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ON SOME FUNCTIONAL EQUATIONS OF JESSEN, KARPF, AND THORUP

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In this paper we shall extend a theorem of B. Jessen, J. Karpf, and A. Thorup which was used in a simplified proof of Sydler's theorem on polyhedra (see [4]). The result is also related to known results in homological algebra and in the measurement of information (cf., e.g. [2; section 3.5]).

In [5], Jessen, Karpf, and Thorup proved the following two theorems.

THEOREM 1. *Let A and X be commutative groups, with X divisible. Then $F\colon A^2 \to X$ satisfies*

$$F(a,b) = F(b,a), \quad \forall a,b \in A, \tag{1}$$

$$F(a,b)+F(a+b,c) = F(a,b+c)+F(b,c), \quad \forall a,b,c \in A, \tag{2}$$

if and only if there is a map $f\colon A \to X$ such that

$$F(a,b) = f(a+b)-f(a)-f(b), \quad \forall a,b \in A. \tag{*}$$

Furthermore, if A is ordered, then the same result holds with A replaced by $A_+ := \{a \in A \mid a > \mathbf{0}\}$.

THEOREM 2. *Let A be an integral domain and X a uniquely A-divisible unitary module over A. Then $F\colon A^2 \to X$ and $G\colon A^2 \to X$ satisfy* (1), (2),

$$G(a,b) = G(b,a), \quad \forall a,b \in A, \tag{3}$$

$$cG(a,b)+G(ab,c) = G(a,bc)+aG(b,c), \quad \forall a,b,c \in A, \tag{4}$$

$$F(ac,bc)-cF(a,b) = G(a+b,c)-G(a,c)-G(b,c), \quad \forall a,b,c \in A, \tag{5}$$

$$\sum_{i=1}^{p} F(\mathbf{1},i\mathbf{1}) = \mathbf{0}, \quad p = \operatorname{char} A, \tag{6}$$

if and only if there is a map $f\colon A \to X$ such that

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$$(*) \qquad F(a,b) = f(a+b) - f(a) - f(b), \quad \forall a,b \in A,$$

$$(**) \qquad G(a,b) = f(ab) - bf(a) - af(b), \quad \forall a,b \in A.$$

Furthermore, if A is ordered, then the same result holds with A replaced by A_+.

It is known (cf. [1], [3]) that Theorem 1 can be extended by removing equation (1) and simultaneously replacing equation (*) by

$$(*') \qquad F(a,b) = f(a+b) - f(a) - f(b) + \psi(a,b),$$

where $\psi: A^2 \to X$ is an arbitrary antisymmetric function which is additive in each variable. In this paper, we shall extend Theorem 2 in a similar way, showing that equation (1) is in fact superfluous. A corollary to this result is also given. The main result is the following.

THEOREM 3. *Let A be an integral domain and X a uniquely A-divisible unitary module over A. Then $F, G: A^2 \to X$ satisfy equations* (2), (3), (4), (5), (6) *if and only if there is a map $f: A \to X$ representing F and G through equations* (*), (**). *Moreover, if A is ordered, then the same result holds with A replaced by A_+.*

PROOF. In one direction, it is trivial.

On the other hand, suppose F and G satisfy (2), (3), (4), (5), (6) (on either A or A_+). We divide this part of the proof into two cases. Note that equation (6) is void if char $A = 0$.

CASE 1. Suppose char $A \neq 2$. Define the canonical symmetric and antisymmetric parts of F, respectively, as follows:

$$(7) \qquad \varphi(a,b) := \tfrac{1}{2}[F(a,b) + F(b,a)], \quad \psi(a,b) := \tfrac{1}{2}[F(a,b) - F(b,a)],$$

for all $a,b \in A$ (or A_+). It is easily checked that equations (1), (2), (3), (4), (5) are satisfied by the pair (φ, G), i.e., with φ in place of F. ((5) follows because of the symmetry of its right-hand side with respect to a and b.) Equation (6) for φ follows from (6) for F and

$$F(\mathbf{1}, i\mathbf{1}) = F(i\mathbf{1}, \mathbf{1}), \quad i = 1, 2, \ldots .$$

This equation, in turn, is established by induction on n with the equation

$$F(\mathbf{1}, n\mathbf{1}) + F((n+1)\mathbf{1}, \mathbf{1}) = F(\mathbf{1}, (n+1)\mathbf{1}) + F(n\mathbf{1}, \mathbf{1}),$$

which is (2) with $a = c = \mathbf{1}$, $b = n\mathbf{1}$.

Thus, by Theorem 2, we have a map f: A (or A_+) $\to X$ such that G is of the form (**), and

$$\varphi(a,b) = f(a+b) - f(a) - f(b).$$

But, by (7), $F=\varphi+\psi$, so F has the form (*′), where ψ is antisymmetric by definition (7).

We show that ψ is also bi-additive. Using (2) three times along with (7), we have

$$\begin{aligned} 2\psi(a+b,c) &= F(a+b,c)-F(c,a+b) \\ &= [F(a,b+c)+F(b,c)-F(a,b)]-[F(c+a,b)+F(c,a)-F(a,b)] \\ &= F(a,c+b)-F(a+c,b)-F(c,a)+F(b,c) \\ &= [F(a+c,b)+F(a,c)-F(c,b)]-F(a+c,b)-F(c,a)+F(b,c) \\ &= F(a,c)-F(c,a)+F(b,c)-F(c,b) \\ &= 2\psi(a,c)+2\psi(b,c)\,. \end{aligned}$$

Moreover, by the antisymmetry of ψ, the additivity of ψ in its second variable follows at once. Hence, in particular,

$$\psi(2a,2b) = 4\psi(a,b), \quad \forall\, a,b \in A \ \ (\text{or } A_+)\,. \tag{8}$$

Finally, we show that $\psi\equiv\mathbf{0}$. It follows immediately from (5) and (7) that

$$\psi(ac,bc)-c\psi(a,b) = \mathbf{0}\,.$$

In particular, if $c=2\cdot\mathbf{1}$, then

$$\psi(2a,2b) = 2\psi(a,b), \quad \forall\, a,b \in A \ \ (\text{or } A_+)\,.$$

Comparing this with (8), we obtain

$$\psi(a,b) = \mathbf{0}, \quad \forall\, a,b \in A \ \ (\text{or } A_+)\,.$$

With this, (*′) becomes (*), and the proof is finished in this case.

CASE 2. Suppose char $A=2$. Now it is impossible to divide F as in (7). But we shall show that in this case F must be (1) symmetric.

For this purpose, we need several preliminary results. Since char $A=2$, equation (6) says

$$F(\mathbf{1},\mathbf{0}) = F(\mathbf{1},\mathbf{1})\,.$$

Equation (4) with $a=b=\mathbf{0}$ becomes

$$G(\mathbf{0},c) = (\mathbf{1}-c)G(\mathbf{0},\mathbf{0})\,.$$

Using these relations with (2) and (5), it can be shown that

$$F(c,c) = F(\mathbf{1},\mathbf{1}),\ F(a,a+c) = F(a,c), \quad \forall\, a,c \in A \ \ (\text{or } A_+)\,.$$

Finally, (2) with $c=a+b$ gives now

$$F(a,b)+F(1,1) = F(1,1)+F(b,a) ,$$

which means that F is (1) symmetric. Therefore, by Theorem 2, the proof is complete.

COROLLARY. *Let V be a vector space over* $\mathbf{R}$ (*the real numbers*). *Then the class of functions* $F\colon]0,\infty[^2 \to V$ *which satisfy the equations*

$$F(a,b)+F(a+b,c) = F(a,b+c)+F(b,c) ,$$

$$F(ac,bc) = cF(a,b)$$

is identical with the class of $F\colon]0,\infty[^2 \to V$ *determined through the equation*

$$F(a,b) = f(a+b)-f(a)-f(b)$$

by means of a function $f\colon]0,\infty[\to V$ *satisfying the equation*

$$f(ab) = bf(a)+af(b) .$$

PROOF. Apply Theorem 3 (on A_+) with $A=\mathbf{R}$, $X=V$, and $G\equiv 0$.

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