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EMBEDDING-OBSTRUCTION FOR PRODUCTS OF NON-SINGULAR, PROJECTIVE VARIETIES

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In [1] Lluis has shown that a projective, non-singular embedded variety $X \hookrightarrow \boldsymbol{P}_k^N$ over an algebraically closed field k can always be embedded in $\boldsymbol{P}_k^{2\dim(X)+1}$ via a projection from $\boldsymbol{P}_k^N$, but not always in $\boldsymbol{P}_k^{2\dim(X)}$. On the other hand it is known (see for instance [2]) that the product of two projective spaces $X=\boldsymbol{P}_k^m\times\boldsymbol{P}_k^n$ embedded by the Segre-embedding in $\boldsymbol{P}_k^N$, $N=mn+m+n$, can be embedded in $\boldsymbol{P}_k^{2\dim(X)-1}$ via a projection from $\boldsymbol{P}_k^N$. This motivates the following general problem: Let X_i be projective, nonsingular embedded varieties of dimension $n_i \geqq 1$, $i=1,\ldots,r$, and let

$$\varphi\colon X = X_1\times\ldots\times X_r \hookrightarrow \boldsymbol{P}_k^N, \qquad N = (n_i+i)\ldots(n_r+1)-1 ,$$

be the Segre-embedding. Find the embedding-dimension $e=e(n_1,\ldots,n_r)\in \mathbf{Z}_+$ such that X can be embedded in $\boldsymbol{P}_k^e$ but not in $\boldsymbol{P}_k^{e-1}$ via a projection from $\boldsymbol{P}_k^N$. In the present note we shall solve this problem by studying the Segre-classes of X, and by using Holme's general result in [4] (see also [5]) which characterizes e in terms of the degree of these classes. We prove that $e(n_1,n_2)=2(n_1+n_2)-1$ if and only if $X_1=\boldsymbol{P}_k^{n_1}$ and $X_2=\boldsymbol{P}_k^{n_2}$, and that otherwise we always have $e(n_1,\ldots,n_r)=2(n_1+\ldots+n_r)+1$.

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0. Preliminaries.

In the following we let k be an algebraically closed field, and X a projective, non-singular k-variety of dimension n embedded in $\boldsymbol{P}_k^N$ by $i\colon X\hookrightarrow \boldsymbol{P}_k^N$. This embedding induces the group-homomorphism of Chow-rings

$$i_*\colon A(X)\to A(\boldsymbol{P}^N) = \mathbf{Z}[T]/T^{N+1}$$

of degree $N-n$, and the ring-homomorphism

$$i^*\colon A(\boldsymbol{P}_k^N)\to A(X)$$

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of degree 0. We may then define the degree deg $(\alpha) \in \mathbf{Z}$ of an element $\alpha \in A^i(X)$ by $i_*(\alpha) = \deg(\alpha) T^{N-n+i}$.

Further, let $c(X) = c(\Omega^{1*}_{X/k})$ be the total Chern-class of X, and let

$$s(X) = c(X)^{-1} = 1 + s_1(X) + \ldots + s_n(X) \in A(X)$$

be the total Segre-class of X. We then have the degree of the kth Segre-class $\deg(s_k(X)) = d_k(X) = d_k \in \mathbf{Z}$.

We now define the "embedding-obstructions" of X by

$$\beta_j(X) = \beta_j = \sum_{k=0}^{j-n} \binom{j+1}{j-n-k} d_k - d_0^2$$

for $n \leqq j \leqq 2n$, $\beta_j = 0$ for $j > 2n$. The main result of [3] is then the following:

THEOREM 0.1. *Let $n \leqq m < N$. Then the projective nonsingular embedded variety $X \hookrightarrow \mathbf{P}_k^N$ can be embedded in $\mathbf{P}_k^m$ via a projection from $\mathbf{P}_k^N$ if and only if $\beta_j = 0$ for every $j \geqq m$.*

We shall need the following result on the embedding-obstructions in section 2. It also gives a stronger version of Theorem 0.1.:

PROPOSITION 0.2. *The embedding-obstructions $\beta_j(X)$ satisfy*

$$0 \geqq \beta_{2n} \geqq \cdots \geqq \beta_n .$$

PROOF. We know from [4] that $-2^{-1}\beta_{2n}$ is *the secant-number* of X, which gives $\beta_{2n} \leqq 0$. To show the proposition we then need to show that $\beta_{m+1} - \beta_m \geqq 0$ when $n \leqq m \leqq 2n-1$. We may assume $m < N$. Then there exists a linear subspace $L \subseteqq \mathbf{P}_k^N$ of dimension $N-m-1$ such that $L \cap X = \emptyset$. Let $\pi \colon X \to \mathbf{P}_k^m$ be induced by the projection with center L. From [6, p. 160], we have further: Let $S_1 \subseteqq X$ be the closed subset

$$\{x \in X \mid \dim_{k(x)} (\Omega^1_{X/\mathbf{P}_k^m} \otimes k(x)) \geqq 1\} .$$

Then S_1 is of pure codimension $m-n+1$, and we may define the cycle $S_1 = \sum v_z Z$, where $v_z = l_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/J)$. Here x is the generic point of the component Z and J is the 0-th Fitting-ideal of $(\Omega^1_{X/\mathbf{P}_k^m})x$. In [6] it is proved that

$$\mathrm{cl}_X(S_1) = \sum_{k=0}^{m-n+1} \binom{m+1}{m-n+1-k} i^*(T^{m-n+1-k}) s_k \in A^{m-n+1}(X) .$$

This immediately gives

$$i_*(\mathrm{cl}_X(S_1)) = \sum_{k=0}^{m-n+1} \binom{m+1}{m-n+1-k} d_k T^{N-2n+m+1}$$

and so we get

$$\deg(\mathrm{cl}_X(S_1)) = \sum_{k=0}^{m-n+1} \binom{m+1}{m-n+1-k} d_k = \beta_{m+1} - \beta_m .$$

On the other hand it is clear from the definition of the cycle S_1 that $\deg(\mathrm{cl}_X(S_1)) \geqq 0$, and the proposition is proved.

1. The embedding-dimension of $P_k^m \times P_k^n$.

Let $m, n \geqq 1$. We have a graded, surjective k-homomorphism

$$S = k[Y_{ij} \mid 0 \leqq i \leqq m,\ 0 \leqq j \leqq n] \to k[X_i\bar{X}_j \mid 0 \leqq i \leqq m,\ 0 \leqq j \leqq n] = T$$

$$Y_{ij} \mapsto X_i\bar{X}_j$$

which induces the closed Segre-embedding

$$\varphi_{m,n}\colon \mathrm{Proj}(T) = P_k^m \times P_k^n \to P_k^N = \mathrm{Proj}(S)$$

where $N = mn + m + n$. Let

$$P_k^m \xleftarrow{\mathrm{pr}_1} P_k^m \times P_k^n \xrightarrow{\mathrm{pr}_2} P_k^n$$

be the projections. Then we have $A(P_k^m \times P_k^n) = \mathbf{Z}[s, t]$ with $s^{m+1} = t^{n+1} = 0$, where s and t are pr_1^* and pr_2^* of the class of a hyperplane in P_k^m and P_k^n respectively.

PROPOSITION 1.1. *The group-homomorphism* $(\varphi_{m,n})_*\colon A(P_k^m \times P_k^n) \to A(P_k^N)$ *is given by*

$$s^{m_1}t^{n_1} \to \binom{m+n-(m_1+n_1)}{m-m_1} T^{mn+m_1+n_1} .$$

PROOF. We have

$$s^{m_1}t^{n_1} = \mathrm{cl}_{P_k^m \times P_k^n}(P^{m-m_1} \times P^{n-n_1}) ,$$

where P^{m-m_1} and P^{n-n_1} are the linear subspaces of P_k^m and P_k^n defined by $X_0 = \ldots = X_{m_1-1} = 0$ and $\bar{X}_0 = \ldots = \bar{X}_{n_1-1} = 0$, respectively. If we identify a linear subspace P^t with P_k^t, $\varphi_{m,n}$ induces the Segre-embedding.

$$\varphi_{m-m_1, n-n_1}\colon P_k^{m-m_1} \times P_k^{n-n_1} \hookrightarrow P_k^M$$

where $M = (m-m_1)(n-n_1) + (m-m_1) + (n-n_1)$. Thus it is enough to show that $(\varphi_{m,n})_*(1) = \binom{m+n}{m} T^{mn}$, or that $P_k^m \times P_k^n$ is of degree $\binom{m+n}{m}$ in P_k^N. For this, see [3, p. 54, exercise 7.1].

We now will compute the embedding-obstructions of $P_k^m \times P_k^n$. We have

$$\Omega^1_{P_k^m \times P_k^n/k} = \mathrm{pr}_1^* (\Omega^1_{P_k^m/k}) \oplus \mathrm{pr}_2^* (\Omega^1_{P_k^n/k})$$

which gives the total Chern-class of $P_k^m \times P_k^n$ as

$$\begin{aligned} c(\Omega^{1*}_{P_k^m \times P_k^n/k}) &= \mathrm{pr}_1 * (c(\Omega^{1*}_{P_k^m/k}))\, \mathrm{pr}_2 * (c(\Omega^{1*}_{P_k^n/k})) \\ &= (1+s)^{m+1}(1+t)^{n+1} \in A(P_k^m \times P_k^n)\,. \end{aligned}$$

By using the identity $(1+X)^{-(n+1)} = \sum_{i=0}^{\infty} (-1)^i \binom{n+i}{i} X^i$ we find the kth Segre-class as

$$s_k(P_k^m \times P_k^n) = (-1)^k \sum_{\substack{i+j=k \\ 0 \leqq i \leqq m,\, 0 \leqq j \leqq n}} \binom{m+i}{i}\binom{n+j}{j} s^i t^j, \quad 0 \leqq k \leqq m+n\,,$$

which together with Proposition 1.1 give the degree

$$d_k = (-1)^k \sum_{\substack{i+j=k \\ 0 \leqq i \leqq m,\, 0 \leqq j \leqq n}} \binom{m+i}{i}\binom{n+j}{j}\binom{m+n-k}{m-i}, \quad 0 \leqq k \leqq m+n\,.$$

Now d_k occurs in the general formula for β_r, $m+n \leqq r \leqq 2(m+n)$, with the contribution $\binom{r+1}{r-(m+n)-k} d_k$, when $0 \leqq k \leqq r-(m+n)$. Further, from the formulas above it is clear that

$$(-1)^{i+j}\binom{m+i}{i}\binom{n+j}{j}\binom{m+n-(i+j)}{m-i}$$

is one of the terms of d_{i+j}, $0 \leqq i \leqq m$, $0 \leqq j \leqq n$. Thus we get β_r by adding terms of the type

$$(-1)^{i+j}\binom{m+i}{i}\binom{n+j}{j}\binom{m+n-(i+j)}{m-i}\binom{r+1}{r-(m+n)-(i+j)}$$

for those i and j that satisfy $0 \leqq i+j \leqq r-(m+n)$. But for $r-(m+n) < i+j \leqq m+n$ the last factor in this term equals 0, and so we may let the summation range over all i and j, $0 \leqq i \leqq m$, $0 \leqq j \leqq n$. This gives

$$\beta_{2(m+n)-t} + \binom{m+n}{m}^2 =$$

$$\sum_{i=0}^{m} (-1)^i \binom{m+i}{i} \sum_{j=0}^{n} (-1)^j \binom{n+j}{j} \binom{m+n-(i+j)}{m-i} \binom{2(m+n)+1-t}{m+n-(i+j)-t},$$

$$0 \leqq t \leqq m+n .$$

We want to look at $\beta_{2(m+n)-t}$ for small values of t, and therefore we will reformulate the expression for $\beta_{2(m+n)-t}$ so that the summation ranges from 0 to t. To do this we need the following combinatorial identities:

1) $\binom{m-t}{n-t} = \sum_{k=0}^{t} (-1)^k \binom{t}{k} \binom{m-k}{n}$

2) $\sum_{k=0}^{n} (-1)^k \binom{r}{k} \binom{m}{n-k} = \binom{m+k-r-1}{n}$ (Vandermonde)

3) $\binom{t}{s}\binom{s}{r} = \binom{t}{r}\binom{t-r}{s-r}$

We get

$$\beta_{2(m+n)-t} + \binom{m+n}{m}^2 =$$

$$\sum_{s=0}^{t} (-1)^s \binom{t}{s} \sum_{i=0}^{m} (-1)^i \binom{m+i}{i} \sum_{j=0}^{n} (-1)^j \binom{n+j}{j} \binom{m+n-(i+j)}{m-i} \binom{2(m+n)+1-s}{m+n-(i+j)} \quad \text{by 1)}$$

$$= \sum_{s=0}^{t} (-1)^s \binom{t}{s} \sum_{i=0}^{m} (-1)^i \binom{m+i}{i} \binom{2(m+n)+1-s}{m-i} \sum_{j=0}^{n} (-1)^j \binom{n+j}{j} \binom{m+2n+1-s+i}{n-j} \quad \text{by 3)}$$

$$= \sum_{s=0}^{t} (-1)^s \binom{t}{s} \sum_{i=0}^{m} (-1)^i \binom{m+i}{i} \binom{2(m+n)+1-s}{m-i} \binom{m+n+i-s}{m+i-s} \quad \text{by 2)}$$

$$= \sum_{s=0}^{t} (-1)^s \binom{t}{s} \sum_{r=0}^{s} (-1)^r \binom{s}{r} \sum_{i=0}^{m} (-1)^i \binom{m+i}{i} \binom{2(m+n)+1-s}{m-i} \binom{m+n+i-r}{m+i} \quad \text{by 1)}$$

$$= \sum_{r=0}^{t} \sum_{s=0}^{t} (-1)^{r+s} \binom{t}{s} \binom{s}{r} \binom{m+n-r}{m}$$

$$\sum_{i=0}^{m} (-1)^i \binom{m+n+i-r}{i} \binom{2(m+n)+1-s}{m-i} \quad \text{by 2)}$$

$$= \sum_{r=0}^{t} \binom{t}{r} \binom{m+n-r}{m} \sum_{s=0}^{t} (-1)^{r+s} \binom{t-r}{s-r} \binom{m+n-s+r}{m} \quad \text{by 3), 2)}$$

$$= \sum_{r=0}^{t} \binom{t}{r} \binom{m+n-r}{m} \sum_{s=0}^{t-r} (-1)^{s} \binom{t-r}{s} \binom{m+n-s}{m}$$

$$= \sum_{r=0}^{t} \binom{t}{r} \binom{m+n-r}{m} \binom{m+n-t+r}{n} \quad \text{by 1)} .$$

This immediately gives $\beta_{2(m+n)} = \beta_{2(m+n)-1} = 0$. For $t=2$ we get

$$\beta_{2(m+n)-2} = 2\left(\binom{m+n}{m}\binom{m+n-2}{m} - \binom{m+n-1}{m}^2\right) < 0$$

when $m, n \geqq 1$. We thus have proved

PROPOSITION 1.2. *Let $\varphi_{m,n}\colon \boldsymbol{P}_k^m \times \boldsymbol{P}_k^n \hookrightarrow \boldsymbol{P}_k^N$, $N = mn+m+n$, be the Segre-embedding, $m, n \geqq 1$, Then $\boldsymbol{P}_n^m \times \boldsymbol{P}_k^n$ can be embedded into $\boldsymbol{P}_k^{2(m+n)-1}$ via a projection from $\boldsymbol{P}_k^N$, but not into $\boldsymbol{P}_k^{2(m+n)-2}$.*

2. The embedding-dimension of $X_1 \times \ldots \times X_r$.

Let X and Y be projective, non-singular varieties of dimension m and n, respectively. We want to express the obstructions of $X \times Y$ by the obstructions of X and Y. To do this, we first express the degrees of the Segre-classes of X by the obstructions of X. We write the obstructions in matrix-form as

$$\begin{bmatrix} \beta_{2m}(X) \\ \vdots \\ \beta_m(X) \end{bmatrix} = M_m \begin{bmatrix} d_0(X) \\ \vdots \\ d_m(X) \end{bmatrix} - \begin{bmatrix} d_0(X)^2 \\ \vdots \\ d_0(X)^2 \end{bmatrix}$$

or

$$\boldsymbol{\beta}(X) = M_m(\boldsymbol{d}(X)) - \boldsymbol{d}_0(X)^2$$

where $M_m = (a_{ij})$ is the $(m+1) \times (m+1)$-matrix with $a_{ij} = \binom{2m+2-i}{m+2-(i+j)}$.

LEMMA 2.1. *The matrix M_m is non-singular with the inverse $M_m^{-1} = (b_{ij})$ where $b_{ij} = (-1)^{i+j-m}\binom{m+i}{i+j-m-2}$.*

PROOF. We find $\det(M_m) = (-1)^{m+1}$. Let M_m^{-1} be defined as above and let $M_m M_m^{-1} = (c_{ik})$. Then we have

$$c_{ik} = \sum_{j=1}^{m+1} (-1)^{j+k-m} \binom{2m+2-i}{m+2-(i+j)} \binom{m+j}{j+k-m-2}$$

$$= (-1)^{k-m+1} \sum_{j=0}^{m} (-1)^j \binom{2m+2-i}{m+1+j} \binom{m+1+j}{2m+2-k}$$

$$= (-1)^{k-m+1} \binom{2m+2-i}{2m+2-k} \sum_{j=0}^{m} (-1)^j \binom{k-i}{k-(m+1)+j}.$$

The factor $\binom{2m+2-i}{2m+2-k}$ gives $c_{ik}=0$ when $i>k$. For $i \leqq k$ we get

$$\sum_{j=0}^{m} (-1)^j \binom{k-i}{k-(m+1)+j} = (-1)^{-k+m-1} \sum_{j=0}^{k-i} (-1)^j \binom{k-i}{j}$$

$$= (-1)^{-k+m-1}(1-1)^{k-i} = (-1)^{-k+m-1}\delta_{i,k}$$

which gives $c_{ik}=\delta_{i,k}$, and so $M_m M_m^{-1}=I$.

By the lemma we can write

(*) $$\boldsymbol{d}(X) = M_m^{-1}(\boldsymbol{\beta}(X)+\boldsymbol{d}_0(X)^2).$$

The next step is to express the degrees of the Segre-classes of $X \times Y$ by the degrees of the Segre-classes of X and Y. We have the embeddings $i\colon X \hookrightarrow \boldsymbol{P}_k^M$ and $j\colon Y \hookrightarrow \boldsymbol{P}_k^N$ and get the product embedded by

$$\psi\colon X \times Y \xrightarrow{i \times j} \boldsymbol{P}_k^M \times \boldsymbol{P}_k^N \xrightarrow{\varphi_{m,n}} \boldsymbol{P}_k^{MN+M+N}.$$

LEMMA 2.2. *We have*

$$d_k(X \times Y) = \sum_{\substack{i+j=k \\ 0 \leqq i \leqq m,\, 0 \leqq j \leqq n}} d_i(X) d_j(Y) \binom{m+n-k}{m-i}, \quad 0 \leqq k \leqq m+n.$$

PROOF. We identify $A(\boldsymbol{P}_k^M)$ and $A(\boldsymbol{P}_k^N)$ with subrings of $A(\boldsymbol{P}_k^M \times \boldsymbol{P}_k^N)$ in the canonical way. Let pr_1 and pr_2 be the projections of $X \times Y$. Since $s(X \times Y) = \mathrm{pr}_1^*(s(X)) \cdot \mathrm{pr}_2^*(s(Y))$, we get

$$\psi_*(s(X \times Y)) = (\varphi_{M,N})_*[i_*(s(X)) j_*(s(Y))].$$

By using Proposition 1.1. on $(\varphi_{M,N})_*$ the lemma follows.

We also write this in matrix-form as

(**) $$\boldsymbol{d}(X \times Y) = M(Y)\boldsymbol{d}(X)$$

where $M(Y)$ is the $(m+n+1) \times (m+1)$-matrix with $(s+1)$-th column

$$\left(0, \underset{s}{\ldots}, 0, d_0(Y)\binom{m+n-s}{m-s}, \ldots, d_n(Y)\binom{m+n-s-m}{m-s}, 0, \ldots 0\right).$$

In $M(Y)$ we now express the $d_k(Y)$'s by the $\beta_j(Y)$'s by (*), so that we finally can use (*) and (**) to find the obstructions of $X \times Y$ in terms of the obstructions of X and Y. We get

$$\begin{aligned}\boldsymbol{\beta}(X \times Y) &= M_{m+n}(\boldsymbol{d}(X \times Y)) - \boldsymbol{d}_0(X \times Y)^2 \\ &= M_{m+n}M(Y)(\boldsymbol{d}(X)) - \boldsymbol{d}_0(X \times Y)^2 \\ &= M_{m+n}M(Y)M_m^{-1}(\boldsymbol{\beta}(X) + \boldsymbol{d}_0(X)^2) - \boldsymbol{d}_0(X \times Y)^2 .\end{aligned}$$

For $\beta_{2(m+n)}(X \times Y)$ we get after rearrangement:

$$\begin{aligned}\beta_{2(m+n)}(X \times Y) = &\left\{\sum_{i=0}^{m}\sum_{j=0}^{n} C_{i,j} - \binom{m+n}{m}^2\right\} d_0(X)^2 d_0(Y)^2 + &\text{(a)}\\ &+\left\{\sum_{j=0}^{n}\left(\left(\sum_{i=0}^{m} C_{i,j}\right)\beta_{2n-j}(Y)\right)\right\} d_0(X)^2 + &\text{(b)}\\ &+\sum_{i=0}^{m}\left(\sum_{j=0}^{n} C_{i,j}\left(\beta_{2n-j}(Y) + d_0(Y)^2\right)\right)\beta_{2m-i}(X) &\text{(c)}\end{aligned}$$

where

$$C_{i,j} = C(m,n,i,j) =$$
$$\sum_{k=0}^{i} (-1)^k \binom{2m+1-i+k}{k}\binom{2m+j-i-1+k}{2m-1-i+k}\binom{2(m+n)+1}{i-k} .$$

Now we find estimates of (a), (b) and (c) from the following lemma.

LEMMA 2.3.

I) $\sum_{i=0}^{m}\sum_{j=0}^{n} C_{i,j} = \binom{m+n}{m}^2$

II) $\sum_{i=0}^{m} C_{i,j} > 0$ *for* $0 \leqq j \leqq n$.

III) $\sum_{j=0}^{n} C_{i,j} > 0$ *for* $0 \leqq i \leqq m$.

IV) $C_{i,0} > 0$ *when* $0 \leqq i \leqq m$, *and if* $C_{i,j_0} \leqq 0$, $1 \leqq j_0 \leqq n$, *we also have* $C_{i,j} < 0$ *when* $j_0 < j \leqq n$.

V) $\sum_{j=0}^{n} C_{i,j}(\beta_{2n-j}(Y) + d_0(Y)^2) > 0, \quad 0 \leqq i \leqq m$.

PROOF. I): Let $X = \boldsymbol{P}_k^m$, $Y = \boldsymbol{P}_k^n$. Then, since the obstructions of a projective space are all trivial, we get (b) = (c) = 0, or $\beta_{2(m+n)} =$ (a). On the other hand $\beta_{2(m+n)} = 0$ from Proposition 1.2., and I) follows.

II): We expand the coefficient $\binom{2m+1-i+k}{k}$ in the expression for $C_{i,j}$ as

$$\binom{2m+1-i+k}{k} = \binom{2m-1-i+k}{k} + 2\binom{2m-1-i+k}{k-1} + \binom{2m-1-i+k}{k-2}.$$

Then we can use (three times) the identities 3) and 2) from section 1 to get

$$C_{i,j} = \binom{2m+j-i-1}{j}\binom{2n-j+i+1}{i} - 2\binom{2m+j-i}{j}\binom{2n-j+i}{i-1} + $$

$$+ \binom{2m+j-i+1}{j}\binom{2n-j+i-1}{i-2}.$$

From this we get, with $\alpha_{i,j} = \binom{2m+j-i-1}{j}\binom{2n-j+i+1}{i}$,

$$\begin{aligned}\sum_{i=0}^{m} C_{i,j} &= \sum_{i=0}^{m} \alpha_{i,j} - 2\sum_{i=0}^{m-1} \alpha_{i,j} + \sum_{i=0}^{m-2} \alpha_{i,j} \\ &= \binom{m+j-1}{j}\binom{2n-j+1+m}{m} - \binom{m+j}{j}\binom{2n-j+m}{m-1} \\ &= \frac{(m+j-1)!\,(2n-j+m)!\,(2(n-j)+1)}{j!\,(m-1)!\,m!\,(2n-j+1)!} = B(m,n,j)\end{aligned}$$

which clearly is >0 when $0 \leqq j \leqq n$.

III) Straightforward calculations on the expression for $C_{i,j}$ found in II) give

$$C_{i,j} = \frac{(2m+j-i-1)!\,(2n-j+i-1)!\,A_{i,j}}{i!\,j!\,(2m-i+1)!\,(2n-j+1)!}$$

where

$$\begin{aligned}A_{i,j} = {} & 2n(2n+1)i^2 - 2n(8mn+2n+4m+1)i + \\ & + 2m(2m+1)j^2 - 2m(8mn+2m+4n+1)j \\ & + 4mn(4mn+2m+2n+2ij+1)\,.\end{aligned}$$

From this we see that $C_{i,j} = C(m,n,i,j) = C(n,m,j,i)$, and so we get

$$\sum_{j=0}^{n} C_{i,j} = B(n,m,i) = \frac{(n+i-1)!\,(2m-i+n)!\,(2(m-i)+1)}{i!\,(n-1)!\,n!\,(2m-i+1)!}$$

which is >0 when $0 \leqq i \leqq m$.

IV): From III) we see that the sign of $C_{i,j}$ only depends on the sign of $A_{i,j}$. $j=0$ gives

$$A_{i,0} = 2n(2n+1)i^2 - 2n(8mn+2n+4m+1)i + 4mn(4mn+2m+2n+1)\ .$$

Clearly $A_{0,0}>0$, and $A_{m,0}=2mn(2mn+m+2n+1)>0$. But $A_{i,0}$ takes the minimum value when $i=(8mn+2n+4m+1)/2(2n+1)$ which is $>m$, and the first part of IV) is clear.

Finally, keeping i fixed, $A_{i,j}$ takes the minimum value when $j=(8mn+2m+4n+1-4in)/2(2m+1)$ which is $>n$ when $0\leqq i\leqq m$, and this means that $A_{i,j}$ is strongly monotonically decreasing with $0\leqq j\leqq n$.

V): Let $b_j=\beta_{2n-j}(Y)+d_0(Y)^2$, $0\leqq j\leqq n$. From Proposition 0.2. we have $b_0 \geqq b_1\geqq\ldots\geqq b_n=d_0(Y)\geqq 1$, and so if $C_{i,j}>0$ for $0\leqq j\leqq n$, we are finished. If not, let j_0 be the *least* index such that $C_{i,j_0}\leqq 0$. Then, using IV), we get

$$\begin{aligned}\sum_{j=0}^{n} C_{i,j}b_j &= C_{i,0}b_0+\ldots+C_{i,j_0}b_{j_0}+\ldots+C_{i,n}b_n\\ &\geqq C_{i,0}b_0+\ldots+b_{j_0}(C_{i,j_0}+\ldots+C_{i,n})\\ &\geqq b_{j_0}\left(\sum_{j=0}^{n} C_{i,j}\right)\end{aligned}$$

and V) follows by using III).

Parts II) and V) of Lemma 2.3. immediately give that $\beta_{2(m+n)}(X\times Y)=0$ if and only if $\beta_{2m}(X)=\ldots=\beta_m(X)=0$ and $\beta_{2n}(Y)=\ldots=\beta_n(Y)=0$, that is if and only if $X=\boldsymbol{P}_k^m$ and $Y=\boldsymbol{P}_k^n$. Since a product of projective varieties is never isomorphic to a projective space, this result also gives $\beta_{2N}(X)<0$ when X is a product of three or more projective varieties, $N=\dim(X)$. Thus we have proved

THEOREM 2.4. *Let X_i be projective, non-singular embedded varieties of dimension $n_i\geqq 1$, $i=1,\ldots,r$, and let $X=X_1\times\ldots\times X_r$ be embedded by the Segre-embedding. Then*:

a) *When $r\geqq 3$, the embedding dimension of X is $2(n_1+\ldots+n_r)+1$.*

b) *When $r=2$, the embedding dimension of X is $2(n_1+n_2)+1$, unless $X=\boldsymbol{P}_k^{n_1}$ and $Y=\boldsymbol{P}_k^{n_2}$, in which case it is $2(n_1+n_2)-1$.*

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