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ON THE STRUCTURE OF SOLVABLE LIE ALGEBRAS

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1. Introduction.

Given two Lie algebras $\mathscr{G}$ and $\mathscr{A}$ where $\mathscr{G}$ is solvable and $\mathscr{A}$ abelian, we shall consider how to classify within isomorphisms all Lie algebras $\tilde{\mathscr{G}}$ which are extensions of $\mathscr{G}$ by $\mathscr{A}$ and for which the centre of the nilradical $\tilde{\mathscr{N}}$ is equal to $\mathscr{A}$. To this end we show that the isomorphism classes of all such Lie algebras $\tilde{\mathscr{G}}$ possessing no abelian direct factors are in bijective correspondence with certain $\mathrm{Aut}\,\mathscr{G} \times \mathrm{Aut}\,\mathscr{A}$ orbits in $\bigcup_\theta H^2(\mathscr{G},\theta)$ where θ runs through a certain family of representations of $\mathscr{G}/\mathscr{N}$ in $\mathscr{A}$ and $\mathscr{N} = \tilde{\mathscr{N}}/\mathscr{A}$. This result gives an inductive method of constructing solvable Lie algebras.

2. Extensions and automorphisms.

2.1. Let $\mathscr{G}$ and $\mathscr{A}$ be Lie algebras, $\mathscr{A}$ abelian, $\theta\colon \mathscr{G} \to \mathrm{End}\,\mathscr{A}$ a representation, $B\colon \mathscr{G} \times \mathscr{G} \to \mathscr{A}$ an anti-symmetric bilinear map satisfying

$$(2.1)\qquad B(X,[Y,Z])+B(Z,[X,Y])+B(Y,[Z,X])+\theta(X)B(Y,Z)$$
$$+\theta(Z)B(X,Y)+\theta(Y)B(Z,X) = 0, \quad \text{all } X,Y,Z \in \mathscr{G},$$

i.e. B is a 2-cocycle on $\mathscr{G}$ with respect to θ. The set of all such 2-cocycles is denoted by $C^2(\mathscr{G},\theta)$. Given $B \in C^2(\mathscr{G},\theta)$ we can construct a Lie algebra $\tilde{\mathscr{G}} = \mathscr{G}(B,\theta)$ which is an extension of $\mathscr{G}$ by $\mathscr{A}$ as follows: $\tilde{\mathscr{G}} = \mathscr{G} \oplus \mathscr{A}$ as vectorspace, and the Lie product is given by

$$(2.2)\qquad [(g,a),(g',a')] = ([g,g'],\theta(g)a'-\theta(g')a+B(g,g'));$$
$$\text{all } a,a' \in \mathscr{A},\ g,g' \in \mathscr{G}.$$

Conversely if $\tilde{\mathscr{G}}$ is an extension of $\mathscr{G}$ by $\mathscr{A}$ there exist $\theta\colon \mathscr{G} \to \mathrm{End}\,\mathscr{A}$ and $B \in C^2(\mathscr{G},\theta)$ such that $\tilde{\mathscr{G}}$ and $\mathscr{G}(B,\theta)$ are isomorphic as Lie algebras.

2.2. Let $\mathscr{N}$ denote the nilradical of $\mathscr{G}$ and $\mathscr{Z}$ the centre of $\mathscr{N}$. (In the sequel we shall only assume that $\mathscr{N}$ is a nilpotent ideal of $\mathscr{G}$, $\mathscr{N} \supset [\mathscr{G},\mathscr{G}]$.) We wish to

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study Lie algebras $\tilde{\mathscr{G}}=\mathscr{G}(B,\theta)$ for which the nilradical $\tilde{\mathscr{N}}$ is a central extension of $\mathscr{N}$ by $\mathscr{A}$. Let $B^0=B|_{\mathscr{N}\times\mathscr{N}}$ and $\theta^0=\theta|_{\mathscr{N}}$. Clearly the extension

$$0 \to \mathscr{A} \to \mathscr{N}(B^0,\theta^0) \to \mathscr{N} \to 0$$

is central if and only if $\ker\theta\supset\mathscr{N}$. In this case $\theta^0=0$ and $\mathscr{N}(B^0,\theta^0)=\mathscr{N}(B^0)\subset\tilde{\mathscr{N}}$. We show that $\mathscr{N}(B^0)=\tilde{\mathscr{N}}$. Obviously $\tilde{\mathscr{N}}$ is an extension of a subalgebra $\mathscr{M}$ of $\mathscr{G}$ by $\mathscr{A}$. It follows that $\mathscr{M}$ is a nilpotent subalgebra of $\mathscr{G}$ containing $\mathscr{N}$. Hence $\mathscr{M}=\mathscr{N}$ proving the assertion. Let $\tilde{\mathscr{Z}}$ denote the centre of $\tilde{\mathscr{N}}$. Assuming $\ker\theta\supset\mathscr{N}$ we have $\tilde{\mathscr{Z}}=(\mathscr{S}_{B^0}\cap\mathscr{Z})\oplus\mathscr{A}$ where

$$\mathscr{S}_{B^0} = \{X\in\mathscr{N} : B^0(X,\mathscr{N})=(0)\} .$$

Thus $\tilde{\mathscr{Z}}=\mathscr{A}$ if and only if $\mathscr{S}_{B^0}\cap\mathscr{Z}=(0)$. We have shown

2.4. Given two such extensions $\tilde{\mathscr{G}}_i=\mathscr{G}(B_i,\theta_i)$, $i=1,2$, of $\mathscr{G}$ by $\mathscr{A}$, and assume *$\mathscr{A}$ is abelian. Then*

a) *The nilradical $\tilde{\mathscr{N}}$ of $\tilde{\mathscr{G}}$ is a central extension of $\mathscr{N}$ by $\mathscr{A}$ if and only if* $\ker\theta\supset\mathscr{N}$.

b) *Let* $\ker\theta\supset\mathscr{N}$. *The centre of $\tilde{\mathscr{N}}$ is $\mathscr{A}$ if and only if* $\mathscr{S}_{B^0}\cap\mathscr{Z}=(0)$.

2.4. Given two such extensions $\tilde{\mathscr{G}}=\mathscr{G}(B_i,\theta_i)$, $i=1,2$, of $\mathscr{G}$ by $\mathscr{A}$, and assume the Lie algebras $\tilde{\mathscr{G}}_1$ and $\tilde{\mathscr{G}}_2$ are isomorphic and that the centres $\tilde{\mathscr{Z}}_i$ of their nilradicals both are equal to $\mathscr{A}$. Let α: $\tilde{\mathscr{G}}_1\to\tilde{\mathscr{G}}_2$ be an isomorphism. Dividing with the common ideal $\mathscr{A}$ we obtain an automorphism α_0: $\mathscr{G}\to\mathscr{G}$. We can realize α as a matrix relative to a suitable basis for $\mathscr{G}\oplus\mathscr{A}$ which is assumed to contain a basis for $\mathscr{G}$ and a basis for $\mathscr{A}$:

$$(2.3)\qquad \alpha = \begin{pmatrix}\alpha_0 & 0\\ \varphi & \psi\end{pmatrix};\qquad \alpha_0\in\operatorname{Aut}G,\ \psi\in\operatorname{Aut}\mathscr{A},\ \varphi\in\operatorname{Hom}(\mathscr{G},\mathscr{A}) .$$

Now α preserves the Lie products and writing $[\cdot,\cdot]$, $[\cdot,\cdot]_i$ for the products of $\mathscr{G}$ and $\mathscr{G}_i$ respectively we have using (2.2)

$$(2.4)\qquad \alpha\left[\begin{pmatrix}g\\a\end{pmatrix},\begin{pmatrix}g'\\a'\end{pmatrix}\right]_1$$

$$= (\alpha_0[g,g'],\varphi[g,g']+\psi B_1(g,g')+\psi\theta_1(g)a'-\psi\theta_1(g')a)$$

and similarly

$$(2.5)\qquad \left[\alpha\begin{pmatrix}g\\a\end{pmatrix},\alpha\begin{pmatrix}g'\\a'\end{pmatrix}\right]_2$$

$$= ([\alpha_0 g,\alpha_0 g'],B_2(\alpha_0 g,\alpha_0 g')+\theta_2(\alpha_0 g)(\varphi g'+\psi a')-\theta_2(\alpha_0 g')(\varphi g+\psi a)) .$$

Hence letting $a=a'=0$ and combining (2.4) and (2.5) we get

$$B_2(\alpha_0 g, \alpha_0 g') = \varphi[g,g'] + \psi\circ B_1(g,g') + \theta_2\circ\alpha_0(g')(\varphi g) - \theta_2\circ\alpha_0(g)(\varphi g')$$

or

$$B_2\circ\alpha_0 = \psi\circ B_1 + d\varphi, \quad d\varphi \in B^2(\mathscr{G}, \theta_2\circ\alpha_0) \tag{2.6}$$

where $B^2(\mathscr{G}, \theta_2\circ\alpha_0)$ denotes the set of coboundaries in $C^2(\mathscr{G}, \theta_2\circ\alpha_0)$, [3, p. 220]. Moreover substitution of (2.6) into (2.4) and (2.5) gives

$$\psi\circ\theta_1(g)a' - \theta_2\circ\alpha_0(g)(\psi a') = \psi\circ\theta_1(g')a - \theta_2(\alpha_0 g')(\psi a),$$

and letting $a'=0$ we obtain

$$\psi\circ\theta_1(g')a = \theta_2\circ\alpha_0(g')\psi(a),$$

thus

$$\psi\circ\theta_1(\cdot)\circ\psi^{-1} = \theta_2\circ\alpha_0, \tag{2.7}$$

in other words ψ must be an intertwining operator for the representations θ_1 and $\theta_2\circ\alpha_0$. Conversely if (2.6) and (2.7) hold it is readily verified that the Lie algebras $\mathscr{G}(B_1,\theta_1)$ and $\mathscr{G}(B_2,\theta_2)$ are isomorphic.

(2.6) can be written $B_2=\psi\circ B_1\circ\alpha_0^{-1}+(d\varphi)\circ\alpha_0^{-1}$. Now we have

$$\begin{aligned}(d\varphi)\alpha_0^{-1}(g,g') &= \varphi\circ[\alpha_0^{-1}g, \alpha_0^{-1}g'] + \theta_2\circ\alpha_0(\alpha_0^{-1}g')(\varphi\alpha_0^{-1}g)\\ -\theta_2\circ\alpha_0(\alpha_0^{-1}g)(\varphi\alpha_0^{-1}g') &= \varphi\circ\alpha_0^{-1}[g,g'] + \theta_2(g')(\varphi\circ\alpha_0^{-1}g)\\ -\theta_2(g)(\varphi\circ\alpha_0^{-1}g').\end{aligned}$$

Hence $(d\varphi)\alpha_0^{-1}=d(\varphi\circ\alpha_0^{-1})\in B^2(\mathscr{G},\theta_2)$. Thus $\mathscr{G}(B_1,\theta_1)$ is isomorphic to $\mathscr{G}(B_2,\theta_2)$ if and only if there exists $\alpha_0\in \operatorname{Aut}\mathscr{G}$ and $\psi\in\operatorname{Aut}\mathscr{A}$ such that

$$B_2 = \psi\circ B_1\circ\alpha_0^{-1} \mod B^2(\mathscr{G},\theta_2)$$

and ψ is an intertwining operator for θ_2 and $\theta_1\circ\alpha_0^{-1}$. We have proved

2.5. Proposition. *Let for $i=1,2$, $\tilde{\mathscr{G}}_i=\mathscr{G}(B_i,\theta_i)$ be an extension of the solvable Lie algebra $\mathscr{G}$ by the abelian Lie algebra $\mathscr{A}$; and assume $\mathscr{A}$ is the centre of the nilradical of $\tilde{\mathscr{G}}_i$, $i=1,2$. Then the Lie algebras $\tilde{\mathscr{G}}_1$ and $\tilde{\mathscr{G}}_2$ are isomorphic if and only if B_1 and B_2 are in the same* $\operatorname{Aut}\mathscr{G}\times\operatorname{Aut}\mathscr{A}$ *orbit in $\bigcup_\theta H^2(\mathscr{G},\theta)$, where θ runs through the family of all representations of $\mathscr{G}$ in $\mathscr{A}$, under the action*

$$((\alpha_0,\psi),B_1)\to\psi\circ B_1\circ\alpha_0\in H^2(\mathscr{G},\psi\theta_1\alpha_0(\cdot)\psi^{-1})$$

In case $B_1=B_2=B$ and $\theta_1=\theta_2=\theta$ we obtain the following description of $\operatorname{Aut}\mathscr{G}(B,\theta)$.

2.6. COROLLARY. *Let $B \in C^2(\mathscr{G},\theta)$, $\theta\colon \mathscr{G} \to \operatorname{End}\mathscr{A}$, and assume $\mathscr{A}$ is the centre of the nilradical of the extended Lie algebra $\mathscr{G}(B,\theta)$. Then the automorphism group of $\mathscr{G}(B,\theta)$ is isomorphic to the group of all matrices*

$$\begin{pmatrix} \alpha_0 & 0 \\ \varphi & \psi \end{pmatrix},$$

where $\alpha_0 \in \operatorname{Aut}\mathscr{G}$, $\varphi \in \operatorname{Hom}(\mathscr{G},\mathscr{A})$, $\psi \in \operatorname{Aut}\mathscr{A}$, and

$$\begin{cases} B\circ\alpha_0 = \psi\circ B + d\varphi, & d\varphi \in B^2(\mathscr{G},\theta) \\ \psi\theta\psi^{-1} = \theta\circ\alpha_0 \end{cases} \tag{2.8}$$

3. The exclusion of abelian direct factors.

3.1. We continue our study of extensions $\mathscr{G}(B,\theta)$ of a solvable Lie algebra $\mathscr{G}$ by an abelian $\mathscr{A}$, and proceed to exclude those 2-cocycles B for which the extended Lie algebra $\mathscr{G}(B,\theta)$ is isomorphic to a direct sum $\mathscr{D}\oplus\mathscr{H}$ where $\mathscr{D}$ and $\mathscr{H}$ are Lie algebras, $\mathscr{D}$ abelian. Obviously any abelian direct factor of $\mathscr{G}(B,\theta)$ must be contained in the nilradical $\tilde{\mathscr{N}}$. Assuming $\tilde{\mathscr{N}}/\mathscr{A} = \mathscr{N}$ and $\ker\theta \supset \mathscr{N}$, $\tilde{\mathscr{N}}$ is a central extension of $\mathscr{N}$ by $\mathscr{A}$ and in order to omit abelian factors in $\tilde{\mathscr{N}}$ it suffices to study the restricted action of $\operatorname{Aut}\mathscr{G}\times\operatorname{Aut}\mathscr{A}$ in $H^2(\mathscr{N},\mathscr{A})$. If $\mathscr{S}_{B^0}\cap\mathscr{Z} = (0)$ the centre of $\tilde{\mathscr{N}}$ is $\mathscr{A}$ and any abelian direct factor $\mathscr{D}$ of $\tilde{\mathscr{N}}$ is contained in $\mathscr{A}$.

Let J be the set of all linear maps $F \in \operatorname{End}\mathscr{A}$ such that there exists $\varphi \in \operatorname{Hom}(\mathscr{N},\mathscr{A})$ with the property

$$F\circ B^0 = \varphi\circ[\cdot,\cdot]_{\mathscr{N}}$$

where $B^0 = B|_{\mathscr{N}\times\mathscr{N}}$ and $[\cdot,\cdot]_{\mathscr{N}}$ denotes the Lie product of $\mathscr{N}$. Then J is a left ideal in $\operatorname{End}\mathscr{A}$ and we have $J = (\operatorname{End}\mathscr{A})\circ\pi$ for some projection π in J. Hence there exists $\varphi_\pi \in \operatorname{Hom}(\mathscr{N},\mathscr{A})$ such that

$$\pi\circ B^0 = \varphi_\pi\circ[\cdot,\cdot]_{\mathscr{N}}\,.$$

Let

$$B' = B^0 - \varphi_\pi\circ[\cdot,\cdot]_{\mathscr{N}} = (I-\pi)\circ B^0\,. \tag{3.1}$$

3.2. LEMMA. *$F\circ B' = \varphi_\pi\circ[\cdot,\cdot]_{\mathscr{N}}$ implies $\varphi_\pi\circ[\cdot,\cdot]_{\mathscr{N}} = 0$.*

PROOF. $F\circ B' = F\circ(I-\pi)\circ B^0 = \varphi_\pi\circ[\cdot,\cdot]_{\mathscr{N}}$ implies $F\circ(I-\pi) \in J$. Hence

$$F\circ(I-\pi) = G\circ\pi \quad \text{for some } G \in \operatorname{End}\mathscr{A}\,.$$

This gives $F = (F+G)\circ\pi$, so that

$$F\circ(I-\pi) = (F+G)\circ\pi\circ(I-\pi) = 0\,.$$

Now $\mathcal{N}(B')=\mathcal{N}(B_\pi^0)\oplus\pi(\mathcal{A})$ where $B_\pi^0=(I-\pi)B^0:\mathcal{N}\times\mathcal{N}\to(I-\pi)\mathcal{A}$. Thus, if $\pi\neq 0$, $\mathcal{N}(B^0)$ contains an abelian direct factor and we have

3.3. LEMMA. *Let $\mathcal{G}(B,\theta)$ be an extension of the solvable Lie algebra $\mathcal{G}$ by the abelian $\mathcal{A}$ where* $\ker\theta\supset\mathcal{N}$ *and $\mathcal{S}_{B^0}\cap\mathcal{Z}=(0)$, $\mathcal{N}=\tilde{\mathcal{N}}/\mathcal{A}$. Assume $\tilde{\mathcal{N}}$ cannot be written as a direct sum $\mathcal{D}\oplus\mathcal{H}$ of Lie algebras where $\mathcal{D}$ is abelian. For any pair $F\in\operatorname{End}\mathcal{A}$, $\varphi\in\operatorname{Hom}(\mathcal{N},\mathcal{A})$ such that $F\circ B^0=\varphi\circ[\cdot,\cdot]_{\mathcal{N}}$ we have $F=0$.*

3.4. Let for $\mathcal{A}$ as above π_i, $1\leqq i\leqq k$, be its coordinate functions relative to some basis. Thus Lemma 3.3. is equivalent to: $\pi_1B^0,\dots,\pi_kB^0$ are linearly independent in $H^2(\mathcal{N},F)$ where F denotes the field of $\mathcal{G}$. We know from section 2 that two extensions $\mathcal{G}(B_1,\theta_1)$ and $\mathcal{G}(B,\theta)$ of $\mathcal{G}$ by $\mathcal{A}$ are isomorphic if and only if $B_1=\psi\circ B\circ\alpha_0+d(\varphi\circ\alpha_0)$ for some $\alpha_0\in\operatorname{Aut}\mathcal{G}$, $\psi\in\operatorname{Aut}\mathcal{A}$, $d(\varphi\circ\alpha_0)\in B^2(\mathcal{G},\theta_1)$, $\psi\theta_1\psi^{-1}=\theta\circ\alpha_0$. This gives by restricting to $\mathcal{N}$:

$$B_1^0=\psi\circ B\circ\beta+\varphi\circ\beta\circ[\cdot,\cdot]_{\mathcal{N}},\qquad \beta=\alpha_0|_{\mathcal{N}},\ B_1^0=B_1|_{\mathcal{N}\times\mathcal{N}}.$$

Such an identity holds if and only if $\pi_1\circ B\circ\beta,\dots,\pi_k\circ B\circ\beta$ and $\pi_1\circ B_1^0,\dots,\pi_k\circ B_1^0$ generate the same subspace of $H^2(\mathcal{N},F)$. Thus the restricted action of $\operatorname{Aut}\mathcal{G}\times\operatorname{Aut}\mathcal{A}$ in $H^2(\mathcal{N},\mathcal{A})$ induces an action of $\operatorname{Aut}\mathcal{G}$ in the set of all k-dimensional subspaces $G_kH^2(\mathcal{N},F)$ of the second cohomology group of $\mathcal{N}$ if and only if $\mathcal{G}(B,\theta)$ contains no abelian direct factor. We say that an $\operatorname{Aut}\mathcal{G}$-orbit Ω in $G_kH^2(\mathcal{N},F)$ *has no kernel in the centre* $\mathcal{Z}$ of $\mathcal{N}$ if $\mathcal{S}_{B^0}\cap\mathcal{Z}=(0)$ for some and hence for all $B^0\in V$, where V runs through Ω. Denote by $H^2(\mathcal{G};\mathcal{G}/\mathcal{N},\mathcal{A})$ the space $\bigcup_\theta H^2(\mathcal{G},\theta)$ where θ runs through those representations of $\mathcal{G}$ in $\mathcal{A}$ which satisfy $\ker\theta\supset\mathcal{N}$ and, for x in the nilradical of $\mathcal{G}$, $\theta(x)$ is nilpotent $\Leftrightarrow x\in\mathcal{N}$ (this ensures $\tilde{\mathcal{N}}/\mathcal{A}\cong\mathcal{N}$).

3.5. PROPOSITION. *Let $\mathcal{G}$ be a solvable Lie algebra over the field F, $\mathcal{N}$ a nilpotent ideal of $\mathcal{G}$, $\mathcal{N}\supset[\mathcal{G},\mathcal{G}]$. The isomorphism classes of solvable Lie algebras $\tilde{\mathcal{G}}$ possessing nilradical $\tilde{\mathcal{N}}$ with k-dimensional centre $\mathcal{A}$ such that $\tilde{\mathcal{G}}/\mathcal{A}\cong\mathcal{G}$, $\tilde{\mathcal{N}}/\mathcal{A}\cong\mathcal{N}$ and such that $\tilde{\mathcal{N}}$ contains no abelian direct factor are in bijective correspondence with those* $\operatorname{Aut}\mathcal{G}\times\operatorname{Aut}\mathcal{A}$*-orbits in $H^2(\mathcal{G};\mathcal{G}/\mathcal{N},\mathcal{A})$ (under the action $(\alpha,\psi,B)\to\psi B\alpha$) which satisfy the following conditions.*

1) *The restricted action of* $\operatorname{Aut}\mathcal{G}\times\operatorname{Aut}\mathcal{A}$ *in $H^2(\mathcal{N},\mathcal{A})$ induces an action of* $\operatorname{Aut}\mathcal{G}$ *in $G_kH^2(\mathcal{N},F)$.*

2) *The induced* $\operatorname{Aut}\mathcal{G}$*-orbits in $G_kH^2(\mathcal{N},F)$ have no kernel in the centre of $\mathcal{N}$.*

If we restrict our attention to the classification of all (isomorphism classes of) *central* extensions of $\mathcal{G}$ by $\mathcal{A}$, we can drop the assumption that $\mathcal{G}$ be solvable. In this case $\theta=0$ and the extended algebra $\tilde{\mathcal{G}}=\mathcal{G}(B)$ is defined by an anti-symmetric bilinear map $B:\mathcal{G}\times\mathcal{G}\to\mathcal{A}$ satisfying the Jacobi-identity.

3.6. Corollary. *Let $\mathscr{G}$ be a Lie algebra over F, $\mathscr{Z}$ its centre. The isomorphism classes of Lie algebra $\tilde{\mathscr{G}}$ with centre $\tilde{\mathscr{Z}}$ of dimension k, $\tilde{\mathscr{G}}/\tilde{\mathscr{Z}} \cong \mathscr{G}$, and without abelian direct factors, are in bijective correspondence with those* Aut$\mathscr{G}$*-orbits Ω in the set of all k-dimensional subspaces of the second cohomology group $H^2(\mathscr{G}, F)$ enjoying the property that $\mathscr{S}_B \cap \mathscr{Z} = (0)$ for all $B \in V$, where V runs through Ω.*

3.7. Suppose $\Omega \subset \bigcup_\theta H^2(\mathscr{G}, \theta)$ is an orbit under $\mathrm{Aut}\,\mathscr{G} \times \mathrm{Aut}\,\mathscr{A}$ and let $B \in \Omega \cap H^2(\mathscr{G}, \theta)$, $S_{B^0} \cap \mathscr{Z} = (0)$. Let $B(\mathscr{N}, \mathscr{N})$ be the range of B^0 in $\mathscr{A}$. Clearly the nilradical $\mathscr{N}$ of the extension $\mathscr{G}(B, \theta)$ contains an abelian direct factor $\mathscr{D} \subset \mathscr{A}$ if and only $B(\mathscr{N}, \mathscr{N}) \neq \mathscr{A}$. Now, let $\mathscr{S}(\theta) = \{a \in \mathscr{A} : \theta(\mathscr{G})a = (0)\}$. We have $\mathscr{G}(B, \theta)$ contains no nonzero, abelian direct factor if and only if $\mathscr{A}$ can not be written $\mathscr{A} = \mathscr{B} \oplus \mathscr{D}$ where $\mathscr{B} \supset B(\mathscr{G}, \mathscr{G})$, $\theta(\mathscr{G})\mathscr{B} \subset \mathscr{B}$, and $(0) \neq \mathscr{D} \subset \mathscr{S}(\theta)$. In view of this observation our main result follows.

3.8. Theorem. *Let $\mathscr{G}$ be a solvable Lie algebra over the field F, $\mathscr{N}$ a nilpotent ideal of $\mathscr{G}$, $\mathscr{N} \supset [\mathscr{G}, \mathscr{G}]$. The isomorphism classes of solvable Lie algebras $\tilde{\mathscr{G}}$ possessing nilradical $\tilde{\mathscr{N}}$ with k-dimensional centre $\mathscr{A}$, such that $\tilde{\mathscr{G}}/\mathscr{A} \cong \mathscr{G}$, $\tilde{\mathscr{N}}/\mathscr{A} \cong \mathscr{N}$, and without nonzero abelian direct factors, are in bijective correspondence with those* $\mathrm{Aut}\,\mathscr{G} \times \mathrm{Aut}\,\mathscr{A}$*-orbits Ω in $H^2(\mathscr{G}, \mathscr{G}/\mathscr{N}, \mathscr{A})$ (under the action $(\alpha, \psi, B) \to \psi B \alpha$) which satisfy the following conditions.*

1) *If $B \in \Omega \cap H^2(\mathscr{G}, \theta)$, then $\mathscr{A}$ can not be written $\mathscr{A} = \mathscr{B} \oplus \mathscr{D}$ where $\mathscr{B} \supset B(\mathscr{G}, \mathscr{G})$, $\theta(\mathscr{G})\mathscr{B} \subset \mathscr{B}$, and $(0) \neq \mathscr{D} \subset \mathscr{S}(\theta)$.*

2) $\mathscr{S}_{B^0} \cap \mathscr{Z} = (0)$.

3.9. Remark. Theorem 3.8 (respectively Corollary 3.6.) gives an algorithm for constructing all solvable (respectively nilpotent) Lie algebras of dimension n, given those algebras of dimension $<n$. Corollary 3.6 was obtained before by T. Skjelbred and the author, and a systematic application to the classification of all real nilpotent Lie algebras of dimension six can be found in [2].

3.10. Applications. Next in table 1 we apply Theorem 3.8 and Corollary 3.6. to the case where $\mathscr{G}$ is real, solvable, $\dim \mathscr{G} = 4$, and $\dim \mathscr{A} = 1$. Note that only those Lie algebras $\tilde{\mathscr{G}}$ which satisfy $\tilde{\mathscr{N}}/\mathscr{A} = \mathscr{N}$ are tabulated. If $(e_i)_{i=1}^4$ is a fixed basis for $\mathscr{G}$, we let B_{ij}: $\mathscr{G} \times \mathscr{G} \to \mathbf{R}$ denote the bilinear form

$$\left(\sum x_k e_k, \sum y_k e_k\right) \to x_i y_j - x_j y_i, \quad 1 \leqq i < j \leqq 4 .$$

The four-dimensional solvable Lie algebras $\mathscr{G}$ not listed ($\mathscr{G}_{4,4}$ etc.) do not yield any extensions of the above type. See P. Bernat et al., *Représentations des groupes de Lie résolubles*, DUNOD, Paris, 1972, pp. 180–182, for notation.

Table 1. *The case* $\dim \mathscr{G}=4$, $\dim \mathscr{A}=1$, $F=\mathbf{R}$.

$\mathscr{G}$	$\mathscr{N}$	Represent. θ Cocycle B	Lie products in $\mathscr{G}$	Lie products in extension $\mathscr{G}(B,\theta)$	$\mathscr{G}(B,\theta)$
$(\mathscr{G}_1)^4$	$(\mathscr{G}_1)^4$	$\theta=0$ $B_{12}+B_{34}$	0	$[e_1,e_2]=e_5$ $[e_3,e_4]=e_5$	$\mathscr{G}_{5,1}$
$\mathscr{G}_{3,1}\times\mathscr{G}_1$	$\mathscr{G}_{3,1}\times\mathscr{G}_1$	$\theta=0$ $B_{14}+B_{23}$	$[e_1,e_2]=e_3$	$[e_1,e_4]=e_5$ $[e_2,e_3]=e_5$	$\mathscr{G}_{5,2}$
$\mathscr{G}_{4,1}$	$\mathscr{G}_{3,1}$	$\theta(e_1)e_5=2e_5$ B_{34}	$[e_1,e_3]=e_3[e_1,e_4]=e_4$	$[e_1,e_5]=2e_5$ $[e_3,e_4]=e_5$	$\mathscr{G}_{5,3}$
	(e_2,e_3,e_4)	$\theta(e_1)e_5=e_5$ B_{24}	$[e_2,e_3]=e_4$	$[e_1,e_5]=e_5$ $[e_2,e_4]=e_5$	$\mathscr{G}_{5,4}$
$\mathscr{G}_{4,2}$	$(\mathscr{G}_1)^2$ (e_3,e_4)	$\theta(e_1)e_5=2e_5$ $\theta(e_2)=0$ B_{34}	$[e_1,e_3]=e_3$ $[e_1,e_4]=e_4$ $[e_2,e_3]=-e_4[e_2,e_4]=e_3$	$[e_1,e_5]=2e_5$ $[e_3,e_4]=e_5$	$\mathscr{G}_{5,5}$
$\mathscr{G}_{4,3}$	$\mathscr{G}_{4,3}$	$\theta=0$ B_{14}	$[e_1,e_2]=e_3$	$[e_1,e_4]=e_5$	$\mathscr{G}_{5,6}$
		$\theta=0$ $B_{14}+B_{23}$	$[e_1,e_3]=e_4$	$[e_1,e_4]=e_5$ $[e_2,e_3]=e_5$	$\mathscr{G}_{5,7}$
$\mathscr{G}_{4,9}(\alpha)$ $0\leqq\alpha\leqq 2$ $\alpha\neq 1$	$\mathscr{G}_{3,1}$ (e_2,e_3,e_4)	$\alpha=0$: $\theta(e_1)e_5$ $=e_5$, B_{34}	$[e_1,e_3]=e_3[e_2,e_3]=e_4$ $[e_1,e_2]=-e_2$	$[e_1,e_5]=e_5$ $[e_3,e_4]=e_5$	$\mathscr{G}_{5,8}$
		$\alpha=2$: $\theta(e_1)e_5$ $=3e_5$, B_{34}	$[e_1,e_3]=e_3[e_2,e_3]=e_4$ $[e_1,e_2]=e_2$ $[e_1,e_4]=2e_4$	$[e_1,e_5]=3e_5$ $[e_3,e_4]=e_5$	$\mathscr{G}_{5,9}$
		$\alpha\neq 0,2$: $\theta(e_1)e_5$ $=(2\alpha-1)e_5$, B_{24}	$[e_1,e_3]=e_3[e_2,e_3]=e_4$ $[e_1,e_2]=(\alpha-1)e_2$	$[e_1,e_5]=(2\alpha-1)e_5$ $[e_2,e_4]=e_5$	$\mathscr{G}_{5,10}(\alpha)$
		$\theta(e_1)e_5$ $=(\alpha+1)e_5$, B_{34}	$[e_1,e_4]=\alpha e_4$	$[e_1,e_5]=(\alpha+1)e_5$ $[e_3,e_4]=e_5$	$\mathscr{G}_{5,11}(\alpha)$
$\mathscr{G}_{4,10}$	$\mathscr{G}_{3,1}$ (e_2,e_3,e_4)	$\theta(e_1)e_5=3e_5$ B_{24}	$[e_2,e_3]=e_4[e_1,e_3]=e_3$ $[e_1,e_2]=e_2+e_3[e_1,e_4]=2e_4$	$[e_1,e_5]=3e_5$ $[e_2,e_4]=e_5$	$\mathscr{G}_{5,12}$

Added in Proof: Table 1 is incomplete.

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