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ENLARGING A SUBSPACE OF $C(X)$ WITHOUT CHANGING THE CHOQUET BOUNDARY

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1.

Let X be a compact metric space, let B be a uniformly closed subspace of $C(X)$, the Banach space of all continuous real-valued functions on X, and suppose that B separates the points of X and contains the constant functions.

For each $x \in X$ let ν_x be a maximal measure on X representing x with respect to B. We are going to study conditions on the subspace B and the selection $x \to \nu_x$ of maximal representing measures which ensure that the space

$$A = \left\{ f \in C(X) : \int f \, d\nu_x = f(x) \text{ all } x \in X \right\}$$

is a simplicial space i.e. the state space for A is a simplex. In [3] it is shown that it is always possible to find a selection $x \to \nu_x$ which is measurable but this is in general not a sufficient condition. The case when the Choquet boundary $\partial_B X$ for B is closed is treated in [2]. There it is shown that if $\partial_B X$ is closed, then the selection $x \to \nu_x$ is continuous if and only if A is simplicial.

In this note conditions on B and on the selection $x \to \nu_x$ are given which imply that A is a simplicial space. These conditions involve separation of maximal representing measures and upper semi-continuity of the selection $x \to \nu_x$ with respeet to the cone of all finite suprema of functions from B.

2.

Let X and B be as above. By $M(X)$ we denote the space of all regular Borel measures on X, by $\partial_B X$ the Choquet boundary for B and by $M(\partial_B X)$ the set of all boundary (or maximal) measures on X; since X is a metric space a measure μ is in $M(\partial_B X)$ if and only if $|\mu|(X \setminus \partial_B X)=0$ ([1, (4.11), p. 35 and II. §2]).

A measurable selection of maximal representing measures is a map $x \to \nu_x$ of X into $M(\partial_B X)$ such that ν_x is a probability measure in $M(\partial_B X)$ for which

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$$b(x) = \int b \, d\nu_x \quad \text{for each } b \in B$$

and such that for each $f \in C(X)$ the function

$$x \to \int f \, d\nu_x$$

is Borel measurable on X. Such a selection gives rise to a map T from $M(X)$ into $M(\partial_B X)$ defined by (cf. [3])

$$\int f \, dT\mu = \int \left(\int f(y) \, d\nu_x(y) \right) d\mu(x), \quad \text{for all } f \in C(X).$$

From the definition of T it is clear that T is a linear map, and that $T\mu = \mu$ if μ is a boundary measure. We are interested in deciding when the space

$$A = \left\{ f \in C(X) : f(x) = \int f \, d\nu_x, \text{ for all } x \in X \right\}$$

is a *simplicial space*, that is when the state space for A is a simplex. In terms of the map T we have a condition which implies that A is simplical:

PROPOSITION 1. *The space A is simplicial, if the null-space for T,*

$$N(T) = \{\mu \in M(X) : T\mu = 0\},$$

is a w^*-closed subspace of $M(X)$ (when $M(X)$ is equipped with the w^*-topology defined by $C(X)$).

PROOF. We look at the space $M(X)/N(T)$ equipped with the quotient topology which is a Hausdorff topological vector space. The set $M_1^+(X)/N(T)$ is a compact convex subset of $M(X)/N(T)$ where $M_1^+(X)$ denotes the set of probability measures on X. Now, if φ denotes the canonical projection of $M(X)$ onto $M(X)/N(T)$ and if $\bar{a}$ is a continuous affine function on $M_1^+(X)/N(T)$ then the restriction of $\bar{a} \circ \varphi$ to $M_1^+(X)$ is a continuous affine function. Since $T\nu = \nu$ if $\nu \in M_1^+(\partial_B X) = M_1^+(X) \cap M(\partial_B X)$, this shows that the continuous affine functions on $M_1^+(X)$, which are constant on each of the sets $\{\mu \in M_1^+(X) \colon T\mu = \nu\}$, where $\nu \in M_1^+(X)$, separate the points of $M_1^+(\partial_B X)$. But these functions are just the functions in A because the continuous affine functions on $M_1^+(X)$ are of the form

$$\mu \to \int f \, d\mu$$

where $f \in C(X)$. If $x \in X \setminus \partial_B X$ then δ_x, the point mass at x, and ν_x represent x with respect to A, which shows that the Choquet boundary for A is contained in that for B. The reverse inclusion is clear since A contains B. Thus A and B have the same Choquet boundary. But then $M_1^+(\partial_B X)=M_1^+(\partial_A X)$ and we conclude that A separates the points of $M_1^+(\partial_A X)$, which shows that each point in the state space for A is represented by a unique measure in $M_1^+(\partial_A X)$. But then the state space for A is a simplex ([1, Thm. II 3.6]).

We now turn to look at conditions given in terms of the space B on selections of representing measures which ensure that $N(T)$ is w^*-closed.

Let P denote the cone of all pointwise suprema of finitely many functions from B. The following condition is necessary for the existence of a simplicial space A containing B and having the same Choquet boundary as B:

(*) There is a measurable selection $x \to \nu_x$ of maximal measures such that for each $f \in P$ the function $x \to \int f\, d\nu_x$ is upper semi-continuous on X.

The necessity of this condition follows from [1, Thm. II 3.7]. If $\partial_B X$ is closed, then the restriction of P to $\partial_B X$ is dense in $C(\partial_B X)$ so that (*) is in that case equivalent to the existence of a measurable selection which is upper semi-continuous with respect to $C(X)$ which is the same thing as saying that the selection is continuous. This is, however known to be a sufficient condition when $\partial_B X$ is closed ([2, Thm. (2.4)]).

The second condition is concerned with separation of measures in $M_1^+(\partial_B X)$ by functions in P:

(**) If $\mu, \nu \in M_1^+(\partial_B X)$ with $\mu \neq \nu$ then there exist functions $f, -g \in P$ with $f \geqq g$ on $\partial_B X$ such that

$$\int f\, d\mu < \int g\, d\nu .$$

We observe that condition (**) is satisfied if $\partial_B X$ is closed, because in that case the restriction of P to $\partial_B X$ is dense in $C(\partial_B X)$. We also note that if (**) is satisfied then each point of $\overline{\partial_B X}$ has a *unique* boundary representing measure with respect to B. This may be seen as follows: Suppose $x \in \overline{\partial_B X}$ has to representing measures $\mu, \nu \in M_1^+(\partial_B X)$. Let $f, -g$ be as in condition (**) relative to μ and ν. Then

$$f(x) \leqq \int f\, d\mu < \int g\, d\nu \leqq g(x) .$$

But $f \geqq g$ on $\partial_B X$ and hence $f(x) \geqq g(x)$.

The following theorem shows how conditions (*) and (**) are related to the existence of a simplicial subspace containing B and having the same Choquet boundary as B.

THEOREM 2. *Let B be a closed subspace of $C(X)$ and suppose that each point in $\overline{\partial_B X}$ has a unique boundary representing measure with respect to B. Then there exists a simplicial space A containing B and having the same Choquet boundary as B if and only if conditions (*) and (**) are satisfied.*

PROOF. Let A be a simplicial subspace of $C(X)$ containing B and suppose that $\partial_A X = \partial_B X$. As noted earlier condition (*) is then satisfied. To see that condition (**) is satisfied let μ and ν be two different elements of $M_1^+(\partial_B X)$ and let a be a function in A such that

$$\int a\,d\mu < \int a\,d\nu\,. \tag{1}$$

Let $\hat{a}$ be the function defined for each $x \in X$ as follows:

$$\hat{a}(x) = \inf\{b(x) : b \in B\ b > a\}\,.$$

It follows from [1, Cor. I. 3.6], that there is a measure η representing x with respect to B such that

$$\hat{a}(x) = \int a\,d\eta\,. \tag{2}$$

Let $\xi \in M_1^+(\partial_B X)$ be such that $\xi - \eta \in A^\perp$. Such a measure exists by Choquets theorem. Then also $\xi - \eta \in B^\perp$ which shows that ξ is a boundary representing measure (with respect to B) for x. Since each point of $\overline{\partial_B X}$ has a unique boundary representing measure with respect to B, we conclude that ξ is a representing measure for x with respect to A, if $x \in \overline{\partial_B X}$. Thus

$$\hat{a}(x) = \int a\,d\eta = \int a\,d\xi = a(x) \tag{3}$$

if $x \in \overline{\partial_B X}$. Therefore the function a can be approximated uniformly on $\overline{\partial_B X}$ by functions in P and applying this result to $-a \in A$ instead of a, we see that a can also be approximated uniformly by functions in $-P$. Taking (1) into account we conclude that condition (**) is satisfied.

Suppose now that conditions (*) and (**) are satisfied and let T be the linear map from $M(X)$ into $M(\partial_B X)$ which the selection $x \to \nu_x$ gives rise to. By Proposition 1 it suffices to show that $N(T)$ is w^*-closed in $M(X)$. By the Krein–Smulian Theorem it suffices to show that the set

$$N_1 = \{\mu \in M(X) : \|\mu\| \leqq 1 \text{ and } T\mu = 0\}$$

is w^*-closed in $M(X)$.

First we observe that each $\mu \in N_1$ can be written as $\mu = t\mu^+ - t\mu^-$ where $\mu^+, \mu^- \in M_1^+(X)$ and $0 \leqq t \leqq 1/2$.

Let μ be in the w^*-closure of N_1 and let $\{\mu_n\}$ be a sequence in N_1 converging to μ. Each μ_n can be decomposed as $\mu_n = t_n\mu_n^+ - t_n\mu_n^-$ where $\mu_n^+, \mu_n^- \in M_1^+(X)$ and where $0 \leqq t_n \leqq 1/2$. Passing to a subsequence if necessary we may suppose that there are $\xi, \eta \in M_1^+(X)$ and a number t with $0 \leqq t \leqq 1/2$ such that $\{\mu_n^+\}$ and $\{\mu_n^-\}$ converge to ξ and η respectively in the w^*-topology and such that $\{t_n\}$ converges to t. Then $\mu = t\xi - t\eta$. We want to show that $T\xi = T\eta$. Suppose this is not the case. Then there are functions $f, -g \in P$ such that $f \geqq g$ on $\partial_B X$ and such that

$$\int f\, dT\eta < \int g\, dT\xi \tag{4}$$

Suppose we knew that for each $h \in P$ the map

$$\eta \to \int h\, dT\eta \qquad \eta \in M_1^+(X) \tag{5}$$

was upper semi-continuous when $M_1^+(X)$ is equipped with the w^*-topology. Then

$$\int f\, dT\eta \geqq \overline{\lim_n} \int f\, dT\mu_n^+ = \overline{\lim} \int f\, dT\mu_n^- \geqq \underline{\lim} \int g\, dT\mu_n^- \geqq \int g\, T\xi \tag{6}$$

contradicting (4). Thus it only remains to show that the map defined by (5) is upper semi-continuous. Since for each $h \in P$ the map

$$\varphi_h \colon x \to \int h\, dv_x, \qquad x \in X$$

is upper semi-continuous there is a family $\{f_i\}_{i \in I}$ of continuous functions, downwards directed in the pointwise ordering, such that $\varphi_h(x) = \inf_{i \in I} f_i(x)$. Now by definition

$$\int h\, dT\eta = \int \varphi_h\, d\eta\ .$$

By Lusin's theorem we can find compact sets to which the restriction of φ_h is continuous, carrying as large a portion of the mass of η as we wish. It then follows that

$$\int h\, dT\eta = \inf_i \int f_i\, d\eta$$

which shows that the map defined by (5) is upper semi-continuous and this concludes the proof of Theorem 2.

3.

We conclude with a couple of examples and some remarks. First an example of a non-simplicial subspace whose Choquet boundary is not closed and where conditions (*) and (**) are satisfied.

EXAMPLE 3. Let $\{P_1, P_2, P_3, P_4\}$ be the 4 corners of a square in $\mathbf{R}^2$, let P_5 denote the midpoint of the segment P_1P_2 and let $\{Q_n\}$ be a sequence of points converging to P_5 from outside the square. Let

$$X = \{P_i : i=1,\ldots,5\} \cup \{Q_n : n \in \mathbf{N}\}$$

and let B be the space of all continuous functions on X which are affine on the set $\{P_i : i=1,\ldots,5\}$. Then the Choquet boundary for B is $X \setminus \{P_5\}$, the space B is not simplicial and it is not hard to see that (*) and (**) are satisfied.

In [2, Counterexample 3.11] an example is given of a subspace for which condition (*) is not satisfied. In that example the Choquet boundary is closed so that condition (**) is satisfied. The second example given here is an example of a subspace B for which points in $\overline{\partial_B X} \setminus \partial_B X$ have more than one boundary representing measure so that condition (**) is not satisfied, but where there exists a simplicial space containing B and having the same Choquet boundary as B.

EXAMPLE 4. Let $X \subseteqq \mathbf{R}^2$ be the set

$$X = \left\{x_n = \left(\frac{1}{n}, 0\right) : n \in \mathbf{N}\right\} \cup \{y_1 = (0,-1),\ y_2 = (0,0),\ y_3 = (0,1)\}$$

and let

$$B = \left\{f \in C(X) : \frac{1}{2}(f(y_1)+f(y_3)) = f(y_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} f(x_n)\right\}.$$

Then $\partial_B X = X \setminus \{y_2\}$ and y_2 has two boundary representing measures so that condition (**) is not satisfied. As a simplicial space A containing B and having the same Choquet boundary as B we can take

$$A = \{f \in C(X) : f(y_2) = \tfrac{1}{2}(f(y_1)+f(y_3))\}.$$

In the general case when points of $\overline{\partial_B X} \setminus \partial_B X$ may have more than one boundary representing measure we do not have a necessary and sufficient condition for the existence of a simplicial space A containing B and having the same Choquet boundary as B. Further, we do not know whether condition (**) is redundant in Theorem 2 i.e. contained in condition (*), or more generally even whether condition (*) is sufficient in the general case when points of $\overline{\partial_B X} \setminus \partial_B X$ may have more than one boundary representing measure.

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