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M-IDEALS OF COMPACT OPERATORS IN CLASSICAL BANACH SPACES

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Abstract.

Let $K(X,X)$ be the space of compact operators on an infinite dimensional Banach space X. It is known that $K(X,X)$ is an M-ideal (in the space of all bounded operators on X) when $X=L_2(\mu)$ for some measure μ. We prove that:

1) If $X=L_1(\mu)$, then $K(X,X)$ is not an M-ideal.
2) If $X^*=L_1(\mu)$ then $K(X,X)$ is an M-ideal iff $X=c_0(\Gamma)$.
3) If $1<p<\infty$, $p\neq 2$, and $X=L_p(\mu)$, then $K(X,X)$ is an M-ideal iff μ is purely atomic.

Introduction.

The object of this paper is to investigate when $K(X, Y)$, the space of compact operators from X to Y, is an M-ideal in $L(X, Y)$, the space of all bounded operators from X to Y.

When Alfsen and Effros [1] introduced the notion of an M-ideal, they knew that in the self-adjoint part of a C*-algebra, the M-ideals coincide with the self-adjoint parts of the closed two-sided ideals. Later Smith and Ward [12] proved that the M-ideals in a C*-algebra are exactly the closed two-sided ideals. In particular, the compact operators on a Hilbert space is an M-ideal in the space of all bounded operators.

Hennefeld [2] and Saatkamp [11] have proved that $K(l_p, l_q)$ are M-ideals when $1<p\leqq q<\infty$ and several authors have observed that $K(X,c_0)$ is an M-ideal for all Banach spaces X [6] [11] [12]. Note that if $1\leqq q<p<\infty$, then $K(l_p, l_q)=L(l_p, l_q)$ [8].

It is known that $K(l_1,l_1)$ and $K(l_\infty,l_\infty)$ are not M-ideals [12] and also in some other cases involving L_p-spaces and preduals of L_1-spaces $K(X, Y)$ is not an M-ideal [9] [11].

The paper consists of two parts. In the first part, we show that the $X=c_0(\Gamma)$ are the only Lindenstrauss spaces such that $K(X,X)$ is an M-ideal. In the second part, we prove some theorems saying that if X and Y have some properties (this is specified later), then $K(X, Y)$ is not an M-ideal.

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When we say that $K(X, Y)$ is an M-ideal, we mean that $K(X, Y)$ is an M-ideal in $L(X, Y)$. A Banach space X is called a Lindenstrauss-space if its dual X^* is isometric to an $L_1(\mu)$-space. The unit ball in X is denoted X_1 and the closed ball in X with center x and radius r is denoted $B(x, r)$.

A closed subspace J of a Banach space A is called an L_p-summand ($l \leqq p < \infty$) if there exists a projection P in A such that $P(A) = J$ and for all $x \in A$ we have

$$\|x\|^p = \|Px\|^p + \|x - Px\|^p .$$

A closed subspace J of A is called an M-summand if J is the range of a projection P in A such for all $x \in A$ we have

$$\|x\| = \max(\|Px\|, \|x - Px\|) .$$

A closed subspace J of A is called an M-ideal if its annihilator J° in A^* is an L_1-summand. Alfsen and Effros [1] characterized M-ideals by intersection properties of balls. In [5] we showed that a closed subspace J of A is an M-ideal iff for all $x \in A_1$, for all $y_1, y_2, y_3 \in J_1$ and for all $\varepsilon > 0$, there exists

$$(*) \qquad y \in J \cap \bigcap_{i=1}^{3} B(x + y_i, 1 + \varepsilon) .$$

If (*) holds when $y_1 = y_2 = -y_3$, then we say that J is a semi M-ideal. We have that J is a semi M-ideal iff for all $x \in A^*$, there exists a unique $y \in J^\circ$ such that $\|x - y\| = d(x, J^\circ)$ and moreover this unique y satisfies $\|x\| = \|y\| + \|x - y\|$. [5; Theorem 6.15 and Theorem 5.6.]

The set of extreme points of a convex set C is denoted $\partial_e C$.

We say that a point $e \in A$ is an order unit for A if $\|e\| = 1$ and

$$\max(\|x + e\|, \|x - e\|) = \|x\| + 1$$

for all $x \in A$. This definition is equivalent to the usual definition of order unit [7; Theorem 4.7]. (See also [6]).

A maximal proper face F of A_1 is called a base if $A_1 = \text{co}(F \cup -F)$. If F is a base for A, then the functional on A which is 1 on F is an order unit for A^*. If e is an order unit in A^* and

$$F = \{x \in A : \|x\| = 1 = e(x)\} ,$$

then $A_1 = \overline{\text{co}}(F \cup -F)$ [6]. (The bar means closure and co means convex hull.)

We consider only the real case, but most of the results are easily extended to the complex case.

1. Characterization of $c_0(\Gamma)$.

Note that if X or X^* is an $L_1(\mu)$-space, then e is an order unit for X^* for all $e \in \partial_e X_1^*$. We will use this property in the first lemma.

LEMMA 1. *Suppose e is an order unit for X^* for all $e \in \partial_e X_1^*$. If $K(X,X)$ is a semi M-ideal in $L(X,X)$, then X is isometric to a subspace of $c_0(\Gamma)$ for some set Γ. If X is separable, then we can take Γ to be countable.*

PROOF. Note that if $f, g \in \partial_e X_1^*$ with $f \neq g$, then $\|f-g\| = 2$ [6; Theorem 2.2]. Choose $e \in \partial_e X_1^*$ and $\varepsilon > 0$ and let $x_1 \in X$ with $\|x_1\| = 1$. Define

$$N = \{f \in \partial_e X_1^* : |f(x_1)| \geqq 2\varepsilon\} .$$

We want to show that N is finite.

Define $S \in K(X,X)$ by $S(x) = e(x)x_1$. Then $\|S\| = 1$. Since $K(X,X)$ is a semi M-ideal there exists an operator

$$U \in K(X,X) \cap B(I-S, 1+\varepsilon) \cap B(I+S, 1+\varepsilon) .$$

Thus

$$\|S+(I-U)\| \leqq 1+\varepsilon, \quad \|S-(I-U)\| \leqq 1+\varepsilon .$$

If $f \in N$, then $S^*f = f(x_1)e$. Hence

$$\begin{aligned} 1+\varepsilon &\geqq \max(\|f(x_1)e+(f-U^*f)\|, \|f(x_1)e-(f-U^*f)\|) \\ &= |f(x_1)| + \|f-U^*f\| \\ &\geqq 2\varepsilon + \|f-U^*f\| \end{aligned}$$

and $1-\varepsilon \geqq \|f-U^*f\|$.

But then we get for $f, g \in N$ with $f \neq g$,

$$\|U^*f-U^*g\| \geqq \|f-g\| - \|f-U^*f\| - \|g-U^*g\| \geqq 2\varepsilon .$$

Since U^* is compact, we get that N is finite, Now we can take $\Gamma = \partial_e X_1^*$. By considering x_1 as a function on $\partial_e X_1^*$, we get $x_1 \in c_0(\Gamma)$. If X is separable, we can take as Γ a countable ω^*-dense subset of $\partial_e X_1^*$.

THEOREM 2. *Suppose X is a Lindenstrauss space. Then the following statements are equivalent.*

1) *X is isometric to $c_0(\Gamma)$ for some set Γ.*
2) *$K(Y,X)$ is an M-ideal in $L(Y,X)$ for all Banach spaces Y.*
3) *$K(X,X)$ is an M-ideal in $L(X,X)$.*
4) *$K(X,X)$ is a semi M-ideal in $L(X,X)$.*

PROOF. 1) $\Rightarrow$ 2) is proved in [6], [11] and [12].

2) $\Rightarrow$ 3) $\Rightarrow$ 4) is trivial.

4) $\Rightarrow$ 1). Let F be a proper maximal face of X_1^* and let $\Gamma = \partial_e X_1^* \cap F$. By the lemma above, we get $X \subseteqq c_0(\Gamma)$ by the natural map. We have $X_1^* = \mathrm{co}\,(F \cup -F)$, so $\partial_e X_1^* = (\Gamma \cup -\Gamma)$ [5]. Since X necessarily is polyhedral, we get $X^* = l_1(\Gamma) = c_0(\Gamma)^*$ [4] [10]. It follows from the Hahn–Banach theorem that $X = c_0(\Gamma)$.

The same method of proof as used to prove the theorem above can be used to prove the following result.

THEOREM 3. *Suppose X is a Lindenstrauss space and assume X is canonically imbedded into X^{**}. Then the following statements are equivalent.*

1) *X is isometric to $c_0(\Gamma)$ for some set Γ.*
2) *X is an* M*-ideal in X^{**}.*
3) *X is a semi* M*-ideal in X^{**}.*

PROOF. 1) $\Rightarrow$ 2). We have that for each finite set $A \subseteqq \Gamma$,

$$l_\infty^A = \{(x(\gamma)) \in l_\infty(\Gamma) : x(\gamma) = 0 \text{ if } \gamma \notin A\}$$

is an M-summand in $X^{**} = l_\infty(\Gamma)$. Hence

$$c_0(\Gamma) = \overline{\bigcup l_\infty^A}$$

(the union taken over all finite subsets A of Γ) is an M-ideal in X^{**} by [5; Proposition 6.20].

2) $\Rightarrow$ 3) is trivial.

3) $\Rightarrow$ 1). Choose $x \in X$ with $\|x\| = 1$ and let $\varepsilon > 0$. First we want to show that

$$N = \{e \in \partial_e X_1^* : |e(x)| \geqq \varepsilon\}$$

is finite. Choose $y \in \partial_e X_1^{**}$. Then $|y(e)| = 1$ for all $e \in \partial_e X_1^*$ [6; Theorem 2.2]. Use the balls $B(y+x, 1)$ and $B(y-x, 1)$ and proceed as in Lemma 1 to show that N is finite. Then argue as in the proof of 4) $\Rightarrow$ 1) in the proof of Theorem 2, and it follows that X is isometric to a $c_0(\Gamma)$ space.

Although it is well known that $K(l_p, l_q)$ is an M-ideal when $1 < p \leqq q < \infty$, we would like to give a simple proof of this using the characterization (*).

THEOREM 4. $K(l_p, l_q)$, $K(l_p, c_0)$ and $K(c_0, c_0)$ are M-ideals when $1 < p \leqq q < \infty$.

PROOF. We write out the details only in the case $K(l_p, l_q)$ with $1 < p \leqq q < \infty$. Let $S_1, S_2, S_3 \in K(l_p, l_q)$ with $\|S_i\| \leqq 1$ and let $T \in L(l_p, l_q)$ with $\|T\| \leqq 1$. Since we have an $\varepsilon > 0$ at our disposal in the formula (*), we may suppose $S_i = Q_m S_i P_n$ for $i = 1, 2, 3$ and some m and n where Q_m and P_n are the projections

$$P_n((x_k)) = Q_n((x_k)) = (x_1, \ldots, x_n, 0, \ldots) .$$

Let $U = Q_m T + T P_n - Q_m T P_n \in K(l_p, l_q)$. Then $T - U = (I - Q_m) T (I - P_n)$. Let $x \in l_p$ with $\|x\| = 1$, and let $y = P_n x$ and $z = (I - P_n)x$. Then

$$1 = \|x\|^p = \|y\|^p + \|z\|^p$$

and

$$\|y\| = \|P_n x\| \geqq \|Q_m S_i P_n x\| \qquad (i = 1, 2, 3)$$

$$\|z\| = \|(I - P_n)x\| \geqq \|(I - Q_m) T (I - P_n)x\| .$$

Hence, since $S_i = Q_m S_i P_n$ for $i = 1, 2, 3$,

$$\begin{aligned} 1 &= \|y\|^p + \|z\|^p \\ &\geqq \|y\|^q + \|z\|^q \\ &\geqq \|Q_m S_i P_n x\|^q + \|(I - Q_m) T (I - P_n) x\|^q \\ &= \|Q_m S_i P_n x + (I - Q_m) T (I - P_n) x\|^q \\ &= \|S_i x + (T - U) x\|^q . \end{aligned}$$

This shows that

$$U \in \bigcap_{i=1}^{3} B(T + S_i, 1) .$$

2. Conditions which ensure that $K(X, Y)$ is not an M-ideal.

The following theorem is an easy consequence of (*) and of [7; Theorem 6.1] and [13]. (We consider here only infinite dimensional spaces.)

THEOREM 5. *Let Y be a Banach space. $K(l_\infty(\Gamma), Y^*)$ is an* M-*ideal for all sets Γ if and only if $K(X, Y^*)$ is an* M-*ideal for all Lindenstrauss spaces X. If $K(X, Y^*)$ is an* M-*ideal for some infinite dimensional Lindenstrauss space X, then $K(c_0, Y^*)$ is an* M-*ideal.*

The theorem remains true if we read semi M-ideals instead of M-ideals. Since $K(c_0, l_\infty)$ is not a semi M-ideal [11], we get that $K(X, l_\infty)$ is not a semi M-ideal for any infinite dimensional Lindenstrauss space X.

It also follows from (*) that if X and Y are 1-complemented in M and N and $K(M, N)$ is an (semi) M-ideal, then $K(X, Y)$ is an (semi) M-ideal.

Since $K(l_1, l_p)$ $(1 \leqq p < \infty)$ and $K(l_p, l_\infty)$ $(1 < p < \infty)$ are not semi M-ideals [11], we get that $K(L_1(\mu), L_p(\nu))$ $(1 \leqq p < \infty)$ and $K(L_p(\nu), l_\infty)$ $(1 < p < \infty)$ are not semi M-ideals in the infinite dimensional cases. [3; Theorem 3].

As we will show now these results are special cases of more general results. Note that all maximal proper faces of the unit balls of $L_1(\mu)$-spaces and Lindenstrauss spaces are bases [5; Corollary 3.6]. These spaces also have the property that every extreme point in the dual unit balls is an order unit for the dual space.

THEOREM 6. *Suppose X^* is an order unit space with order unit f and suppose Y is an order unit space with order unit e. If $K(X, Y)$ is a semi* M-*ideal in $L(X, Y)$, then $K(X, Y) = L(X, Y)$.*

PROOF. Let $F = \{x \in X : \|x\| = 1 = f(x)\}$. Then $\overline{\text{co}}\,(F \cup -F) = X_1$ [6]. Hence, the compact operator S defined by $S(x) = f(x)e$, has norm 1. Suppose $T \in L(X, Y)$ with $\|T\| = 1$ and let $\varepsilon > 0$. Then there exists

$$U \in K(X, Y) \cap B(T+S, 1+\varepsilon) \cap B(T-S, 1+\varepsilon)\,.$$

We get

$$\max \|S \pm (T-U)\| \leqq 1+\varepsilon\,.$$

Let

$$G = \{y^* \in Y^* : \|y^*\| = 1 = y^*(e)\}\,.$$

Then $Y_1^* = \text{co}\,(G \cup -G)$. If $y^* \in \partial_e G$, then $S^*y^* = y^*(e)f = f$. Hence we get for $y^* \in \partial_e G$

$$\begin{aligned} 1+\varepsilon &\geqq \max_{\pm} \|f \pm (T^*y^* - U^*y^*)\| \\ &= 1 + \|(T^* - U^*)y^*\|\,. \end{aligned}$$

Thus

$$\|T-U\| \leqq \varepsilon\,.$$

Since $U \in K(X, Y)$ and $\varepsilon > 0$ is arbitrary, we get $T \in K(X, Y)$.

THEOREM 7. *Suppose X^* is an order unit space with order unit f. Let Y be a Banach space that contains a proper L_p-summand for some $1 \leqq p < \infty$. If $K(X, Y)$ is a semi* M-*ideal in $L(X, Y)$, then $K(X, Y) = L(X, Y)$.*

PROOF. Write $Y = E \oplus_p F$. For simplicity, assume $p = 1$. Let $T \in L(X, Y)$ with

$\|T\|=1$, and let P be the L_p-projection in Y with range E. Choose $x_1 \in E$ with $\|x_1\|=1$ and let $\varepsilon>0$. Define $S \in K(X,Y)$ by $S(x)=f(x)x_1$. Then there exists

$$U \in K(X,Y) \cap B(T+S, 1+\varepsilon) \cap B(T-S, 1+\varepsilon) .$$

Hence

$$\max \|S \pm (T-U)\| \leqq 1+\varepsilon .$$

For $x \in X$ with $\|x\| \leqq 1$, we get

$$\begin{aligned} 1+\varepsilon &\geqq \|f(x)x_1-(T-U)x\| \\ &= \|f(x)x_1-P(T-U)x-(I-P)(T-U)x\| \\ &= \|f(x)x_1-P(T-U)x+(I-P)(T-U)x\| \end{aligned}$$

and we also have

$$1+\varepsilon \geqq \|f(x)x_1+P(T-U)+(I-P)(T-U)x\| .$$

Hence we get when $x \in H=\{x \in X : \|x\|=1=f(x)\}$

$$\begin{aligned} 2(1+\varepsilon) &\geqq \|f(x)x_1-P(T-U)x+(I-P)(T-U)x\| \\ &\quad + \|f(x)x_1+P(T-U)x+(I-P)(T-U)x\| \\ &\geqq 2\|f(x)x_1+(I-P)(T-U)x\| \\ &= 2\|x_1\|+2\|(I-P)(T-U)x\| . \end{aligned}$$

This together with $X_1=\overline{\mathrm{co}}\,(H \cup -H)$ yields

$$\varepsilon \geqq \|(I-P)(T-U)\| = \|(I-P)T-(I-P)U\| .$$

Thus $(I-P)T \in K(X,Y)$. Similarly, we get $PT \in K(X,Y)$ by choosing $x_1 \in F$, so $T \in K(X,Y)$.

A proof similar to the proof of Theorem 7 shows that we also have the following result.

THEOREM 8. *Assume X contains a proper* M-*ideal or a proper L_p-summand for some $1<p<\infty$, and assume Y is an order unit space. If $K(X,Y)$ is a semi* M-*ideal in $L(X,Y)$, then $K(X,Y)=L(X,Y)$.*

A bounded subset A of a Banach space Y is said to be *dentable* if for all $\varepsilon>0$, there exists $t>0$ and $f \in Y^*$ with $\|f\|=1$ such that the slice

$$S(f,t) = \left\{x \in A : f(x)>\sup_{a \in A} f(a)-t\right\}$$

has diameter less than ε. In reflexive spaces and separable dual spaces, all bounded sets are dentable. [14].

THEOREM 9. *Assume X^* has an order unit e and assume Y_1 is dentable. If $K(X, Y)$ is a semi* M*-ideal, then $K(X, Y)=L(X, Y)$.*

PROOF. Assume $K(X, Y)$ is a semi M-ideal and let $T \in L(X, Y)$ with $\|T\|=1$.
Assume for contradiction that $d(T, K(X, Y)) > \varepsilon$ and $\varepsilon > 0$. Let $S(f, t)$ be a slice of Y_1 with $\operatorname{diam} S(f, t) < \varepsilon$. By the Bishop–Phelps theorem, we may assume $\|f\|=1=f(y)$ for some $y \in S(f, t) \subseteq Y_1$ [14]. Define $S \in K(X, Y)$ by $S(x)=e(x)y$. Choose $0<\delta<1$ such that $(1-\delta)(1+\delta)^{-1} > 1-t$. Since $K(X, Y)$ is a semi M-ideal, there exists $U \in K(X, Y)$ such that for both $\pm$:

$$\|S \pm (T-U)\| \leqq 1+\delta .$$

Since $\|T-U\| > \varepsilon$, there exists $x \in X_1$ with $e(x)=1$ such that $\|(T-U)x\| > \varepsilon$. Then $S(x)=y$. Let $z=(T-U)x$. Then

$$\max \|y \pm z\| \leqq 1+\delta$$

such that

$$1+\delta > f(y \pm z) = 1 \pm f(z)$$

and

$$|f(z)| \leqq \delta .$$

But then

$$f\left(\frac{y \pm z}{1+\delta}\right) \geqq \frac{1-\delta}{1+\delta} > 1-t$$

so $(y \pm z)(1+\delta)^{-1} \in S(f, t)$. $\operatorname{diam} S(f, t) < \varepsilon$ implies that

$$2\varepsilon < 2\|z\| = \|(y+z)-(y-z)\| < \varepsilon(1+\delta)$$

such that $\delta > 1$. This is a contradiction. Hence $K(X, Y)=L(X, Y)$.

THEOREM 10. *If l_1 is isomorphic to a subspace of X, then $K(X, X)$ is not a semi* M*-ideal.*

PROOF. Assume for contradiction that l_1 is isomorphic to a subspace of X and that $K(X, X)$ is a semi M-ideal. Let $\varepsilon > 0$. Then there exists a linear operator $T\colon l_1 \to X$ such that for all $x \in l_1$

$$\|x\| \leqq \|Tx\| \leqq \|x\|(1+\varepsilon)$$

[15; Proposition 2.e.3]. In order to avoid technical complications, we will assume $\|x\| = \|Tx\|$ for all $x \in l_1$. Let $Y = T(l_1)$. We will identify Y with l_1. Let $e = (1, 1, \ldots) \in \partial_e Y_1^*$ and let $\tilde{e}$ be a normpreserving extension of e to X. Let $x_0 = (1, 0, \ldots) \in \partial_e Y_1 \subseteqq X_1$. Define $S \in K(X, X)$ by

$$S(x) = \tilde{e}(x) x_0 .$$

Then $S^*(f) = f(x_0)\tilde{e}$. For each $g \in \partial_e Y_1^*$, let $\tilde{g}$ be a norm-preserving extension to X.

Suppose $U \in K(X, X)$ is such that

$$U \in B(I+S, 1+\varepsilon) \cap B(I-S, 1+\varepsilon) .$$

Then $\|S \pm (I-U)\| \leqq 1+\varepsilon$, and if $g \in \partial_e Y_1^*$, then

$$\begin{aligned} 1+\varepsilon &\geqq \max \|S^*(\tilde{g}) \pm (\tilde{g} - U^*\tilde{g})\| \\ &= \max \|\tilde{g}(x_0)\tilde{e} \pm (\tilde{g} - U^*\tilde{g})\| \\ &= \max \|\tilde{e} \pm (\tilde{g} - U^*\tilde{g})\| \\ &\geqq \max \|e \pm (g - U^*\tilde{g}|_Y)\| \\ &= 1 + \|g - U^*\tilde{g}|_Y\| . \end{aligned}$$

Therefore

$$\|g - U^*\tilde{g}|_Y\| \leqq \varepsilon .$$

But then if $g_1, g_2 \in \partial_e Y_1^*$ and $g_1 \neq g_2$, then

$$\begin{aligned} &\|U^*\tilde{g}_1 - U^*\tilde{g}_2\| \\ &\geqq \|U^*\tilde{g}_1|_Y - U^*\tilde{g}_2|_Y\| \\ &\geqq \|g_1 - g_2\| - \|g_1 - U^*\tilde{g}_1|_Y\| - \|g_2 - U^*\tilde{g}_2|_Y\| \\ &\geqq 2 - 2\varepsilon . \end{aligned}$$

If now $\varepsilon < \frac{1}{2}$, then this clearly contradicts that U^* is compact.

We conclude with the following theorem.

THEOREM 11. *Let* $1 < p < \infty$ *and* $p \neq 2$. *Let* $X = L_p(\mu)$ *for some measure* μ. *The following statements are equivalent*:

1) μ *is purely atomic.*
2) $K(X, X)$ *is an* M-*ideal in* $L(X, X)$.
3) $K(X, X)$ *is a semi* M-*ideal in* $L(X, X)$.

PROOF. 1) ⇒ 2) is proved in Theorem 4.

2) ⇒ 3) is trivial.

3) ⇒ 1). Assume μ is not purely atomic. We will show that $K(X,X)$ is not a semi M-ideal. Then it is well known that $L_p(0,1)$ is 1-complemented in $L_p(\mu)$. Hence it follows from (*) that it is enough to show that $K(X,X)$ is not a semi M-ideal when $X=L_p(0,1)$. We can also assume $p>2$.

Let $(\chi_n)_{n=1}^\infty$ be the Haar basis in $L_p(0,1)$, and let $P_n(\sum_{k=1}^\infty a_k\chi_k)=\sum_{k=1}^n a_k\chi_k$ be the natural projections. $(\chi_n)_{n=1}^\infty$ is a monotone basis so $\|P_n\|=1$ for all n.

Let $1>\varepsilon>0$. Since $L_p(0,1)$ is uniformly convex, there exists $\varepsilon \geqq \delta>0$ such that if $\|x\|\leqq 1$, $\|y\|\leqq 1$ and $\|x-y\|>\varepsilon$, then $\|x+y\|\leqq 2(1-\delta)$ [8].

Assume for contradiction that $K(X,X)$ is a semi M-ideal. Then there exists $U \in K(X,X)$ such that

$$\|I-U+P_1\| < 1+\frac{\delta}{2}, \qquad \|I-U-P_1\| < 1+\frac{\delta}{2}.$$

Since $\|\chi_1\|=1$, we get

$$\|U\chi_1\| < 1+\frac{\delta}{2}, \qquad \|2\chi_1-U\chi_1\| < 1+\frac{\delta}{2}.$$

If $2\|\chi_1-U\chi_1\| \geqq \varepsilon(1+\frac{1}{2}\delta)$, then by the uniform convexity, we get

$$\left(1+\frac{\delta}{2}\right)2(1-\delta) \geqq \|(2\chi_1-U\chi_1)+U\chi_1\| = 2.$$

This is a contradiction, hence

$$\|\chi_1-U\chi_1\| < \frac{1}{2}\varepsilon\left(1+\frac{\delta}{2}\right) < \varepsilon.$$

Hence we may assume $U\chi_1=\chi_1$ and

$$\|I-U-P_1\| < 1+2\varepsilon.$$

We define a sequence $(y_k)_{k=1}^\infty$ in $L_p(0,1)$ by

$$y_k = \sum (\chi_{2^{k-1}+j}+\chi_{2^k+2j-1})$$

where j runs from 1 to 2^{k-1}. Let μ be the Lebesgue measure on $(0,1)$. Then we have for all k

$$\mu(\{x : y_k(x)=-1\}) = \tfrac{1}{2}$$

and

$$\mu(\{x : y_k(x)=2\}) = \tfrac{1}{4} = \mu(\{x : y_k(x)=0\}).$$

Define $K>0$ by $K^p=\frac{1}{4}2^p+\frac{1}{2}3^p$. We have $\|-2\chi_1+y_k\|=K$ for all k. Since $p>2$, we have $\frac{1}{2}4^p+1>3^p$. Hence there exists $c>0$ such that for all k

$$\|2\chi_1+y_k\| = (\tfrac{1}{4}4^p+\tfrac{1}{4}2^p+\tfrac{1}{2})^{1/p} = K+c\,.$$

$$\begin{aligned}&(1+2\varepsilon)\|-2\chi_1+y_k\|\\ \geqq\;& \|(I-U-P_1)(-2\chi_1+y_k)\|\\ \geqq\;& \|2\chi_1+y_k\|-\|Uy_k\|\\ =\;& c+\|-2\chi_1+y_k\|-\|Uy_k\|\end{aligned}$$

such that

$$c \leqq \|Uy_k\|+2\varepsilon K\,.$$

Choosing ε small enough, we get $\|Uy_k\|\geqq\frac{1}{2}c$. This is impossible since U is compact, so $K(X,X)$ can not be a semi M-summand.

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