.

LEMMA 5.1. *Let N be a von Neumann algebra on the Hilbert space H with a cyclic and separating vector ξ_0. Let $\xi \in H$. Then the following two conditions are equivalent:*

(i) *$\xi = t\xi_0$ for some closed operator $t \,\eta\, N$.*
(ii) *There exists a vector ξ_1 in $P^{\#}_{sa}$ such that $\omega'_\xi=\omega'_{\xi_1}$.*

If these conditions are satisfied for $\xi \in H$, then the element ξ_1 in (ii) *is uniquely determined by ξ.*

PROOF. (i) $\Rightarrow$ (ii): We may assume that $t=\pi(\xi)$. Let $\pi(\xi)=uh$ be the polar decomposition of $\pi(\xi)$, where initial $(u)=\mathscr{R}(h)^-$ and final $(u)=\mathscr{R}(\pi(\xi))^-$. As $\xi \in \mathscr{R}(\pi(\xi))$ we have $uu^*(\xi)=\xi$. Now $\pi(u^*\xi)$ is the closure of the map

$$x'\xi_0 \to x'u^*\xi = u^*x'\xi = u^*uhx'\xi_0 = hx'\xi_0, \qquad x' \in N' .$$

Hence $\pi(u^*\xi)$ is the closure of h restricted to $N'\xi_0$. As $N'\xi_0$ is a core for $\pi(\xi)$, and so for h, we get that $\pi(u^*\xi)=h$. Hence $u^*\xi \in P^{\#}_{sa}$. Set $\xi_1=u^*\xi$. If $x' \in N'$, we have

$$\langle x'\xi_1,\xi_1\rangle = \langle x'uu^*\xi,\xi\rangle = \langle x'\xi,\xi\rangle .$$

Hence $\omega'_{\xi_1}=\omega'_\xi$.

(ii) $\Rightarrow$ (i): The condition stated in (ii) gives rise to a partial isometry u in N such that $u\xi_1=\xi$ and initial $(u)=[N'\xi_1]=\mathscr{R}(\pi(\xi_1))^-$. If $x' \in N'$ we have

$$x'\xi = x'u\xi_1 = ux'\xi_1 = u\pi(\xi_1)x'\xi_0 .$$

Since initial $(u)=\mathscr{R}(\pi(\xi_1))^-$, this implies that the map $x'\xi_0 \to x'\xi$, $x' \in N'$, is closable, i.e. $\pi(\xi)$ exists, and so (i) is established. Furthermore, $\pi(\xi)=u\pi(\xi_1)$.

Now let $\xi_1,\xi_2 \in P^{\#}_{sa}$ such that $(\langle x'\xi,\xi\rangle=)\ \langle x'\xi_1,\xi_1\rangle=\langle x'\xi_2,\xi_2\rangle$, $x' \in N'$. By the proof (ii) $\Rightarrow$ (i) above we get that $\pi(\xi_2)=u\pi(\xi_1)$ for some partial isometry $u \in N$ with initial $(u)=\mathscr{R}(\pi(\xi_1))^-$. Now both $\pi(\xi_1)$ and $\pi(\xi_2)$ are positive and self-adjoint. Hence

$$\pi(\xi_2)^2 = \pi(\xi_2)^*\pi(\xi_2) = \pi(\xi_1)u^*u\pi(\xi_1) = \pi(\xi_1)^2 .$$

By uniqueness of positive square root we get that $\pi(\xi_2)=\pi(\xi_1)$. Hence $\xi_2=\xi_1$. This completes the proof of the lemma.

LEMMA 5.2. *Let M be a von Neumann algebra on the Hilbert space H and let ξ, γ be two non-zero vectors in H. Assume $\gamma=t\xi$ for some closed operator $t\ \eta\ M$. Then there exists $\varphi' \neq 0$ in $(M'_*)_+$ such that $\varphi' \leqq \omega'_\xi$, $\varphi' \leqq \omega'_\gamma$.*

PROOF. Let $t=u|t|$ be the polar decomposition of t and let $|t|=\int_0^\infty \lambda E(d\lambda)$ be the spectral resolution of $|t|$. Set $|t|_n=\int_0^n \lambda E(d\lambda)$ and $t_n=u|t|_n$. Then $t_n \in M$ and $t_n\xi \to t\xi=\gamma$. If $x' \in M'$ we have,

$$(*) \qquad \langle x't_n\xi, t_n\xi\rangle = \langle x't_n^*t_n\xi, \xi\rangle = \langle x'|t|_n u^*u|t|_n\xi, \xi\rangle$$
$$= \langle x'|t|_n^2\xi, \xi\rangle$$

(Recall that initial $(u)=\mathscr{R}(|t|)^- \supset \mathscr{R}(|t|_n)^-$. By (*) we get that

$$\omega'_{t_n\xi} \leqq K_n\omega'_\xi ,$$

where K_n is a suitable positive constant. We also get from (*) that

$$\omega'_{t_n\xi}(x') \nearrow \omega'_\gamma(x') \qquad \text{for } x' \in M'_+ .$$

Hence for sufficiently large n we have that $\omega'_{t_n\xi} \neq 0$ and

$$\omega'_{t_n\xi} \leqq \omega'_\gamma, \qquad \omega'_{t_n\xi} \leqq K_n\omega'_\xi .$$

By multiplying $\omega'_{t_n\xi}$ with a suitable positive number we obtain φ' which satisfies the conditions of the lemma.

LEMMA 5.3. *Let φ be a faithful normal positive linear functional on $\mathscr{B}(K)$, where K is a Hilbert space of countable infinite dimension. Then there exists another faithful normal positive linear functional ψ on $\mathscr{B}(K)$ such that, if $0 \leqq \theta \leqq \varphi$, $\theta \leqq \psi$,* then $\theta=0$.

PROOF. Let Tr be the canonical trace on $\mathscr{B}(K)$. There exists a positive trace class operator h in $\mathscr{B}(K)$ such that $\varphi(x)=\mathrm{Tr}\,(hx)$, $x \in \mathscr{B}(K)$ [17, Theorem 1.15.3]. Since φ is faithful h is non-singular. Now $\mathscr{R}(h^{\frac12})$ is the domain of $h^{-\frac12}$, which is a (densely defined) closed unbounded operator in K. Hence there exists a unitary operator u in $\mathscr{B}(K)$ such that

$$\mathscr{R}(h^{\frac12}) \cap \mathscr{R}(u^*h^{\frac12}u) = \{0\} ,$$

cf. [9, Satz 18] (see also [4, Lemme 8.3] for a simpler proof and a more general result). Set $k=u^*hu$ and let $\psi(x)=\mathrm{Tr}\,(kx)$, $x \in \mathscr{B}(K)$. Then ψ is faithful since k is non-singular. Now assume $\theta \geqq 0$ is dominated by both φ and ψ. Let

$$\theta(x) = \mathrm{Tr}\,(lx), \qquad x \in \mathscr{B}(K),$$

where l is a positive trace class operator in $\mathscr{B}(K)$. Then $l \leqq h$ and $l \leqq k$. Hence $l^{\frac{1}{2}} = h^{\frac{1}{2}}a$, $l^{\frac{1}{2}} = k^{\frac{1}{2}}b$ for some $a, b \in \mathscr{B}(K)$ [3; I, § 1, Lemme 2]. So

$$\mathscr{R}(l^{\frac{1}{2}}) \subset \mathscr{R}(h^{\frac{1}{2}}) \cap \mathscr{R}(k^{\frac{1}{2}}) = \{0\}.$$

Hence $l=0$ and so $\theta=0$.

PROPOSITION 5.4. *Let N be a von Neumann algebra on the Hilbert space H with a cyclic and separating vector ξ_0. If the T-theorem holds for N with respect to ξ_0, that is, for every vector $\xi \in H$ there exists a closed operator $t \ \eta \ N$ such that $\xi_0 \in \mathscr{D}(t)$ and $\xi = t\xi_0$, then N is finite.*

PROOF. Assume to the contrary that N is not finite. Then we may decompose N by a central projection as a direct sum of its finite portion and its properly infinite portion. Thus, as is easily seen, we may assume that there is a properly infinite von Neumann algebra N with a cyclic and separating vector ξ_0 such that the T-theorem holds for N with respect to ξ_0. We show that this is impossible, thus proving the proposition by a reductio ad absurdum argument.

Now N' is also properly infinite since N has a cyclic and separating vector. By Proposition 2.2 we may identify N' with $\mathscr{B}(K)\otimes N'$ acting on $K\otimes H$, where K is a Hilbert space of countable infinite dimension. Let φ be ω'_{ξ_0} restricted to $\mathscr{B}(K)\otimes I_H$. By identifying $\mathscr{B}(K)$ and $\mathscr{B}(K)\otimes I_H$ in the obvious manner we then get a faithful normal positive linear functional φ on $\mathscr{B}(K)$. Choose ψ as in Lemma 5.3 and let τ' be any normal state on N'. Then $\psi\otimes\tau'$ is a non-zero normal positive linear functional on $\mathscr{B}(K)\otimes N'$ (cf. [3; I, § 4, Exercise 6]). We have $\psi\oplus\tau' = \omega'_\xi$ for some vector ξ in $K\otimes H$ [3; III, § 1, Théorème 4]. Now let θ' be a normal positive linear functional on $\mathscr{B}(K)\otimes N'$ such that $\theta' \leqq \omega'_{\xi_0}$, $\theta' \leqq \omega'_\xi$. Let θ denote the restriction of θ' to $\mathscr{B}(K)\otimes I_H$ (identified with $\mathscr{B}(K)$). Then $\theta \geqq 0$ and $\theta \leqq \varphi$, $\theta \leqq \psi$. Thus $\theta=0$ by the choice of ψ. This implies that $\theta'=0$, a fact that is readily verified. By Lemma 5.2 we get a contradiction to the assumption that $\xi = t\xi_0$ for some closed operator $t \ \eta \ N$. This completes the proof.

PROOF OF THEOREM 3.1. (1) $\Rightarrow$ (2) follows from Theorem 2.3.

(2) $\Rightarrow$ (1) is Proposition 5.4.

(2) $\Leftrightarrow$ (3) follows from Lemma 5.1 by noting that every $\varphi' \in (N'_*)_+$ is of the form ω'_ξ for some $\xi \in H$ (cf. [3; III, § 1, Théorème 4]). Besides we know that if $\gamma \in P^\#$, then $\gamma = h\xi_0$ for some closed operator $h \ \eta \ N$; for example, let $h = \pi(\gamma)$.

(4) $\Leftrightarrow$ (5) is an immediate consequence of Lemma 5.1.

(5) $\Rightarrow$ (6) is obvious.

(6) $\Rightarrow$ (5): Assume $P^\#_{\mathrm{sa}} \neq P^\#$ and let $\xi \in P^\# \setminus P^\#_{\mathrm{sa}}$. Hence $\pi(\xi)$ is a closed

positive operator which is not self-adjoint. Set $\xi_1 = \xi_0 + \xi$. Then $\xi_1 \in P^\#$ and it is easily verified that $\pi(\xi_1) = I + \pi(\xi)$. Clearly $\pi(\xi_1)$ is not self-adjoint and also $m(\pi(\xi_1)) \geqq 1$. By Theorem 4.5 we get that ξ_1 is not in $P^\#_{\mathrm{un}}$.

(1) $\Rightarrow$ (5) follows from Theorem 2.3.

To complete the proof of Theorem 3.1 we will prove that (5) $\Rightarrow$ (1). This is Lemma 5.5. However, before we prove Lemma 5.5 we make some preparatory observations that will be useful in the sequel.

Let M be a von Neumann algebra on the Hilbert space H and let e' be a projection in M' with central support equal to I. Then the induction map $x \to xe'$, $x \in M$, is an isomorphism of M onto Me'. This map also establishes a 1–1 correspondence between the set of closed operators affiliated with M and the set of closed operators affiliated with Me'. Specifically, if $t\ \eta\ M$ is a closed operator then te', i.e. t restricted to $e'(H) \cap \mathscr{D}(t) = e'(\mathscr{D}(t)) \subset e'(H)$, is a closed operator affiliated with Me'. If $t = v|t|$ is the polar decomposition of t and if $|t| = \int_0^\infty \lambda E(d\lambda)$ is the spectral resolution of $|t|$, then the polar decomposition of te' is $te' = (ve')(|t|e')$ and the spectral resolution of $|t|e'$ in $e'(H)$ is $|t|e' = \int_0^\infty \lambda F(d\lambda)$, where $F(\omega) = E(\omega)e'$ (ω Borel subset of $\mathbf{R}_+$). All this is readily verified.

We also observe that this correspondence preserves the adjoint operation, i.e. the adjoint of te' (acting in $e'(H)$) is t^*e'. Obviously symmetric, positive, and self-adjoint operators correspond to symmetric, positive, and self-adjoint operators, respectively, by the above map. Also, $t_1 \subset t_2$ if and only if $t_1e' \subset t_2e'$.

Now let L be another Hilbert space and let $M \otimes I_L$ be the amplification of M acting on $H \otimes L$. The amplification map $x \to x \otimes I_L$, $x \in M$, is an isomorphism between M and $M \otimes I_L$. Let $\{f_\nu\}_{\nu \in \Gamma}$ be an orthonormal basis for L with card $(\Gamma) = \dim(L)$. By the standard unsymmetric realization of $H \otimes L$ as the Hilbert sum of $\dim(L)$ copies of H, we may consider $M \otimes I_L$ as copies of operators in M "along the diagonal". The elements of $H \otimes L$ are represented as $\{\xi_\nu\}_{\nu \in \Gamma}$, where $\xi_\nu \in H$ ($\nu \in \Gamma$) and $\sum_{\nu \in \Gamma} \|\xi_\nu\|^2 < \infty$ (cf. [3; I, § 2, 3]). Let t be a closed operator in H, with $t\ \eta\ M$. We denote by $t \otimes I_L$ the operator in $H \otimes L$ whose domain of definition is

$$\{\{\xi_\nu\} \in H \otimes L \mid \xi_\nu \in \mathscr{D}(t), \text{ all } \nu, \text{ and } \{t\xi_\nu\} \in H \otimes L\}.$$

and

$$t \otimes I_L(\{\xi_\nu\}) = \{t\xi_\nu\} \quad \text{for } \{\xi_\nu\} \in \mathscr{D}(t \otimes I_L).$$

It is easily verified that $t \otimes I_L$ is a closed operator in $H \otimes L$ that is affiliated with $M \otimes I_L$. In fact, the map $t \to t \otimes I_L$ is a 1–1 correspondence between the set of closed operators affiliated with M and the set of closed operators affiliated with $M \otimes I_L$, having analogous properties as the map considered in the preceding paragraph. In particular, if $t = v|t|$ is the polar decomposition of t, then

$$t\otimes I_L = (v\otimes I_L)(|t|\otimes I_L)$$

is the polar decomposition of $t\otimes I_L$. Also, if $|t|=\int_0^\infty \lambda E(d\lambda)$ is the spectral resolution of $|t|$, then the spectral resolution of $|t|\otimes I_L$ is

$$|t|\otimes I_L = \int_0^\infty \lambda F(d\lambda),$$

where $F(\omega)=E(\omega)\otimes I_L$ (ω Borel subset of $\mathbf{R}_+$).

We also note at this stage, since we shall be needing this later on, that if $v=(s-iI)(s+iI)^{-1}$ is the Cayley transform of the closed symmetric operator s in H, then $v\otimes I_L$ is the Cayley transform of the closed symmetric operator $s\otimes I_L$ in $H\otimes L$.

From the preceding discussion it follows that if $\Phi\colon M\to N$ is an isomorphism between the von Neumann algebras M and N, then we can establish a 1–1 correspondence between the set of closed operators affiliated with M and the set of closed operators affiliated with N. In fact, Φ is the composition of an amplification map, an induction map, and a spatial isomorphism ([3; I, § 4, 4]). Specifically, let $t\ \eta\ M$ be a closed operator with polar decomposition $t=v|t|$, and let $|t|=\int_0^\infty \lambda E(d\lambda)$ be the spectral resolution of $|t|$. Then the corresponding operator affiliated with N, denoted by $\Phi(t)$, has polar decomposition $\Phi(t)=\Phi(v)\Phi(|t|)$, and the spectral resolution of $\Phi(|t|)$ is $\Phi(|t|)=\int_0^\infty \lambda F(d\lambda)$, where $F(\omega)=\Phi(E(\omega))$ (ω Borel subset of $\mathbf{R}_+$).

LEMMA 5.5. *Retaining the same notation as in Theorem* 3.1, *let us assume that* $P^\sharp_{sa}=P^\sharp$. *Then N is finite.*

PROOF. Assume to the contrary that N is not finite. Arguing similarly as in the proof of Proposition 5.4, we may assume that there is a properly infinite von Neumann algebra N having the property stated in the lemma. We will then reach a contradiction, thus proving the lemma.

By Proposition 2.2 we may identify N with $\mathscr{B}(K)\otimes N$, acting on $K\otimes H$, where K is a Hilbert space of countable infinite dimension. Let $e'=[(\mathscr{B}(K)\otimes I_H)\xi_0]$. Then e' is a projection in the commutant of $\mathscr{B}(K)\otimes I_H$ with central support equal to I (since ξ_0 clearly is separating for $\mathscr{B}(K)\otimes I_H$). So $(\mathscr{B}(K)\otimes I_H)e'$ is isomorphic to $\mathscr{B}(K)\otimes I_H$, and hence to $\mathscr{B}(K)$. Now ξ_0 is a cyclic and separating vector for $(\mathscr{B}(K)\otimes I_H)e'$, acting on $e'(K\otimes H)$. Let $Q^\sharp$ denote the cone $\{(\mathscr{B}(K)\otimes I_H)e'\}_+\xi_0^-$ in $e'(K\otimes H)$. Then there is a vector ξ in $Q^\sharp$ such that $\xi=h_1\xi_0$ for some positive self-adjoint and *unbounded* operator h_1 affiliated with $(\mathscr{B}(K)\otimes I_H)e'$, with $\xi_0\in\mathscr{D}(h_1)$. [In fact, if this is not the case, then by [20, Theorem 15.1] every normal positive linear functional on $M=(\mathscr{B}(K)\otimes I_H)e'$ is of the form $x\to\omega_{\xi_0}(k_1xk_1)$, $x\in M$, for some $k_1\in M_+$. Since

M is isomorphic to $\mathscr{B}(K)$, we transport ω_{ξ_0} to a faithful normal positive linear functional φ on $\mathscr{B}(K)$. Let $\varphi(x) = \mathrm{Tr}\,(tx)$ for a non-singular trace class operator t in $\mathscr{B}(K)_+$ ([17, Theorem 1.15.3]). Then every ψ in $(\mathscr{B}(K)_*)_+$ must be of the form

$$\psi(x) = \mathrm{Tr}\,(tkxk) = \mathrm{Tr}\,(ktkx), \qquad x \in \mathscr{B}(K)\,,$$

for some $k \in \mathscr{B}(K)_+$. Hence $\{ktk \mid k \in \mathscr{B}(K)_+\}$ is the set of all trace class operators in $\mathscr{B}(K)_+$. This, however, can easily be seen to contradict [11]].

By the discussion preceding this lemma we have that $h_1 = he'$ for some positive self-adjoint and unbounded operator h affiliated with $\mathscr{B}(K)\otimes I_H$. Observe that $\xi_0 \in \mathscr{D}(h)$. Also, by the preceding discussion, we have that $h = k\otimes I_H$, where k is a positive self-adjoint and unbounded operator in K. Let $\gamma \in K\setminus \mathscr{D}(k)$, $\gamma \neq 0$. Extend k to an operator $\tilde{k}$, whose domain of definition is $\mathscr{D}(k)+\mathbf{C}\gamma$, and where

$$\tilde{k}(\xi+\alpha\gamma) = k\xi+\alpha\gamma_0 \qquad \text{for } \xi \in \mathscr{D}(k),\ \alpha \in \mathbf{C}\,,$$

and where γ_0 is an arbitrary vector in K. Then $\tilde{k}$ is a closed operator ([19, p. 48]) which is a strict extension of k. Let $\tilde{k} = v|\tilde{k}|$ be the polar decomposition of $\tilde{k}$, and let s be the restriction of $|\tilde{k}|$ to $\mathscr{D}(k)$. Then it is easy to see that s is a closed positive operator such that $s \subsetneqq |\tilde{k}|$, hence s is not self-adjoint. Set $t = s\otimes I_H$. Then t is a closed positive operator affiliated with $\mathscr{B}(K)\otimes I_H$, hence affiliated with $\mathscr{B}(K)\otimes N$, which is not self-adjoint. Now $\mathscr{D}(t) = \mathscr{D}(h)$, since $\mathscr{D}(s) = \mathscr{D}(k)$ and $\|s\xi\| = \|k\xi\|$ for $\xi \in \mathscr{D}(k)$. So $\xi_0 \in \mathscr{D}(t)$. Set $\theta = t\xi_0$. Then θ is not in $P^{\#}_{\mathrm{sa}}$ since $\pi(\theta) \subset t$, and t is not self-adjoint. Clearly $\theta \in P^{\#}$ since t is positive, hence has a positive self-adjoint extension affiliated with $\mathscr{B}(K)\otimes N$.

This completes the proof of the lemma.

Proof of Theorem 3.2. (i) $\Rightarrow$ (ii) follows from Theorem 2.3.

(ii) $\Rightarrow$ (i): Assume to the contrary that N is not finite. Arguing similarly as in the proof of Proposition 5.4, we may assume that there is a properly infinite von Neumann algebra N satisfying condition (ii). We will then reach a contradiction, thus proving the theorem.

Since N has a cyclic and separating vector, we have that N' is also properly infinite and so by Proposition 2.2 N' is spatially isomorphic to $N'\otimes\mathscr{B}(K)$, acting on $H\otimes K$, where K is a Hilbert space of countable infinite dimension. So N is spatially isomorphic to $N\otimes I_K$, acting on $H\otimes K$. Again by Proposition 2.2 we get that N is spatially isomorphic to $(\mathscr{B}(K)\otimes N)\otimes I_K$, acting on $(K\otimes H)\otimes K$, and so we may identify N with $(\mathscr{B}(K)\otimes N)\otimes I_K$. (Henceforth we drop the parentheses since the tensor product is associative).

Let t_1 be a closed positive operator in K which is not self-adjoint, but has a unique positive self-adjoint extension. (Such operators exist as we established in (C) in Section 4). Let t be the closed, positive (non self-adjoint) operator $t_1 \otimes I_H$ in $K \otimes H$. By Theorem 4.5 we get that t has a unique positive self-adjoint extension. In fact, t is affiliated with $\mathscr{B}(K) \otimes I_H$, and every self-adjoint extension of t that is affiliated with $\mathscr{B}(K) \otimes I_H$ must be of the form $t_2 \otimes I_H$, where t_2 is a self-adjoint operator in K such that $t_1 \subset t_2$. Applying this argument again we get that the closed, positive (non self-adjoint) operator $s = t \otimes I_K = t_1 \otimes I_H \otimes I_K$ in $K \otimes H \otimes K$ has a unique positive self-adjoint extension. Since s is affiliated with $\mathscr{B}(K) \otimes I_H \otimes I_K$ we obviously have that s is affiliated with $\mathscr{B}(K) \otimes N \otimes I_K$.

In order to get a more suitable notation we set $M = \mathscr{B}(K) \otimes N$, acting on the Hilbert space $L = K \otimes H$. Note that M has a cyclic and separating vector. So we have that the operator $s = t \otimes I_K$ is affiliated with $M \otimes I_K$ and t is affiliated with M. To get the desired contradiction we would like to find a cyclic and separating vector ξ_0 for $M \otimes I_K$ such that $\xi_0 \in \mathscr{D}(s)$ and $s = \pi(\xi)$, where $\xi = s\xi_0$; in other words, $(M \otimes I_K)'\xi_0$ is a core for s. If we can establish this, we conclude namely that

$$\xi \in P^{\#}_{\mathrm{un}} \quad \text{and} \quad \xi \notin P^{\#}_{\mathrm{sa}}, \quad \text{where } P^{\#} = (M \otimes I_K)_+ \xi_0^- .$$

We are going to achieve this by choosing a slightly different s, as will emerge from the following.

Let $t = v|t|$ be the polar decomposition of t. Then the polar decomposition of s is

$$s = (v \otimes I_K)(|t| \otimes I_K),$$

with $\mathscr{D}(s) = \mathscr{D}(|t| \otimes I_K)$. Now s and $|t| \otimes I_K$ have the same cores (cf. Section 2). Let $\eta_0 \in L$ be a cyclic and separating vector for M and set

$$\eta_n = \frac{1}{n}(E[n-1, n\rangle)\eta_0 \quad \text{for } n = 1, 2, \ldots,$$

where $|t| = \int_0^\infty \lambda E(d\lambda)$ is the spectral resolution of $|t|$. Let θ_0 be the vector $\{\eta_n\}_{n=1,2,\ldots}$ in $L \otimes K$, where we identify $L \otimes K$ with the Hilbert sum of copies of L a countable number of times. It is easily verified that θ_0 is a separating vector for $M \otimes I_K$. We also observe that $\theta_0 \in \mathscr{D}(|t| \otimes I_K)$. We claim that $(M \otimes I_K)'\theta_0$ is a core for $|t| \otimes I_K$. In fact, $M'\eta_n$ is dense in the range of the projection $E[n-1, n\rangle$ for all n. Besides, $I_L \otimes \mathscr{B}(K)$, which is contained in $(M \otimes I_K)' = M' \otimes \mathscr{B}(K)$, will permute the summands in the Hilbert sum $\oplus \sum L_n \cong L \otimes K$ (where $L_n = L$ for all n). Since the linear span of the ranges of $E[n-1, n\rangle$ for $n = 1, 2, \ldots$ is a core for $|t|$, the claim follows.

So (by the remark above) we have found a separating vector θ_0 for $M \otimes I_K$ such that $(M \otimes I_K)'\theta_0$ is a core for s. Set $e' = [(M \otimes I_K)\theta_0]$. Then e' is a projection in $(M \otimes I_K)'$ whose central support is I. Hence $M \otimes I_K$ is isomorphic to $(M \otimes I_K)e'$ by the induction map. Clearly θ_0 is a cyclic and separating vector for the von Neumann algebra $(M\otimes I_K)e'$ acting on $e'(L\otimes K)$. Let se' be the restriction of s to $e'(L\otimes K)\cap\mathscr{D}(s)$. Then se', like s, is a closed, positive (non self-adjoint) operator which has a unique positive self-adjoint extension in $e'(L\otimes K)$. Besides, se' is affiliated with $(M\otimes I_K)e'$. (All this follows from the discussion preceding Lemma 5.5). It is also easily seen that $\{(M\otimes I_K)e'\}'\theta_0$ is a core for se'. Now $M\otimes I_K$ and $(M\otimes I_K)e'$ are *spatially* isomorphic since both have a cyclic and separating vector [3; III, § 1, Théorème 3]. Hence we can transport θ_0 and se', respectively, to a cyclic and separating vector ξ_0 for $M\otimes I_K$ and an operator s_0 affiliated with $M\otimes I_K$ with corresponding properties as se', respectively. As remarked above we have thus reached a contradiction to the assumption that $M\otimes I_K$ satisfies condition (ii), and so the proof of (ii) $\Rightarrow$ (i) is complete.

6. Concluding remarks.

(a) Let N be a von Neumann algebra acting on the Hilbert space H, and let ξ_0 be a cyclic and separating vector for N that we fix for the time being. Let ξ be a vector in H. Then the conjugate linear map $a\colon x'\xi_0\to x'^*\xi$, $x'\in N$, is closable. In fact, the (densely defined) conjugate linear map $b\colon x\xi_0\to x^*\xi$, $x\in N$, is easily seen to be contained in the adjoint of a, that is, $b\subset a^*$. (See [2, Section 2] where such maps are studied). However, as we have shown in Theorem 3.1, when N is not finite there exists a vector ξ in H such that the map $\pi_0(\xi)\colon x'\xi_0\to x'\xi$, $x'\in N'$, formally very similar to the map a above, is not closable. Let us denote by $\mathscr{F}$ the set of those ξ in H such that $\pi_0(\xi)$ is closable. Recall that $\mathscr{D}^\#$, which is the domain of definition of the closure S of the conjugate linear map $x\xi_0\to x^*\xi_0$, $x\in N$, is a subset of $\mathscr{F}$ [20, p. 19]. (One shows quite simply that $\mathscr{D}^\#=\mathscr{K}+i\mathscr{K}$, where $\mathscr{K}=\overline{N_h\xi_0}$; N_h denotes the hermitian elements in N. We also have that $\mathscr{D}^\#=\mathscr{D}(\Delta^{\frac12})$, where $S=J\Delta^{\frac12}$ is the polar decomposition of S).

Now $\mathscr{D}^\#$ may be different from $\mathscr{F}$. Indeed, we claim that $\mathscr{D}^\#=\mathscr{F}$ if and only if Δ is bounded, which in turn is equivalent to ω_{ξ_0} being minorized and majorized, respectively, by a positive multiple of a faithful trace on N. In particular, for this to occur N has to be finite.

To prove this claim we first observe that the linear span of $\mathscr{F}$ is H. In fact, if $\xi\in H$ then by the BT-theorem (cf. [18, Lemma 3.4, Corollary 1]) there exist a closed operator $t\;\eta\;N$, $\xi_0\in\mathscr{D}(t)$, and an operator $b\in N$ such that $\xi=bt\xi_0$. Now b is the linear sum of four unitary operators in N, and it follows easily from this that ξ is the linear sum of four elements in $\mathscr{F}$. Hence we have that $\mathscr{D}^\#=\mathscr{F}$ if and only if $\mathscr{D}^\#=H$. Now by the closed graph theorem the latter is equivalent to Δ being bounded, and so the claim is proved.

(b) As in (a) let N be a von Neumann algebra on H with a cyclic and separating vector ξ_0. Let $\mathscr{K} = N_h\xi_0^-$, where N_h denotes the hermitian part of N. We have that $\mathscr{K} = P^\# - P^\#$, where $P^\#$ as before denotes $N_+\xi_0^-$ (cf. [20, Lemma 15.2]).

Let $\xi \in \mathscr{K}$. Then $\pi(\xi)$ is easily seen to be a closed *symmetric* operator affiliated with N. Conversely, let $\xi \in H$ such that $\pi(\xi)$ is a symmetric operator. Then we claim that $\xi \in \mathscr{K}$. In fact, let $x' \in N_h'$. Then

$$\langle \xi, x'\xi_0 \rangle = \langle x'\xi, \xi_0 \rangle = \langle \pi(\xi)x'\xi_0, \xi_0 \rangle = \langle x'\xi_0, \pi(\xi)\xi_0 \rangle = \langle x'\xi_0, \xi \rangle .$$

By the characterization given of $\mathscr{K}$ in [15] we have that $\xi \in \mathscr{K}$.

We know that if $\xi \in P^\#$, then $\pi(\xi)$ can always be extended to a (positive) self-adjoint operator affiliated with N. However, this is not generally true if $\xi \in \mathscr{K}$. In fact, if N is not finite there exists a cyclic and separating vector ξ_0 for N and a vector $\xi \in \mathscr{K} = N_h\xi_0^-$, such that the symmetric operator $\pi(\xi)$ has no self-adjoint extension even in H.

To see this we argue similarly as in the proof of (ii) $\Rightarrow$ (i) of Theorem 3.2. We may identify N with $\mathscr{B}(K)\otimes N \otimes I_K$ acting on $K\otimes H\otimes K$, where K is a Hilbert space of countable infinite dimension. Let s_0 be a closed symmetric operator in K with deficiency indices $(0,1)$. Then the Cayley transform v_0 of s_0 has initial space K, hence v_0 is an isometry on K. Set $s = s_0 \otimes I_H \otimes I_K$. Then s is a closed symmetric operator affiliated with $\mathscr{B}(K)\otimes N\otimes I_K$, and the Cayley transform of s is $v = v_0 \otimes I_H \otimes I_K$, hence v is an isometry on $K\otimes H\otimes K$ and v is not a unitary operator. Hence s has no self-adjoint extension in $K\otimes H\otimes K$. The rest of the argument proceeds exactly as in the proof of (ii) $\Rightarrow$ (i) of Theorem 3.2.

We now want to give an example of a $\pi(\xi)$ for $\xi \in \mathscr{K}$, such that $\pi(\xi)$ has a self-adjoint extension in H but has no self-adjoint extension affiliated with N. In fact, let K be a Hilbert space of countable infite dimension. Let s be a closed symmetric operator in K with deficiency indices equal to $(1,2)$ and let v be the Cayley transform of s. Let $s_1 = s\otimes I_K$ in $H = K\otimes K$. Then s_1 is a closed symmetric operator with Cayley transform $v_1 = v\otimes I_K$, and it is easily seen that s_1 has deficiency indices (∞,∞). Hence s_1 has a self-adjoint extension in H. We also have that s_1 is affiliated with the von Neumann algebra $N = \mathscr{B}(K)\otimes I_K$. However, s_1 has no self-adjoint extension which is affiliated with N. In fact, if $\tilde{s}_1$ is a self-adjoint extension of s_1 such that $\tilde{s}_1 \,\eta\, N$, then the Cayley transform of $\tilde{s}_1$ would have to have the form $u\otimes I_K$, where u is a unitary operator in $\mathscr{B}(K)$ which is an extension of v. However, this is impossible.

One can show quite easily that there is a cyclic and separating vector ξ_0 for $N = \mathscr{B}(K)\otimes I_K$ such that $s_1 = \pi(\xi)$ for some $\xi \in \mathscr{K} = N_h\xi_0^-$.

In conclusion one can say that the situation for symmetric operators affiliated with a von Neumann algebra N is very different from the situation for positive operators affiliated with N.

(c) We retain the notation from (a). For α a real number between 0 and $\frac{1}{2}$ we consider the cone $V^\alpha = \Delta^\alpha(N_+\xi_0)^-$. Then V^0 is the cone $P^\sharp = N_+\xi_0^-$, and $V^{\frac{1}{2}}$ is the cone $P^\flat = N'_+\xi_0^-$ of [20]. For $\alpha = 1/4$ we have the cone $P^\natural$ of [2]. Araki [6, Theorem 4.1, p. 79] has proved that if $\varphi \in (N_*)_+$ is dominated by a positive multiple of ω_{ξ_0}, then for every $0 \leqq \alpha \leqq \frac{1}{2}$, there exists a $\xi \in V^\alpha$ such that $\varphi = \omega_\xi$. For $\alpha = 0$ ($\alpha = 1/4$) we have that for *any* $\varphi \in (N_*)_+$ there exists a unique vector $\xi \in V^0$ ($\xi \in V^{1/4}$) such that $\varphi = \omega_\xi$. It seems to be a reasonable conjecture that the corresponding result is true for any α between 0 and 1/4. In light of Theorem 3.1 (interchanging the roles of N and N') it seems reasonable to conjecture that if the same is true for some α such that $1/4 < \alpha < 1/2$, then N is finite, and conversely.

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REFERENCES

1. N. I. Achieser and I. M. Glasmann, *Theorie der linearen operatoren im Hilbert-raum*, Akademie-Verlag, Berlin, 1965.
2. A. Connes, *Caractérisation des espaces vectoriels ordonnés sous-jacent aux algèbres de von Neumann*, Ann. Inst. Fourier, Grenoble, 24 (1974), 121–155.
3. J. Dixmier, *Les algèbres d'opératuers dans l'espace hilbertien* (Cahier Scientifique 25), Gauthier–Villars, Paris, 2nd. ed., 1969.
4. J. Dixmier, *Étude sur les variétés et les opérateurs de Julia, avec quelques applications*, Bull. Soc. Math. France, 77 (1949), 11–101.
5. H. A. Dye, *The Radon–Nikodym theorem for finite rings of operators*, Trans. Amer. Math. Soc. 72 (1952), 243–280.
6. D. Kastler (editor), C*-*algebras and their applications to statistical mechanics and quantum field theory*, Proceedings of the International School of Physics "Enrico Fermi", North-Holland, Amsterdam, 1976.
7. M. G. Krein, *The theory of self-adjoint extensions of semi-bounded Hermitian operators and its applications* I, Mat. Sb. 20 (62), (1947), 431–495.
8. F. J. Murray and J. von Neumann, *On rings of operators*, Ann of Math. 37 (1936), 116–229.
9. J. von Neumann, *Zur theorie der unbeschränkten matrizen*, J. Reine Angew. Math. 161 (1929), 208–236.
10. G. K. Pedersen and M. Takesaki, *The Radon–Nikodym theorem for von Neumann algebras*, Acta Math. 130 (1973), 53–87.
11. G. K. Pedersen and M. Takesaki, *The operator equation $THT = K$*, Proc. Amer. Math. Soc. 36 (1972), 311–312.
12. F. Perdrizet, *Elements positifs relatifs à une algèbre Hilbertienne à gauche*, Compositio Math. 23 (1971), 25–47.
13. M. Reed and B. Simon, *Methods of modern mathematical physics*; II: *Fourier analysis, self-adjointness*, Academic Press, New York, 1975.
14. M. A. Rieffel, *Square-integrable representations of Hilbert algebras*, J. Functional Analysis 3 (1969), 265–300.

15. M. A. Rieffel and A. van Daele, *The commutation theorem for tensor products of von Neumann algebras*, Bull. London Math. Soc. 7 (1975), 257–260.
16. F. Riesz and B. Sz.-Nagy, *Functional Analysis*, Ungar, New York, 10 M (Translated from the 2nd French edition).
17. S. Sakai, *C*-algebras and W*-algebras* (Ergebnisse Math. 60), Springer-Verlag, Berlin - Heidelberg - New York, 1971.
18. C. S. Skau, *Finite subalgebras of a von Neumann algebra*, J. Functional Analysis, 25 (1977), 211–235.
19. M. H. Stone, *Linear transformations in Hilbert spaces and their applications to analysis*, Amer. Math. Soc. Colloquium Publications 15, New York, 1932.
20. M. Takesaki, *Tomita's theory of modular Hilbert algebras and its applications* (Lecture Notes in Math. 128), Springer-Verlag, Berlin - Heidelberg - New York, 1970.
21. N. Dunford and J. T. Schwartz, *Linear Operators* II (Interscience Tracts in Pure and Applied Mathematics 7) Interscience Publ., New York, 1963.
22. W. G. Faris, *Self-adjoint operators* (Lecture Notes in Math. 433), Springer-Verlag, Berlin - Heidelberg - New York, 1975.
23. T. Kato, *Perturbation theory for linear operators* (Grundlehren Math. Wissensch. 132), Springer-Verlag, Berlin - Heidelberg - New York, 1966.

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