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A STRONGLY ANNULAR FUNCTION WITH COUNTABLY MANY SINGULAR VALUES

F. W. CARROLL

0. Introduction.

Let D denote the unit disk $\{z : |z|<1\}$, and $H(D)$ the set of functions holomorphic on D. A function f in $H(D)$ is said to be *strongly annular* if there is a sequence $0<r_1<r_2<\ldots, r_n\uparrow 1$, such that

$$\lim_{n\to\infty} \min_{|z|=r_n} |f(z)| = \infty . \tag{0.1}$$

For any finite complex number a, let $Z'(f,a)$ denote the closed subset of the unit circle C consisting of limit points of the set $\{z : f(z)=a\}$. It is known that, for f strongly annular, $Z'(f,a)$ is non-empty, and that the open sets $C\setminus Z'(f,a)$ and $C\setminus Z'(f,b)$ do not overlap for $a\neq b$, [3, Lemma 4.9], [5, p. 491]. Hence, the set $S(f)$ consisting of those a for which $Z'(f,a)\neq C$ is at most countable. With A. Osada [5], we call such a "singular values" for f.

In [3, Problem 6.4], D. D. Bonar asked what cardinalities are possible for $S(f)$. At the time, the set $S(f)$ was empty for all known strongly annular functions. Subsequently, examples were obtained with $S(f)=\{0\}$, (see [2] and [4]), and recently Osada [5] has constructed a strongly annular function f with $S(f)=\{0,1\}$. In this paper, we use his methods to prove

THEOREM 2. *There is a strongly annular function f such that $S(f)$, the set of singular values of f, is countably infinite.*

1. Geometry, and Lemma A.

Let ζ_k be the point on C whose argument is $2\pi(1-2^{-k})$. Thus $\zeta_0=1, \zeta_1=-1, \zeta_2=-i$, etc. Let Π_n denote the polygon $[\zeta_0,\zeta_1,\ldots,\zeta_n,\zeta_0]$ $(n=2,3,\ldots)$. The portion of D outside Π_n consists of $n+1$ segments of D, denoted by $G_0,G_1,\ldots,G_{n-1}$ and G_n'. Here, G_0 is the upper semidisk, G_1 lies in the third quadrant, etc. We note that G_{n-1} and G_n' are congruent, but that when we replace Π_n by Π_{n+1}, G_n' is replaced by two segments G_n and G_{n+1}' whose union

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is strictly smaller than G_n'. We also note that the distance from $z=0$ to the segment G_n' is $\cos(\pi/2^n)$.

Now suppose that $\{r_2, r_3, \ldots\}$ is a sequence of positive numbers such that the circle $|z|=r_n$ intersects G_n in an open arc but does not intersect the closure of G_{n+1}. (The choice of the r_n will be made inductively in the course of the construction.) For $n \geqq 3$ and $0 \leqq j \leqq n$, let $T_{n,j}$ denote the small "triangular" region that is inside Π_n and outside $|z|=r_{n-1}$, and that has a vertex at ζ_j. We denote by $B_{n,j}$ the arc in which $T_{n,j}$ intersects the circle $|z|=r_n$.

LEMMA A. *Let n and j be fixed. There exists a subarc $\sigma_{n,j}$ of the interior of the arc $B_{n,j}$, with the following property. For each pair of positive numbers ε_j and M_j, there is a function h_j in $H(D)$ such that*

$$|h_j(z)| < \varepsilon_j \quad \text{if } z \in D \setminus T_{n,j} \tag{1.1}$$

$$|h_j(z)| > M_j \quad \text{if } z \in \sigma_{n,j} \tag{1.2}$$

$$\operatorname{Re} h_j(z) > 0 \quad \text{if } z \in B_{n,j} \setminus \sigma_{n,j}\,. \tag{1.3}$$

Lemmas of this sort have been used to "close the gaps" in building strongly annular functions: when a function has been constructed that has desired properties on most of the disk and is large on most of the circle $|z|=r_n$, a function h_j can be added such that the sum is large on $\sigma_{n,j}$, and inherits the other properties. Proofs of similar lemmas are in [2] and [4]; for completeness we shall include an outline of the proof of Lemma A at the end of the last section. For the present, we emphasize that the arcs $\sigma_{n,j}$ are independent of M and ε, and we use them as we continue the geometrical discussion.

If we remove the $n+1$ small arcs $\sigma_{n,j}$ $(j=0,1,\ldots,n)$ from the circle $|z|=r_n$, we are left with $n+1$ large arcs $A_{n,j}$ $(j=0,1,\ldots,n)$. For $j=0,1,\ldots,n-1$, $A_{n,j}$ is basically $G_j \cap \{z : |z|=r_n\}$, except that at each end a short extension protrudes into the adjacent triangular region. Similarly, $A_{n,n}$ is the component that meets G_n'.

2. Construction of the functions.

THEOREM 1. *There exist an increasing sequence $0=a_0<a_1<\ldots$ of real numbers, a sequence $\{r_k\}$ in $(0,1)$ increasing to 1, and a sequence of functions $f_1, f_2, f_3, \ldots$ in $H(D)$ such that for $k=2,3,\ldots$, we have*

$$|f_k(z)-f_{k-1}(z)| < 2^{-k} \quad \textit{for } |z| \leqq r_{k-1}\,, \tag{2.1}$$

$$|f_k(z)| > 2^k \quad \textit{for } |z|=r_k\,, \tag{2.2}$$

$$f_k(z)-a_j \textit{ is bounded away from 0 in } G_j \ (0 \leqq j \leqq k-1)\,, \tag{2.3}$$

(2.4) $$f_k(z) - a_k \text{ is bounded away from } 0 \text{ in } G'_k\,,$$

(2.5) $$f_k \text{ is bounded in } G'_k \cup \bigcup_{j=1}^{k-1} G_j\,.$$

Assuming the truth of Theorem 1 for a moment, we present the proof of Theorem 2.

Proof of Theorem 2. It follows from (2.1) that $\{f_k\}$ is a Cauchy sequence in the space $H(D)$; let f be its limit. The inequalities (2.1) and (2.2) show that f is strongly annular. Moreover, (2.3), together with Hurwitz's theorem, shows that $f(z) \neq a_j$ for $z \in G_j$ $(j=0,1,\ldots)$.

Proof of Theorem 1. We take $a_0=0$, $a_1=1$, $a_2=2$,

$$(\cos)(\pi/4) < r_1 < \cos(\pi/8) < r_2 < \cos(\pi/16)\,,$$

and $f_1(z)=f_2(z)\equiv 5$. Thus, all the conditions (2.1)–(2.5) are satisfied for $k=2$. Suppose we have obtained $a_0, a_1, \ldots, a_{n-1}$, $r_1, r_2, \ldots, r_{n-1}$, $f_2, f_3, \ldots, f_{n-1}$ satisfying the requirements of the theorem. Since f_{n-1} is bounded in G'_{n-1}, we may and do choose a real number $a_n > a_{n-1}$ such that $f_{n-1}(z) - a_n$ is bounded away from zero on G'_{n-1}. Next, we choose a number $r_n > r_{n-1}$ such that, in the first place,

$$\cos(\pi/2^{n+1}) < r_n < \cos(\pi/2^{n+2})\,,$$

and secondly $f_{n-1}(z) - a_j$ has no zeros on the circle $|z| = r_n$ for $j=0,1,\ldots,n$. By Runge's theorem, there is a polynomial P_j such that the following approximations hold: we use the notation $\doteq$ to mean "is approximately equal to".

(2.6) (i) $P_j(z) \doteq 0$ on $|z| \leq r_{n-1}$, and also for z in $G_0, G_1, \ldots, G_{j-1}$, $G_{j+1}, \ldots, G'_n$.

(ii) $\{f_{n-1}(z) - a_j\} \exp\{P_j(z)\} + a_j \doteq R$, R a large positive number, for z on $A_{n,j}$.

The square matrix of size $(n+1) \times (n+1)$ with 0's on the diagonal and 1's elsewhere is nonsingular, having a determinant $(-1)^n n$. Hence the system

$$\sum_{i \neq j} \alpha_i = a_j \qquad (j=0,1,\ldots,n)$$

has a unique solution. Using these values for $\alpha_0, \ldots, \alpha_n$, we define

$$g(z) = \left\{ f_{n-1}(z) - \sum_{i=0}^{n} \alpha_i \right\} \exp\left(\sum_{i=0}^{n} P_i(z) \right) + \sum_{i=0}^{n} \alpha_i \exp(P_i(z))\,.$$

On $|z| \leqq r_{n-1}$, we see that $g(z)$ approximates $f_{n-1}(z)$, and the tolerance on the P_j's can be chosen so that

$$|g(z) - f_{n-1}(z)| < 2^{-n} \quad \text{for } |z| \leqq r_{n-1} .$$

On the region G_j (respectively G_n'), only P_j (respectively P_n) can differ greatly from zero, so that there $g(z)$ is approximately

$$\text{(2.7)} \qquad \Big\{ f_{n-1}(z) - \sum_{i \neq j} \alpha_i \Big\} \exp (P_j(z)) + \sum_{i \neq j} \alpha_i$$
$$= \{ f_{n-1}(z) - a_j \} \exp (P_j(z)) + a_j, \qquad (z \in G_j \cup A_{n,j}) .$$

Since $f_{n-1}(z)$ is bounded away from a_j on G_j (from a_n on G_n'), the same will be true of $g(z)$ provided that the P_i's $(i \neq j)$ are close enough to zero there. In view of (2.6 (ii)), Re $(g(z))$ will be large on $\bigcup \{A_{n,j} : j = 0, 1, \ldots, n\}$ provided R is large enough. We shall assume that it has been chosen so that

$$\text{(2.8)} \qquad \operatorname{Re}[g(z)] > 2^n, \quad \text{for } z \in \bigcup_{j=0}^{n} A_{n,j} .$$

Now g is a holomorphic function having all the desired properties of the function f_n we seek, with one exception: it is not known to have large modulus on the gaps $\sigma_{n,j}$ between the arcs $A_{n,j}$.

We invoke Lemma A to close each gap successively. The ε_j in (1.1) are chosen so small that the new function

$$f_n(z) = g(z) + h_0(z) + \ldots + h_n(z)$$

remains bounded away from a_j in G_j $(0 \leqq j \leqq n-1)$ and away from a_n in G_n', and is large on $A_{n,j} \cap G_j$ $(0 \leqq j \leqq n-1)$ and on $A_{n,n} \cap G_n'$. On $\sigma_{n,j}$, the function f_n will have large modulus if we choose the M_j in (1.2) to be sufficiently large. Finally, on the protrusions of the $A_{n,j}$ into the triangular regions, f_n will be large in view of (1.3) and (2.8). This concludes the proof of Theorem 1.

PROOF OF LEMMA A. If Ω denotes the extended complex plane, then $\Omega \setminus B_{n,j}$ is simply connected and hence conformally equivalent to $\Omega \setminus K$, where K is the union of the closed disk $\{\zeta : |\zeta| \leqq 1\}$ and the line segment $[-2, 2]$. The conformal mapping φ is continuous at the boundary (in the sense of prime ends) and can and will be chosen so that the two endpoints of $B_{n,j}$ map into -2 and 2. We take $\sigma_{n,j}$ to be the arc of $B_{n,j}$ whose two sides correspond under φ to the top and bottom semicircles of $|\zeta| = 1$. (Symmetry justifies this assertion). Now φ maps the boundary to $T_{n,j}$ onto a closed curve in $|\zeta| > 1$. Choose a number ϱ, $1 < \varrho < 2$, such that $|\varphi(z)| > \varrho$ for all z on $\partial T_{n,j}$. Then for all sufficiently large positive integers m, the function

$$\Psi(z) = (\varrho/\varphi(z))^{2m}$$

is holomorphic on $\Omega \setminus B_{n,j}$, is small on $D \setminus T_{n,j}$ and large in modulus on $\sigma_{n,j}$, and it has positive real part on $B_{n,j} \setminus \sigma_{n,j}$. Let T be the interior of $T_{n,j} \cap \{z : |z| > r_n\}$ and let E be $D \setminus T$. Then E is a relatively closed subset of D, and ψ is continuous on E and holomorphic on its interior.

For every point z in $D \setminus E$ (that is, for every z in T), there is a continuous curve τ_z mapping $[0, 1)$ into $D \setminus E$ which satisfies (a) $\tau_z(0) = z$ and $\lim_{t \to 1} |\tau_z(t)| = 1$, and (b) for every $\varepsilon > 0$, there is a $\delta > 0$ such that $1 > |z| > 1 - \delta$ implies $|\tau_z(t)| > 1 - \varepsilon$, $0 \leqq t < 1$. That is, E belongs to Arakelian's class K_D. Arakelian's Theorem [1] states that for a relatively closed subset E of D, the condition $E \in K_D$ is (necessary and) sufficient in order that every complex function continuous on E and holomorphic in its interior can be uniformly approximated on E by a function holomorphic on D. Hence ψ can be approximated uniformly on $D \setminus T$ by a function h_j in $H(D)$, which will then have the desired properties (1.1), (1.2), and (1.3).

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THE OHIO STATE UNIVERSITY
COLUMBUS, OHIO 43210
U.S.A.