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THE MINIMAL DENSE TWO-SIDED IDEAL OF A C*-ALGEBRA WITH CONTINUOUS TRACE

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1. Introduction.

Let A be a C*-algebra with continuous trace, let $\hat{A}$ be the spectrum of A, and let J_A be the set of all $x \in A$ such that

$$\sup\{\dim \operatorname{Range} \pi(x) \ : \ \pi \in \hat{A}\} < \infty ,$$

and $\pi(x)=0$ for all π outside some compact subset of $\hat{A}$. In [2, 4.7.24, p. 100], Dixmier asked whether J_A is the minimal dense two-sided ideal of A. Pedersen and Petersen answered the question negatively in [12, Prop. 3.6, p. 202], and the authors showed in [3, Theorem 3.2] that the answer is negative even in case the continuous field of C*-algebras generated by A is trivial.

The purpose of this note is to give sufficient conditions on $\hat{A}$ in order that J_A will be the minimal dense two-sided ideal of A. The main result is the following:

THEOREM. *Let A be a C*-algebra with continuous trace. If each compact subset of the spectrum $\hat{A}$ has finite covering dimension, then J_A is the minimal dense two-sided ideal of A.*

This theorem applies to any continuous trace C*-algebra having finite dimensional spectrum because a closed subset of a finite dimensional space is finite-dimensional [7, C, p. 196]. The theorem also applies to a large class of C*-algebras having infinite-dimensional spectra, including those whose spectra are disjoint unions of open, compact finite-dimensional subspaces. The disjoint union of the complex projective space $P^1, P^2, P^3, \ldots$, is an instance of this theorem. This spectrum is countable-dimensional in the strong sense [7, Def. VI.4, p. 162]. The one-point compactification of this spectrum is an example of an infinite dimensional spectrum associated with a continuous trace C*-algebra for which the conclusion of the theorem fails [12, Prop. 3.6, p. 202], [3, Theorem 3.2].

In [5, 2, p. 168], Laursen and Sinclair show that there exists a minimal dense

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two-sided ideal in every C*-algebra A and this minimal dense ideal is the Pedersen ideal of A (the minimal dense two-sided *hereditary* ideal of A). Pedersen notes in [11, § 5, p. 12] that concrete descriptions of the minimal dense hereditary ideal are known for only a few non-commutative C*-algebras. Our result provides a large class of non-trivial examples of C*-algebras for which the Pedersen ideal is explicitly known.

For basic concepts and definitions we refer the reader to [2], [8], [9].

2. The minimal dense two-sided ideal.

Let T be a locally compact Hausdorff space, let $\mathscr{A}=(A(t),\Theta)$ be a continuous field of C*-algebras defined on T, and let A be the C*-algebra defined by $\mathscr{A}$. (The algebra A consists of all $x \in \Theta$ such that $\|x(t)\|$ vanishes at infinity on T.) For each subset E of T, let $\Theta \mid E$ denote the set of vector fields on E that are continuous with respect to Θ [2, 10.1.6, p. 188]. The continuous field of C*-algebras $((A(t))_{t\in E}, \Theta \mid E)$ will be denoted $\mathscr{A} \mid E$, and the C*-algebra defined by $\mathscr{A} \mid E$ will be denoted $A \mid E$. For any C*-algebra A, $K(A)=K_A$ will denote the minimal dense two-sided ideal of A; that is, the Pedersen ideal of A.

2.1. Proposition. *If $x \in A^+$ and x has compact support, then $x \in K_A^+$ if and only if, for each $t \in T$ there is a compact neighborhood V of t such that $x \mid V \in K_{A\mid V}^+$.*

Proof. The proof is a straightforward "partition of unity" type argument.

In 2.2 below, $\mathscr{H}=(H(t),\Gamma)$ will be a continuous field of Hilbert spaces on T, $\mathscr{A}=\mathscr{A}(\mathscr{H})$ will denote the continuous field of elementary C*-algebras generated by $\mathscr{H}$ [2, 10.7.2, p. 205], and, for each $e, f \in \Gamma$, $\theta(e,f)=\theta_{e,f} \in \mathscr{A}$ will be defined as in [2, 10.7.2, p. 205].

2.2. Lemma. *Let Ω be the set of all $e \in \Gamma$ having compact support, and let A be the C*-algebra defined by $\mathscr{A}(\mathscr{H})$. If $\mathscr{H}$ admits a continuous, non-vanishing vector field, then K_A^+ is the set of all elements in A of the form*

$$\theta(e_1,e_1)+\theta(e_2,e_2)+\ldots+\theta(e_n,e_n)\,, \tag{2.1}$$

where each e_i belongs to Ω.

Proof. The proof is a straightforward adaptation of the proof of Theorem 2.2 in [3] to the continuous field setting.

2.3. Lemma. *Suppose that A is a C*-algebra, and suppose that $|z| \in K_A$ for some $z \in A$. Then $z \in K_A$.*

PROOF. According to the polar decomposition theorem, there is a $u \in A''$ such that $z = u|z|$. By [1, 1.1.1, p. 166], $u|z|^{\frac{1}{2}} \in A$, and since $|z|^{\frac{1}{2}} \in K_A$, we conclude that $z = u|z|^{\frac{1}{2}}|z|^{\frac{1}{2}}$ must belong to K_A.

The proof of the next lemma follows the proof of [4, Prop. 5.4, p. 30].

2.4. LEMMA. *Suppose that S is a paracompact space having finite covering dimension*: $\dim S = n-1$. *Then each open covering $\mathscr{V}$ of S admits a locally refinement $\mathscr{U} = \mathscr{U}_1 \cup \ldots \cup \mathscr{U}_n$ which is an open covering of S such that the members of $\mathscr{U}_k$ are pairwise disjoint for $k = 1, \ldots, n$.*

PROOF. Let $\mathscr{V} = \{V_i\}_{i \in I}$ be an open covering of S. Since S is paracompact we may assume $\mathscr{V}$ is locally finite. Since $\dim S = n-1$, there is a locally finite refinement of $\mathscr{V}$ by an open covering of S having order at most n [7, Theorem II.6, p. 22]. (The order of a family $\{A_i\}$ is at most n provided that if F is any set of $n+1$ distinct indices, then $\bigcap \{A_i : i \in F\} = \emptyset$). Thus we may assume that the covering $\{V_i\}_{i \in I}$ is locally finite and has order at most n.

Let $\{\alpha_i\}_{i \in I}$ be a partition of unity on S such that support $\alpha_i \subset V_i$ for each $i \in I$. For each finite subset F of I, $F \neq I$, let $U(F)$ be the set of all $t \in S$ such that $\alpha_i(t) > \alpha_j(t)$ whenever $i \in F$ and $j \in I - F$. If $F = I$, let $U(F)$ be the set of all $t \in S$ such that $\alpha_i(t) > 0$ whenever $i \in F$. Since the supports of the α_i constitute a locally finite family, each $U(F)$ is an open subset of S. If U is an open set which meets the support of α_i only for the indices $i = i_1, \ldots, i_k$, and if U meets $U(F)$, then F is a subset of $i_1, \ldots, i_k$. Thus the family of all $U(F)$ is a locally finite family. Moreover, this family covers S.

Let $\mathscr{U}_m$ be the family of all $U(F)$ such that $|F| = m$. If F and F' are distinct sets of m indices, then $U(F)$ and $U(F')$ are disjoint. Because the family $\{V_i\}$ has order at most n, $U(F) = \emptyset$ whenever $|F| = m > n$. Thus $\mathscr{U}_1 \cup \ldots \cup \mathscr{U}_n$ is a covering of S. Since $U(F)$ is contained in the supports of the α_i for which $i \in F$, this covering is a refinement of $\mathscr{V}$. This completes the proof of the lemma.

2.5. LEMMA. *Suppose that $\mathscr{H} = (H(t), \Gamma)$ is a continuous field of Hilbert spaces on the locally compact Hausdorff space T, and let A be the* C*-*algebra defined by* $\mathscr{A}(\mathscr{H})$. *If T has finite covering dimension*, $\dim T = n-1$, *then for each $z \in J_A^+$ there exist $z_1, \ldots, z_n \in J_A^+$ and $e_1, \ldots, e_n \in \Gamma$ satisfying the following conditions*:

(2.6) $z = z_1 + \ldots + z_n$.

(2.7) *For each k and each t there is a scalar $\alpha \geqq 0$ such that $z_k(t) = \alpha z(t)$.*

(2.8) *Each e_k has compact support, and for each $t \in T$, $\|z_k(t)\| = \|e_k(t)\|$ and $e_k(t) \in$* Range $z_k(t)$.

PROOF. Let $S=\{t \in T : z(t) \neq 0\}$. Note that $\mathrm{Cl}_T S$ is compact, and since S is σ-compact, S is paracompact. Since $\dim E \leqq n-1$ for any closed subset E of T [7, C, p. 196], S admits a countable closed covering by sets of dimension at most $n-1$. It follows that $\dim S \leqq n-1$ [7, Cor., p. 195].

Given $t_0 \in S$, choose $u_0 \in \mathrm{Range}\, z(t_0)$ with $\|u_0\|=1$. There is a neighborhood U of t_0 with $\mathrm{Cl}_T U \subset S$ on which $z(t)u(t)$ is never zero where $u(t_0)=u_0$. Choose a neighborhood V of t_0 such that $\mathrm{Cl}_T V \subset U$, and let α be a continuous function on T which is 1 on $\mathrm{Cl}_T V$, and 0 off U. Then the vector field e defined by

$$e(t) = \begin{cases} \alpha(t)\|z(t)u(t)\|^{-1}z(t)u(t), & \text{when } t \in U\,, \\ 0, & \text{otherwise}\,, \end{cases}$$

belongs to Γ and has compact support (see [2, 10.1.9, p. 188] and [2, 10.7.3, p. 205]). Furthermore, $e(t) \in \mathrm{Range}\, z(t)$ for all $t \in T$, and $\|e(t)\|=1$ for all $t \in \mathrm{Cl}_T V$. In this way one can construct an open covering $\mathscr{V}=\{V_i\}_{i\in I}$ of S and a family $\{e_i\}_{i\in I}$ of members of Γ having compact support such that for each $i \in I$ the following conditions are satisfied:

$$\mathrm{Cl}_T V_i \subset S;$$

$$e_i(t) \in \mathrm{Range}\, z(t) \quad \text{for all } t \in T;$$

$$\|e_i(t)\| = 1 \quad \text{for all } t \in V_i\,.$$

Let $\mathscr{U}=\mathscr{U}_1 \cup \ldots \cup \mathscr{U}_n$ be a locally finite open covering of S which refines $\mathscr{V}$ and for which the mebers of $\mathscr{U}_k$ are pairwise disjoint (Lemma 2.4). Letting $\mathscr{U}_k=\{U_{k,j} : j \in I(k)\}$, there is a partition of unity $\{\alpha_{k,j}\}$ on S with the support of $\alpha_{k,j}$ contained in $U_{k,j}$. Since $\mathrm{Cl}_T U_{k,j} \subset S$, we may assume that $\alpha_{k,j}$ is defined and continuous on all of T. Let z_k be defined as follows:

$$z_k(t) = \begin{cases} \alpha_{k,j}(t)z(t), & \text{if } t \in U_{k,j} \text{ for some } j \in I(k); \\ 0, & \text{otherwise}\,. \end{cases}$$

Then z_k belongs to A since $\mathscr{U}$ is locally finite, $\alpha_{k,j}$ tends to zero at the boundary of $U_{k,j}$, and z tends to zero at the boundary of S.

From the definition of z_k it is clear that (2.7) is satisfied, and (2.6) follows from the fact that the $\{\alpha_{k,j}\}$ form a partition of unity. To define e_k satisfying (2.8), choose $i(k,j) \in I$ such that $U_{k,j} \subset V_{i(k,j)}$ for each k, j and let e_k be given by the following formula:

$$e_k(t) = \begin{cases} \|z_k(t)\|e_{i(k,j)}(t), & \text{if } t \in U_{k,j} \text{ for some } j \in I(k); \\ 0, & \text{otherwise}\,. \end{cases}$$

Since z_k tends to zero at the boundary of each $U_{k,j}$ and at the boundary of S, e_k belongs to Γ. The support of e_k is contained in $\mathrm{Cl}_T S$ and is therefore compact. If $j \in I(k)$ and $t \in U_{k,j}$, then $e_{i(k,j)} \in \mathrm{Range}\, z(t)$; and if in addition $z_k(t) \neq 0$, then $\mathrm{Range}\, z_k(t)=\mathrm{Range}\, z(t)$. Consequently (2.8) is satisfied.

2.6. LEMMA. *Suppose that* $\mathscr{H}=(H(t),\Gamma)$ *is a continuous field of Hilbert spaces on the locally compact Hausdorff space T, and let A be the* C**-algebra defined by* $\mathscr{A}(\mathscr{H})$. *If* $\beta z\in K_A^+$, *where* $z\in A^+$, $\beta\in C_c(T)^+$ *and the support of z is contained in the support of* β, *then* $z\in K_A^+$.

PROOF. The hereditary C*-algebra B generated by βz contained in K_A [6, 3.3, p. 8]. Since $\beta(t)z(t)\neq 0$ if $z(t)\neq 0$, it is clear that z is in the $\sigma(A'',A')$ closure of B taken in the bidual A''. Hence $z\in B$.

2.7. THEOREM. *Suppose that* $\mathscr{H}=(H(t),\Gamma)$ *is a continuous field of Hilbert spaces on the locally compact Hausdorff space T. Suppose also that* $\mathscr{H}$ *admits a continuous, non-vanishing vector field, and T has finite covering dimension. If A is the* C**-algebra defined by* $\mathscr{A}(\mathscr{H})$, *then* $K_A=J_A$.

PROOF. Since J_A is dense, we have $K_A\subset J_A$. (See [2, 10.4.4, p. 196]). By Lemma 2.3, it will suffice to show that $J_A^+\subset K_A^+$. (Alternately, one can observe that $J_A=\operatorname{Span} J_A^+$). Let $J(m)^+$ consists of all z in J_A^+ such that

$$\dim \operatorname{Range} z(t) \leqq m$$

for each $t\in T$. We will show by induction that $J(m)^+\subset K_A$ for each $m=1,2,\dots$. It will be convenient to let $\dim T=n-1$.

Suppose $z\in J(1)^+$, and let z_k, e_k satisfy (2.6)–(2.8) of Lemma 2.5 ($k=1,2,\dots,n$). Clearly for each k and each $t\in T$.

$$\|z_k(t)\|^2 z_k(t) = z_k(t)\theta(e_k(t),e_k(t)) . \tag{2.9}$$

According to Lemma 2.2 the right-hand side of (2.9) belongs to K_A, and so by Lemma 2.6, z_k belongs to K_A. Now by (2.6), $z\in K_A$. This completes the proof that $J(1)^+\subset K_A$.

Next suppose that $J(m-1)^+\subset K_A$. Fix $z\in J(m)^+$, and let z_k,e_k satisfy (2.6)–(2.8) of Lemma 2.5. For a given $k=1,\dots,n$, define y by the formula

$$y(t) = \|z_k(t)\|^2 z_k(t)-z_k(t)\theta(e_k(t),e_k(t)), \qquad (t\in T) .$$

For a fixed $t\in T$, we have by (2.7) of Lemma 2.5, $z_k(t)=\alpha z(t)$, where $\alpha\geqq 0$. Let P be the projection of $H(t)$ onto Range $z(t)$, and let

$$w(t) = [\|z_k(t)\|^2 P-\theta(e_k(t),e_k(t))] .$$

From (2.8) of Lemma 2.5 we see that $e_k(t)$ is in the kernel of $w(t)$ (recalling that $e_k(t)$ is in the range of both $\theta(e_k(t),e_k(t))$ and P). Clearly $w(t)$ is self-adjoint, and therefore $e_k(t)$ is orthogonal to $\operatorname{Range} w(t)$. Since $\operatorname{Range} w(t)\subset \operatorname{Range} z(t)$, $\dim\operatorname{Range} w(t)\leqq m-1$. Since $y(t)=z_k(t)w(t)$, $\dim\operatorname{Range} y(t)\leqq m-1$, and therefore $\dim\operatorname{Range}|y|(t)\leqq m-1$. But t was an arbitrary point of T, so by the inductive hypothesis, $|y| \in K_A$. Hence $y \in K_A$ by Lemma 2.3. Since $z_k\theta(e_k, e_k) \in K_A$ by Lemma 2.2, it follows that $\|z_k(\cdot)\|^2 z_k \in K_A$, and by Lemma 2.6, $z_k \in K_A$. Consequently $z \in K_A$ by (2.6) of Lemma 2.5 and the proof is complete.

2.8. COROLLARY. *Let A be a* C**-algebra with continuous trace. If each compact subset of the spectrum $\hat{A}$ has finite covering dimension, then J_A is the minimal dense two-sided ideal of A.*

PROOF. The proof will follow from a straightforward application of 2.1, 2.7 and the results in [2, Chap. 10].

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