MATH. SCAND. 44 (1979), 289–294

PLÜCKER RELATIONS FOR $p(e^{i\theta})$

J. R. QUINE

Introduction.

Let p be a polynomial of degree n. In connection with the problem of determining whether p is univalent in the unit disc, we showed in [4] that the closed curve $p(e^{i\theta})$ has at most $(n-1)^2$ vertices, proving a conjecture of Titus [7]. In a later paper [5], we showed that $p(e^{i\theta})$ may be considered as a real algebraic, rational curve, and its algebraic completion f in PC^2 has certain interesting properties, most notable of which is that the circular points on the line at infinity are (imaginary) points of order n on f. In this paper we derive the four Plücker relations for f. Since Plücker's relations are classically for curves with simple singularities, we must examine carefully the contribution of the multiple singularities on the line at infinity. We discover that the above mentioned result in [4] is a consequence of one of the Plücker relations. We further show that the number μ of zeros of p' symmetric in $|z|=1$ is closely related to the Plücker characteristics of f.

Finally, we show as a consequence of Plücker's relations that f, and therefore $p(e^{i\theta})$, has at most $(n-1)(2n-3)$ double tangents.

1. $p(e^{i\theta})$ is algebraic.

Let $p(z)=\sum_{k=0}^{n} a_k z^k$ where for the duration of the paper we will assume that $a_n \neq 0$, $a_1 \neq 0$, and $n \geqq 1$. Let

$$\bar{p}(z) = \sum_{k=0}^{n} \bar{a}_k z^k \quad \text{and} \quad p^*(z) = z^n \bar{p}(1/z) = \sum_{k=0}^{n} \bar{a}_{n-k} z^k .$$

We will study $p(e^{i\theta})$ using the method of inversive geometry (see Morley [3]). We write $\eta=p(e^{i\theta})$ in $(\eta,\bar{\eta})$ coordinates: $\eta=p(z)$, $\bar{\eta}=\bar{p}(1/z)$ where $z=e^{i\theta}$. Thus we consider the rational curve $z \to (p(z),\bar{p}(1/z))$. To study the algebraic completion in PC^2, we let $[\eta,\xi,\zeta]$ be the point in PC^2 with the given homogeneous coordinates, with the line $\zeta=0$ being the line at infinity. We now look at the rational curve in PC^2 determined by $(p(z),\bar{p}(1/z),1)=(\eta,\xi,\zeta)$. This determines a map f from the extended complex plane $\bar{\mathbf{C}}$ into PC^2 whose image is the algebraic completion of our original curve. The map

Received September 12, 1978.

$z \to (z^n p(z), p^*(z), z^n)$ from $\mathbf{C}$ into $C^3 - \{0\}$ determines f in local coordinates about $z=0$. The map $w \to (\bar{p}^*(w), w^n \bar{p}(w), w^n)$ with $w = 1/z$ likewise determines f in local coordinates about $z = \infty$. We note that $f(0) = [0,1,0]$ and $f(\infty) = [1,0,0]$. These are the circular points in classical projective geometry and are the only points of f on the line at infinity. The subset of $P\mathbf{C}^2$ given by $\bar{\eta} = \xi$ and ζ real is identified with $P\mathbf{R}^2$. We see that $f(e^{i\theta})$ is in $P\mathbf{C} \cap f(\bar{\mathbf{C}})$, i.e., in the real part of f, and is essentially the point $p(e^{i\theta})$ given in conjugate coordinates. We will often write $p(e^{i\theta})$ for $f(e^{i\theta})$.

2. Plücker's relations.

The classical Plücker relations for algebraic curves in $P\mathbf{C}^2$ with simple singularities are

(1) $$2 - 2g + M + k = 2N$$

(2) $$2 - 2g + N + i = 2M$$

(3) $$(N-1)(N-2) = 2(\delta + k + g)$$

(4) $$(M-1)(M-2) = 2(\tau + i + g)$$

where N is the order, M is the class, i is the number of inflection points, k is the number of cusps, δ is the number of double points, τ is the number of double tangents, and g is the genus (Coolidge [1]). We note that (1) and (2) are dual; (3) and (4) are dual. For curves with multiple singularities such as the one we are considering, things are more complicated; however, if the curve is rational the analysis is easier. The theory of meromorphic curves as in Weyl [9] or Cowen and Griffiths [2] is useful. Suppose, then, that $f\colon \bar{\mathbf{C}} \to P\mathbf{C}^2$ is a rational curve. Then $g = 0$ and Plücker's formulas become

(1′) $$M + \sum_{z \in \bar{\mathbf{C}}} (v_1)_z = 2(N-1)$$

(2′) $$N + \sum_{z \in \bar{\mathbf{C}}} (v_2)_z = 2(M-1)$$

(3′) $$(N-1)(N-2) = \sum_{(z,\zeta) \in \bar{\mathbf{C}} \times \bar{\mathbf{C}}} (\delta_1)_{(z,\zeta)}$$

(4′) $$(M-1)(M-2) = \sum_{(z,\zeta) \in \bar{\mathbf{C}} \times \bar{\mathbf{C}}} (\delta_2)_{(z,\zeta)}$$

where the integers v_1, v_2, δ_1 and δ_2 will be explained in the next paragraph.

The integer $(v_1)_z$ is called the first stationary index at $f(z)$, or the ramification index of f at z. Suppose $\varphi(t) \in \mathbf{C}^3 - \{0\}$ gives f in local coordinates with $t = 0$ corresponding to $z \in \bar{\mathbf{C}}$, then $(v_1)_z$ is the smallest integer a such that $\varphi^{(1+a)}(0)$ and $\varphi(0)$ are linearly independent. Since $(v_1)_z$ is a projective invariant, we may characterize it another way. By a projective linear transformation assume that φ is in the normalized form

$$\varphi(t) = (1+O(t), t^\alpha+O(t^{\alpha+1}), t^\beta+O(t^{\beta+1}))$$

where $0<\alpha<\beta$. Then $(v_1)_z=\alpha-1$. If $f(z)$ is a simple cusp then $(v_1)_z=1$, thus in the special case that f has only simple singularities, (1′) becomes (1) with $g=0$. We define $(\delta_1)_{(z,\zeta)}=0$, unless $f(z)=f(\zeta)$. If $f(z)=f(\zeta)$ with $z\neq\zeta$, then let $\varphi(t)$ and $\psi(s)$ give f in local coordinates about z and ζ respectively as above, and let (a,b,c) be Plücker coordinates of $\varphi(t)\wedge\psi(s)$. Then $(\delta_1)_{(z,\zeta)}$ is defined to be the algebraic intersection number of $b=0$ and $c=0$ at $s=t=0$. If $z=\zeta$, then let (a,b,c) be Plücker coordinates of $\varphi(s)\wedge\varphi(t)/(s-t)$. Then $(\delta_1)_{(z,z)}$ is defined to be the algebraic intersection number of $b=0$ and $c=0$ at $s=t=0$. Note that $(\delta_1)_{(z,z)}>0$ only if $(v_1)_z>0$, $(\delta_1)_{(\zeta,z)}=(\delta_1)_{(z,\zeta)}=1$ if $f(z)=f(\zeta)$ is a simple node, and $(\delta_1)_{(z,z)}=2$ if $f(z)$ is a simple cusp. In the special case, (2′) becomes (2) with $g=0$.

To define v_2 and δ_2, we need the notion of first associated, or dual curve. If $\varphi(t)$ is a local equation for f near z as above and $a=(v_1)_z$, then $\psi(t)=t^{-a}\varphi(t)\wedge\varphi'(t)$ is a local equation for $f_1\colon \bar{\mathbf{C}}\to P\mathbf{C}^2$, the first associated, or dual curve. Now $(v_2)_z$ for f is defined to be $(v_1)_z$ for the dual curve. Likewise $(\delta_2)_{(z,\zeta)}$ for f is defined to be $(\delta_1)_{(z,\zeta)}$ for the dual curve. We note that $(v_2)_z=1$ at a simple inflection point, $(\delta_2)_{(z,\zeta)}=(\delta_2)_{(\zeta,z)}=1$ if $f(z)$ is a simple double tangent where $f_1(\zeta)=f_1(z)$, and $(\delta_2)_{(z,z)}=2$ if $f(z)$ is a simple inflection point.

3. Plücker's relations for $p(e^{i\theta})$.

Let f be the completion of $p(e^{i\theta})$ as in section 1. From the representation $z\to(z^n p(z), p^*(z), z^n)$ for f in $\mathbf{C}$, we see that $N=2n$. By a linear transformation of the image space, this representation can be transformed to one of the form

$$z\to(1+O(z), z^n+O(z^{n+1}), z^{n+1}+O(z^{n+2}))\,.$$

Thus we see that $(v_1)_0=n-1$. Likewise by symmetry $(v_1)_\infty=n-1$. From the representation $z\to(p(z),\bar{p}(1/z),1)$ in $\mathbf{C}-\{0\}$ we see that if $z\neq 0,\infty$, then $(v_1)_z$ is the largest integer a such that z and $1/\bar{z}$ are zeros of order a of p'. We say that p' has μ zeros symmetric in the unit circle if $p'(z)$ and $p'^*(z)=z^{n-1}\overline{p}'(1/z)$ have μ zeros in common including multiplicity. Thus we see that if p' has μ zeros symmetric in the unit circle, then $\mu=\sum_{0<|z|<\infty}(v_1)_z$. Combining this with $(v_1)_0=(v_1)_\infty=n-1$ and (1′) we get

$$M+\mu = 2n \tag{1″}$$

and this is the first Plücker relation for $p(e^{i\theta})$.

We mention another way of deriving (1″). We compute that

$$z \to (p'^*(z), z^{n+1}p'(z), p'^*(z)p(z)+zp^*(z)p'(z))$$

defines f_1 near $z=0$. Let h be the greatest common divisor of p' and p'^* and write

$$p'(z) = h(z)q(z) \quad \text{and} \quad p'^*(z) = h(z)r(z) .$$

Then

$$z \to (r(z), z^{n+1}q(z), r(z)p(z)+zp^*(z)q(z))$$

defines f_1 in C. Since $\mu=\deg h$ we see that the class of $f_1=M=2n-\mu$, and this is exactly (1″).

In connection with (1″) we mention an interesting example. Suppose $\mu=n-1$, i.e. all zeros of p' are symmetric. Here $p'(z)=cp'^*(z)$ where $|c|=1$, and it easily follows as in Suffridge [6] that $d\arg e^{i\theta}p'(e^{i\theta})/d\theta=(n+1)/2$ for $p'(e^{i\theta})\neq 0$. Thus the argument of the tangent to $p(e^{i\theta})$ changes at a constant rate with respect to θ. The total change is $(n+1)\pi$, showing that the apparent class of f, i.e. the class of the real part (see Coolidge [2]), is the same as the class of f which by (1″) is $n+1$.

Now (2′) becomes

$$(2'') \qquad \sum_{z\in\bar{C}} (v_2)_z = 2(n-1)-2\mu .$$

As a simple consequence, there are at most $2(n-1)$ simple inflection points on $p(e^{i\theta})$. In general p' has no symmetric zeros, $\mu=0$, and f has $2(n-1)$ inflection points, some of which may be imaginary.

Next we look at (3′) and compute the contribution to $\sum(\delta_1)_{(z,\zeta)}$ by the singular points $f(0)$ and $f(\infty)$ on the line at infinity. As before, after a linear transformation f is given near $z=0$ by

$$\varphi(z) = (1+O(z), z^n+O(z^{n+1}), z^{n+1}+O(z^{n+2})) .$$

We compute that the last two Plücker coordinates of $\varphi(s)\wedge\varphi(t)/(s-t)$ are of the form $s^n+s^{n-1}t+\ldots+t^n+$ (higher order terms) and $s^{n-1}+s^{n-2}t+\ldots+t^{n-1}+$ (higher order terms). From elementary properties of intersection numbers (Walker [8, p. 114, Theorem 5.11]) these intersect $n(n-1)$ times at $s=t=0$. Thus $(\delta_1)_{(0,0)}=n(n-1)$. Likewise $(\delta_1)_{(\infty,\infty)}=n(n-1)$. Since $f(0)\neq f(z)$ for $z\neq 0$ and $f(\infty)\neq f(z)$ for $z\neq\infty$, (3′) becomes

$$(3'') \qquad \sum_{(z,\zeta)\in C\times C-\{0\}} (\delta_1)_{(z,\zeta)} = 2(n-1)^2 .$$

Thus (3′) gives an alternate proof of the theorem proved in Quine [4] that $p(e^{i\theta})$ has at most $(n-1)^2$ nodes. Taking $p(z)=(rz)^n-n(rz)$ for r large shows that the theorem is sharp, i.e. there is a polynomial p for which all the nodes are on the real part of f.

We now apply Plücker's relation (4′) to f. There are no contributions to $\sum (\delta_2)_{(z,\zeta)}$ from the singular points of f on the line at infinity, since these are not in general singular points of f_1. Combining (2″) and (4′) we get

$$\sum_{(z,\zeta)\in\bar{C}\times\bar{C}} (\delta_2)_{(z,\zeta)} = 2(2n-1)(n-1)+\mu(3-4n)+\mu^2 . \tag{4''}$$

We now use (4″) to prove

THEOREM *The curve $p(e^{i\theta})$ has at most $(2n-3)(n-1)$ simple double tangents, and this bound is sharp.*

PROOF. We prove the theorem in the generic case $\mu=0$. The general case follows either by a similar argument keeping track of terms involving μ, or simply by a continuity argument. If $\mu=0$, we have by (4″) that

$$\sum_{(z,\zeta)\in\bar{C}\times\bar{C}} (\delta_2)_{(z,\zeta)} = 2(2n-1)(n-1)$$

and by (2″) that $\sum_{z\in\bar{C}} (v_2)_z = 2(n-1)$. Now since $(v_2)_z \geqq 1$ implies $(\delta_2)_{(z,z)} \geqq 2$, it follows from the latter equation that

$$\sum_{z\in\bar{C}} (\delta_2)_{(z,z)} \geqq 4(n-1) .$$

Now combining this with the first equation gives

$$\sum_{(z,\zeta)\in\bar{C}\times\bar{C}-\Delta} (\delta_2)_{(z,\zeta)} \leqq 2(2n-3)(n-1) ,$$

where Δ is the diagonal of $\bar{C}\times\bar{C}$. The conclusion of the theorem now follows. To show that the theorem is sharp we look at $p(z)=(rz)^n-n(rz)$ for r slightly larger than 1, and note that this curve has exactly $(2n-3)(n-1)$ double tangents. For example if $n=4$, the curve looks as in Figure 1. This curve has 15 double tangents.

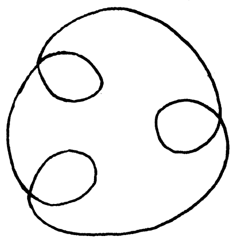

Fig. 1

REFERENCES

1. J. Coolidge, *A Treatise on Algebraic Plane Curves*, Oxford University Press, Oxford, 1931.
2. M. Cowen and P. Griffith, *Holomorphic curves and metrics of negative curvature*, J. Analyse Math. 29 (1976), 93–153.
3. F. Morley and F. V. Morley, *Inversive Geometry*, Ginn, Boston, 1933.
4. J. R. Quine, *On the self-intersections of the image of the unit circle under a polynomial mapping*, Proc. Amer. Math. Soc. 39 (1973), 135–140.
5. J. R. Quine, *The geometry of* $p(S^1)$, Pacific J. Math. 64 (1976), 551–557.
6. T. J. Suffridge, *On univalent polynomials*, J. London Math. Soc. 44 (1969), 496–504.
7. C. J. Titus, *The combinatorial topology of analytic functions on the boundary of a disk*, Acta Math. 106(1961), 45–64.
8. R. J. Walker, *Algebraic Curves* (Princeton Mathematical series 13), Princeton University Press, Princeton, 1950.
9. H. Weyl and J. Weyl, *Meromorphic Functions and Analytic Curves*, Princeton University Press, Princeton, New Jersey, 1943.

THE FLORIDA STATE UNIVERSITY
TALLAHASSEE
FLORIDA 32306

AND

THE UNIVERSITY OF MARYLAND
COLLEGE PARK
MARYLAND 20742
MD78-26-JQ