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A RESULT ON SPECTRAL SYNTHESIS OF DIRECT PRODUCTS

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It is well known that for locally compact abelian groups G a closed subset S of $\hat{G}$ with scattered boundary ∂S is a C-set, see [1, Theorem 2.5.1] or [2, Ch. 2, § 5.3]. We shall show that under certain conditions the product $S \cdot T$ of a closed subset S with scattered boundary ∂S in a closed subgroup $(G/H)\hat{} \subseteq \hat{G}$ and a compact C-set T in $\hat{G}$ is an S-set, respectively C-set, if the product $(G/H)\hat{} \cdot T$ is direct. ($(G/H)\hat{} \cdot T$ is a direct product, if every element λ of $(G/H)\hat{} \cdot T$ has a unique representation $\lambda = \beta\gamma$, $\beta \in (G/H)\hat{}$, $\gamma \in T$. This is equivalent to the injectivity of r on T, where $r \colon \hat{G} \to \hat{H}$ denotes the restriction map of $\hat{G}$ on $\hat{H}$). More precisely we shall prove the following theorem, the demonstration of which is based on an appropriate modification of the technique of local membership. We shall use the notation of [1].

THEROEM. *Let G be a locally compact abelian group, H a closed subgroup and $r \colon \hat{G} \to \hat{H}$ the restriction map of $\hat{G}$ on $\hat{H}$. Let $T \subseteq \hat{G}$ be a compact* C-set, *such that r is one-to-one on T, and let $S \subseteq (G/H)\hat{}$ be a closed set with scattered boundary ∂S.*

a) *If $(G/H)\hat{} \cdot T$ is an* S-set, *then $S \cdot T$ is an* S-set.

b) *If $(G/H)\hat{} \cdot T$ is a* C-set *and the approximate identity of $k((G/H)\hat{} \cdot T)$ is bounded in the "multiplier norm" on $k(S \cdot T)$, (i.e. $\|f\| = \sup\{\|f * g\|_1 : g \in k(S \cdot T),\ \|g\|_1 = 1\}$), then $S \cdot T$ is a* C-set.

PROOF. a) Suppose that I is a closed ideal in $L^1(G)$ with $Z(I) = S \cdot T$, and let $f \in L^1(G)$.

(i) We prove that if f satisfies the following 2 conditions, then $f \in I$:

(1) there exist a compact set $C \subseteq (G/H)\hat{}$ and a $g \in I$ such that $f - g \in k(((G/H)\hat{} \setminus C) \cdot T)$;

(2) for each $\lambda \in (G/H)\hat{}$ there exist an open neighbourhood V of λ in $(G/H)\hat{}$ and a $g \in I$ such that $f - g \in k(V \cdot T)$.

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(This is the modification of the concept of "belonging locally" referred to in the first paragraph.)

Let $C \subseteq (G/H)\hat{}$ be such a compact set and $g_0 \in I$ such that $f-g_0 \in k(((G/H)\hat{} \setminus C)\cdot T)$. Then there are open $V_i \subseteq (G/H)\hat{}$, $g_i \in I$ such that $f-g_i \in k(V_i \cdot T)$, $i=1,\ldots,n$, for which $C \subseteq \bigcup_{i=1}^n V_i$. Denote

$$J = k(((G/H)\hat{} \setminus V_1)\cdot T)+\ldots+k(((G/H)\hat{} \setminus V_n)\cdot T).$$

Then

$$Z(J) = \bigcap_{i=1}^{n} ((G/H)\hat{} \setminus V_i)\cdot T = \left(\bigcap_{i=1}^{n} (G/H)\hat{} \setminus V_i\right)\cdot T \subseteq ((G/H)\hat{} \setminus C)\cdot T$$

by the injectivity of r on T. Since r is one-to-one on T, $C\cdot T \cap Z(J)=\emptyset$. Hence there is a $u \in L^1(G)$ such that $\hat{u}=1$ in a compact neighbourhood of $C\cdot T$ and $\operatorname{supp}\hat{u}$ is compact and disjoint from $Z(J)$. Consequently, $u=\sum_{i=1}^n u_i$, $u_i \in k(((G/H)\hat{} \setminus V_i)\cdot T)$, and u is an identity modulo $k(C\cdot T)$. Thus

$$f = \left(g_0 - u * g_0 + \sum_{i=1}^{n} u_i * g_i\right) + \left((f-g_0) - u*(f-g_0) + \sum_{i=1}^{n} u_i * (f-g_i)\right)$$
$$\in I + k((G/H)\hat{} \cdot T),$$

because

$$u_i * (f-g_i) \in k\big(((G/H)\hat{} \setminus V_i)\cdot T\big) \cap k(V_i\cdot T) = k((G/H)\hat{} \cdot T) \quad \text{for } i=1,\ldots,n,$$

and

$$(f-g_0)-u*(f-g_0) \in k(C\cdot T) \cap k(((G/H)\hat{} \setminus C)\cdot T) = k((G/H)\hat{} \cdot T).$$

Since $(G/H)\hat{} \cdot T$ is an S-set, we see that $k((G/H)\hat{} \cdot T) = \overline{J((G/H)\hat{} \cdot T)} \subseteq \overline{J(S\cdot T)} \subseteq I$. Consequently $f \in I$.

(ii) Let $f \in k(S\cdot T)$ and

$$\Delta(f,I) = \{\lambda \in (G/H)\hat{}, \text{ for which there do not exist the } V \text{ and } g \text{ of (i)}\}.$$

$\Delta(f,I)$ is closed and $\Delta(f,I) \subseteq \partial S$. To show this, let $\lambda \in (G/H)\hat{} \setminus S$. Then there is a compact neighbourhood W of λ such that $W \cap S = \emptyset$. Hence $W\cdot T \cap Z(I) = \emptyset$ and therefore f belongs to I locally at all $\lambda \in W\cdot T$, [1, Prop. 2.5.1]. By the usual conclusions [1, Proof of Theorem 2.4.1] it follows that $f \in I + k(W\cdot T)$. In particular, $\lambda \notin \Delta(f,I)$. Finally $f \in k((\operatorname{int} S)\cdot T)$, and hence $\Delta(f,I) \subseteq \partial S$. We shall show that $\Delta(f,I)=\emptyset$, by proving that the closed set $\Delta(f,I)$ has no isolated points. Suppose λ is an isolated point of $\Delta(f,I)$. Then there exists an open neighbourhood U of λ in $(G/H)\hat{}$ such that

$$U \setminus \{\lambda\} \subseteq (G/H)\hat{} \setminus \Delta(f,I).$$

Since $I+k(\lambda\cdot T)=k(\lambda\cdot T)\supseteqq k(S\cdot T)$, there is a $g\in I$ such that $f-g\in k(\lambda\cdot T)$. Further $\lambda\cdot T$ is a C-set, because T is a C-set, hence there exist $f_n\in k(W_n)$, W_n open, $W_n\supseteqq\lambda\cdot T$, for which

$$f-g = \lim f_n*(f-g) .$$

Then there are open $V_n\subseteqq(G/H)\hat{}$ such that $\lambda\in V_n$ and $V_n\cdot T\subseteqq W_n$, therefore $f_n\in k(V_n\cdot T)$. Choose open neighbourhoods V, W in $(G/H)\hat{}$ such that $\lambda\in V$, $\bar{V}\subseteqq W$, $\bar{W}\subseteqq U$ and $\bar{V},\bar{W}$ compact. Then

$$\bar{V}\cdot T\cap((G/H)\hat{}\setminus W)\cdot T = \emptyset$$

and thus there is an $h\in L^1(G)$ such that $\hat{h}(\alpha)=1$ for all $\alpha\in\bar{V}\cdot T$ and $\hat{h}(\gamma)=0$ for all $\gamma\in((G/H)\hat{}\setminus W)\cdot T$. It is immediate that we can now apply part (i); thus $h*f_n*(f-g)\in I$ for each $n\in\mathsf{N}$, and therefore $h*f\in I$. Now

$$f = h*f-(h*f-f)\in I+k(V\cdot T) ,$$

hence $\lambda\notin\Delta(f,I)$, a contradiction.

(iii) By using (i) again we can prove a) for open H. Therefore suppose that $(G/H)\hat{}$ is not compact. For $f\in k(S\cdot T)$ there are $g_n\in L^1(G)$ such that $\operatorname{supp}\hat{g}_n$ is compact and $\lim f*g_n=f$. Further for each $n\in\mathsf{N}$ there is a compact $C\subseteqq(G/H)\hat{}$ such that

$$\operatorname{supp}\hat{g}_n\cap(G/H)\hat{}\cdot T\subseteqq C\cdot T,$$

hence $f*g_n\in k(((G/H)\hat{}\setminus C)\cdot T)$. By (ii), $\Delta(f*g_n,I)$ is empty, this yields $f*g_n\in I$, so that $f\in I$. Consequently $S\cdot T$ is an S-set.

b) Let $f\in k(S\cdot T)$, and let

$$L = \overline{f*k(S\cdot T)+k((G/H)\hat{}\cdot T)} .$$

Using the fact that $k((G/H)\hat{}\cdot T)\subseteqq L$, it follows by the arguments of a) that:

(i) If f satisfies the following 2 conditions, then $f\in L$:

(1) there exist a compact set $C\subseteqq(G/H)\hat{}$ and a $g\in L$ such that $f-g\in k(((G/H)\hat{}\setminus C)\cdot T)$;

(2) for each $\lambda\in(G/H)\hat{}$ there exist an open neighbourhood V of λ and a $g\in L$ such that $f-g\in k(V\cdot T)$.

(ii) Define $\Delta(f,L)$ as in a). $\Delta(f,L)$ is closed and $\Delta(f,L)\subseteqq\partial S$. If $\lambda\in(G/H)\hat{}\setminus S$, and if W is a compact neighbourhood of λ in $(G/H)\hat{}$ such that $W\cdot T\cap S\cdot T=\emptyset$, then there is a function $h\in L^1(G)$, which is an identity modulo $k(W\cdot T)$ and $h\in k(S\cdot T)$. Hence $f-f*h\in k(W\cdot T)$ and $f*h\in L$, i.e. $\lambda\notin\Delta(f,L)$, and thus $\Delta(f,L)\subseteqq\partial S$. Again, let λ be an isolated point of $\Delta(f,L)$. By applying the fact that $L+k(\lambda\cdot T)\supseteqq k(S\cdot T)$, we show exactly as in a) that $\Delta(f,L)=\emptyset$, and the arguments of a) (iii) yield $f\in L$.

(iii) Hence for $\varepsilon>0$ there are $h \in k(S \cdot T)$ and $g \in k((G/H)^{\wedge} \cdot T)$ such that $\|f - f*h - g\|_1 < \varepsilon$. There is by hypothesis a $k \in k((G/H)^{\wedge} \cdot T)$ such that $\|g - g*k\|_1 < \varepsilon$ and

$$\sup\{\|k*u\|_1 : u \in k(S \cdot T), \|u\|_1 = 1\} < C,$$

where C does not depend on g and ε. Thus

$$\begin{aligned}\|f - f*(h + k - k*h)\|_1 &\leqq \|f - f*h - g\|_1 + \|g - g*k\|_1 \\ &\quad + \|g*k - (f - f*h)*k\|_1 < 2\varepsilon + C\varepsilon .\end{aligned}$$

This and a) yield the fact that $S \cdot T$ is a C-set.

COROLLARY. *Let G be a locally compact abelian group and let $G = H \cdot N$, where H, N are closed subgroups. If $T \subseteqq G/N^{\wedge}$ is a compact* C-*set and $S \subseteqq (G/H)^{\wedge}$ is a closed set with scattered boundary, then $S \cdot T$ is an* S-*set.*

PROOF. Since $r\colon \hat{G} \to \hat{H}$ is one-to-one on $G/N^{\wedge}$, Theorem 3.1.9 of [1] applies to $\alpha = r \mid G/N^{\wedge}$, the restriction of r to $G/N^{\wedge}$. Hence $\alpha(T) = r(T)$ is an S-set and by Reiter's inverse projection theorem [2, Ch. 7, § 3.1] $r^{-1}(r(T)) = (G/H)^{\wedge} \cdot T$ is an S-set. T is a C-set in $\hat{G}$ by [2, Ch. 7, § 4.5] and r of course one-to-one on T. Thus the assertion follows.

REFERENCES

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