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POSITIVE MAPS OF C*-ALGEBRAS COMMUTING WITH A REAL CONTINUOUS FUNCTION*

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Abstract.

We prove a conjecture of M. D. Choi to the effect that a positive unitial linear map of C*-algebras which commutes with a non-affine, real continuous function on an interval must be a Jordan homomorphism.

0. Conventions and terminology.

All of the C*-algebras $\mathscr{A}, \mathscr{B} \ldots$ considered have unit element $I_{\mathscr{A}}$ or, $I_{\mathscr{B}}, \ldots$, or just I. A map $\Phi\colon \mathscr{A} \to \mathscr{B}$ is *unital* if $\Phi(I_{\mathscr{A}}) = I_{\mathscr{B}}$, *positive* if $x \geq 0$ implies $\Phi(x) \geq 0$, *self-adjoint* if $\Phi(x^*) = \Phi(x)^*$ for all x in $\mathscr{A}$, 2-*positive* if $\Phi \otimes \mathrm{id}$ is positive on $\mathscr{A} \otimes M_2(\mathsf{C})$ to $\mathscr{B} \otimes M_2(\mathsf{C})$. A linear self-adjoint map $\Phi\colon \mathscr{A} \to \mathscr{B}$ is a *Jordan homomorphism* (or C*-*homomorphism*) if $\Phi(x^2) = \Phi(x)^2$ for $x = x^*$ in $\mathscr{A}$. An equivalent condition is that on each commutative subalgebra (or *-subalgebra of $\mathscr{A}$, the restriction of Φ is a *-homomorphism. The *absolute value* of $x \in \mathscr{A}$ is $(x^*x)^{\frac{1}{2}}$.

1.

In [2, Theorem 6], Kadison proved that if a (necessarily positive) unital linear map of C*-algebras preserves the absolute value on self-adjoint elements, it is a Jordan homomorphism.

In his paper [1, Theorem 2.5], M. D. Choi proves that if $\Phi\colon \mathscr{A} \to \mathscr{B}$ is a positive, unitial linear map of C*-algebras, and if f is a non-affine real, operator-convex function on the real interval $(-a, a)$ satisfying $\Phi(f(x)) = f(\Phi(x))$ for all self-adjoint x in $\mathscr{A}$ with spectrum contained in $(-a, a)$, then Φ is a Jordan homomorphism. Choi conjectures that "operator-convex" can be replaced by "continuous" in the hypothesis. Here we give a proof of Choi's conjecture.

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Our proof uses an elementary lemma on the geometry of the graph of a continuous, non-affine function, and the fact that f can be pre- and post-composed with affine functions while the identity $\Phi \circ f = f \circ \Phi$ persists on a suitable domain, to tailor an f which tames projections. Then we lift the argument to W*-algebras via the double-dual device due to Sherman [6], to get hold of the projections.

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2.

LEMMA. *Let f be a non-affine continuous real-valued function on the real interval (a, b). Then there is a chord of the graph of f which has non-zero slope and intersects the graph of f only at its endpoints.*

PROOF. Since f is not affine, it is not constant. Pick $p<q$ in (a, b) so that $f(p) \neq f(q)$, and call the corresponding chord PQ. If PQ coincides with the graph Γ of f on $[p, q]$, let $\mathscr{S}$ be a maximal line segment, $PQ \subset \mathscr{S} \subset \Gamma$. Then since $\mathscr{S} \neq \Gamma$, $\mathscr{S}$ has at least one endpoint, E, and F can be chosen in $\Gamma \setminus \mathscr{S}$ arbitrarily close to E so that with $M=\frac{1}{2}(P+Q)$, the chord MF does not contain the point E of Γ and has non-zero slope. Thus in any case we are assured that there exist a chord $\mathscr{C}$ of Γ of non-zero slope, and a point E of Γ above or below $\mathscr{C}$; we'll say that $\mathscr{C}$ straddles E. Fix such a pair $\mathscr{C}$ and E, and let $\mathscr{D}$ be the intersection of all subchords of $\mathscr{C}$ which straddle E. Then $\mathscr{D}$ is a chord of Γ of non-zero slope which intersects Γ only at the endpoints of $\mathscr{D}$.

THEOREM. *Let $\Phi \colon \mathscr{A} \to \mathscr{B}$ be a positive, unital linear map of C*-algebras, and suppose that f is a non-affine, real continuous function on the real open interval $\mathscr{J}$ such that for all self-adjoint x in $\mathscr{A}$ with spectrum $\sigma(x)$ contained in $\mathscr{J}$, $\Phi \circ f(x) = f \circ \Phi(x)$. Then Φ is a Jordan homomorphism.*

PROOF. It follows from the lemma that there exist real affine bijections l_1 and l_2 of R such that with $g = l_2 \circ f \circ l_1$ on $l_1^{-1}(\mathscr{J})$, g is defined on $\mathscr{J}' = (-\varepsilon, 1+\varepsilon)$ for some $\varepsilon > 0$, $g(0)=0$, $g(1)=1$, and $g(t)=t$ for no $t \in (0,1)$, while if $x = x^* \in \mathscr{A}$, with $\sigma(x) \subset \mathscr{J}'$, $\Phi \circ g(x) = g \circ \Phi(x)$.

Now replace g by h defined by

$$h(t) = \begin{cases} 0\;, & t \leqq 0\,, \\ g(t), & t \in [0,1]\,, \\ 1\;, & t \geqq 1\,. \end{cases}$$

Then h is bounded and continuous on all of R, $h(0)=0$, $h(1)=1$, $h(t)=t$ for no

$t \in (0,1)$, and if $x = x^* \in \mathscr{A}$ with $\sigma(x) \subset [0,1]$, $\Phi \circ h(x) = h \circ \Phi(x)$. For convenience, we will assume $\mathscr{A}$ is commutative, and prove that the existence of h as above forces Φ to be a *-homomorphism. We use Sherman's theorem to the effect that the second adjoint space $(\mathscr{A}^{\mathrm{d}})^{\mathrm{d}} = \mathscr{A}^{\mathrm{dd}}$ of $\mathscr{A}$ is a W*-algebra. We note that the second transpose map $\Phi^{\mathrm{dd}}: \mathscr{A}^{\mathrm{dd}} \to \mathscr{B}^{\mathrm{dd}}$ is strongly continuous: In fact, by [7, Theorem 4], Φ^{dd} is completely positive and unital, so an easy consequence of Stinespring's theorem [7], see e.g. [8, Theorem 3.1], assures us that

$$\Phi^{\mathrm{dd}}(x)^* \Phi^{\mathrm{dd}}(x) \leq \Phi^{\mathrm{dd}}(x^*x) \qquad \text{for all } x \in \mathscr{A}^{\mathrm{dd}};$$

then for η in the universal representation space of $\mathscr{B}$,

$$\begin{aligned} \|\Phi^{\mathrm{dd}}(x)\eta\|^2 &= \langle \Phi^{\mathrm{dd}}(x)^*\Phi^{\mathrm{dd}}(x)\eta, \eta\rangle = \omega_\eta(\Phi^{\mathrm{dd}}(x)^*\Phi^{\mathrm{dd}}(x)) \\ &\leqq \omega_\eta(\Phi^{\mathrm{dd}}(x^*x)) = (\Phi^{\mathrm{d}}(\omega_\eta))(x^*x) = \omega_\xi(x^*x) = \|x\xi\|^2 \end{aligned}$$

for some ξ in the universal representation space of $\mathscr{A}$. Since also Kaplansky's paper [3] shows us that the unit interval $\{x \in \mathscr{A} : 0 \leqq x \leqq I\}$ in $\mathscr{A}$ is strongly dense in that of $\mathscr{A}^{\mathrm{dd}}$, and that h is strongly continuous on self-adjoint elements of $\mathscr{A}^{\mathrm{dd}}$ (respectively $\mathscr{B}^{\mathrm{dd}}$), we conclude that for $x \in \mathscr{A}^{\mathrm{dd}}$ with $0 \leqq x \leqq I$,

$$\Phi^{\mathrm{dd}} \circ h(x) = h \circ \Phi^{\mathrm{dd}}(x).$$

This reduces the problem to the case of commutative W* domain $\mathscr{A}$, and Φ commuting with h on the unit interval of $\mathscr{A}$. To solve the reduced problem, we first note that Φ maps projections to projections. For if e is a projection in $\mathscr{A}$, $\Phi(h(e)) = \Phi(e) = h(\Phi(e))$, which forces $\sigma(\Phi(\mathrm{e})) \subset \{0,1\}$. Since $\Phi(e)$ is self-adjoint, it's a projection. Next, since for projections e, f in $\mathscr{A}$ (respectively $\mathscr{B}$) $ef = 0$ if and only if $e+f$ is a projection, Φ preserves disjointness of projections: $ef = 0 \Rightarrow \Phi(e)\Phi(f) = 0$. It then follows easily that for all projections e, f in $\mathscr{A}$, $\Phi(ef) = \Phi(e)\Phi(f)$, so that Φ preserves products of elements of the form $\sum_{j=1}^n \alpha_j e_j$, $\alpha_j \in \AA$, e_j pairwise orthogonal projections. Since by the spectral theorem such elements form a norm-dense subset of $\mathscr{A}$, Φ preserves products, which we wished to show. The theorem is proved.

2.

Among the consequences of the theorem, aside from the theorems of Kadison and Choi quoted in the introduction, we mention the following:

(1) Choi [1, Lemma 2.4] asserts that if $\Phi: \mathscr{A} \to \mathscr{B}$ is a positive linear unital map of C*-algebras such that $\Phi(A^{-1}) = \Phi(A)^{-1}$ for all positive invertible A in $\mathscr{A}$, then (and only then) Φ is a Jordan homomorphism. This follows directly from our theorem with $f(t) = t^{-1}$ for $t \in (0,1) = (a,b)$.

(2) If S is any line segment in the complex plane $\mathbf{C}$, f a continuous complex function on S which is not real-affine on S, and $\Phi: \mathscr{A} \to \mathscr{B}$ a positive unital linear map of C*-algebras which satisfies $\Phi \circ f(x) = f \circ \Phi(x)$ for all normal $x \in \mathscr{A}$ with $\sigma(x) \subset S$, then Φ is a Jordan homomorphism. To see this, we first apply a non-singular real affine transformation τ to $\mathbf{C}$ transforming S into $\mathbf{R}$, and observe that if $x \in \mathscr{A}$ is normal with $\sigma(x) \subset S$, then $x' = \tau(x) \in \mathscr{A}$ is normal with $\sigma(x') \subset \tau(S) = S' \subset \mathbf{R}$, so x' is self-adjoint, and conversely. Also, $f \circ \tau^{-1}$ is continuous, complex-valued, and not real-affine on S'; if f' is either the real or the imaginary part of $f \circ \tau^{-1}$ (choose one which is not real affine), then it's routine to check that $f' \circ \Phi(x') = \Phi \circ f'(x')$ for x' self-adjoint with $\sigma(x') \subset S'$. Now the theorem applies.

(3) If $\Phi: \mathscr{A} \to \mathscr{B}$ is a unital, linear, self-adjoint map of C*-algebras satisfying $\Phi \circ \exp = \exp \circ \Phi$ on skew-hermitian elements of $\mathscr{A}$, then Φ is a Jordan homomorphism. This follows from (3) as soon as we verify that Φ is positive, or equivalently, that Φ is contractive. Since Φ maps skew-hermitian elements to skew-hermitian elements, Φ maps $E_{\mathscr{A}} = \{\exp(ia) : a \in \mathscr{A}, a \text{ self-adjoint}\}$ into $E_{\mathscr{B}}$, a subset of the unit ball in $\mathscr{B}$. Now by Palmer's refinement [4], Lemma 1, of the main result of Russo and Dyc [5], the norm-closed convex hull of $E_{\mathscr{A}}$ is the unit ball of $\mathscr{A}$. So Φ is contractive. It is also easy to prove (4) directly.

(4) Finally, we remark that if the hypothesis of the theorem is strengthened by replacing "positive" by "2-positive", the conclusion is strengthened by replacing "Jordan homomorphism" by "*-homomorphism". This is an immediate corollary of the theorem together with Choi's Corollary 3.2, [1]. Of course, the same observation applies to remarks (1) and (3) above.

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