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BANACH–SAKS PROPERTIES AND SPREADING MODELS

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Introduction.

Our aim in this paper will be to study several versions of the Banach–Saks properties for a Banach space E: how they are characterized, how they can be compared, and also to show that their negation can be described by the existence of a bounded sequence of points satisfying a certain geometric property (formally analogous, for example, to R. C. James's characterization of non reflexivity [12]).

Such a study was initiated both by H. P. Rosenthal who in [12] gave a characterization of the spaces which do not possess the property called by us Banach–Saks–Rosenthal, and by A. Brunel and L. Sucheston ([8], [9]), who, by associating to any bounded sequence a space with interesting properties, obtained, among other results, several informations about the Banach–Saks and the Alternate-Signs Banach–Saks properties.

The idea of using Brunel–Sucheston's space (called a "spreading model") to obtain H. P. Rosenthal's result was first suggested by L. Tzafriri (see H. P. Rosenthal [16]). It is carried out in this paper, in which we not only obtain, as a corollary, H. P. Rosenthal's result [16], but also a complete description of the Alternate-signs Banach–Saks property, answering a question of Brunel–Sucheston [9]. Also, the connections between these properties and the properties of the spreading model lead to some results which, we think, are noteworthy.

The organization of the paper will be the following: after recalling the definitions and basic facts about the Banach–Saks properties which we study, we introduce, in section I, the Brunel–Sucheston spreading model; most of the results of this section are well-known and are given without proof. In section II, we study the following question: when is the spreading model isomorphic to l_1? The answer, as a corollary, provides H. P. Rosenthal's result [16], with some slight improvements. In section III, we turn to the Alternate-Signs Banach–Saks property and give a complete description of it, both in terms of geometric conditions and in terms of spreading models.

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In section IV, we turn to operators between Banach spaces: for them, Banach–Saks properties can also be defined, and we prove for these properties several factorization results, following, with some simplifications and improvements, the lines initiated by us in [4].

In section V, using the tools preceedingly defined, we build an example of a reflexive space which does not possess Banach–Saks property: the first example of such a space was given by A. Baerstein [1], but our construction is much simpler and much more transparent.

The section VI is devoted to the question, asked by H. P. Rosenthal, of the reiteration of spreading models: given a spreading model of a spreading model of E, is it a spreading model of E itself? We give an example (due to B. Maurey) which proves that this is not the case in general.

Let us also mention, to end with this introduction, that we benefitted from several talks with W. B. Johnson about these questions.

NOTATION. Besides the usual Banach–Saks property, there are several other properties, formally analogous, which can be defined for a Banach Space E. Let us recall the most common of them:

—A Banach space E has the (usual) Banach–Saks property if every bounded sequence $(x_n)_{n\in\mathbf{N}}$ contains a subsequence $(x'_n)_{n\in\mathbf{N}}$ such that the Cesaro means $n^{-1}\sum_1^n x'_k$ are norm-convergent.

—A Banach space E has the Alternate-Signs Banach–Saks property if every bounded sequence $(x_n)_{n\in\mathbf{N}}$ contains a subsequence $(x'_n)_{n\in\mathbf{N}}$ such that the alternate signs Cesaro means $n^{-1}\sum_1^n(-1)^k x'_k$ are norm-convergent.

—A Banach space E has the Banach–Saks–Rosenthal property (sometimes also called Weak Banach–Saks property) if every weakly null sequence $(x_n)_{n\in\mathbf{N}}$ contains a subsequence $(x'_n)_{n\in\mathbf{N}}$ such that the Cesaro means $n^{-1}\sum_1^n x'_k$ are norm-convergent.

We shall denote in short by B.S., A.B.S., B.S.R. these three properties. Concerning them, the following facts are well known:

a) Every uniformly convex, or more generally super-reflexive, space has B.S. property (Kakutani [11]); clearly, this property implies B.S.R.
b) Every Banach space which does not contain $l^1_{(n)}$ uniformly (such a space is called B. convex; see A. Beck [6]) has A.B.S. This was proved by A. Brunel–L. Sucheston [9].
c) Every B. convex Banach space has B.S.R. This was proved by H. P. Rosenthal [16].
d) B.S. property implies reflexivity, but not conversely (A. Baernstein [1]); A.B.S. and B.S.R. do not imply reflexivity: c_0 has A.B.S. (Brunel–Sucheston [9]), and l^1 has B.S.R. (since every weakly null sequence in l^1 is norm convergent to zero).

Our tool for the study of these properties will be the Invariant under Spreading space introduced by Brunel–Sucheston ([7], [8]), which will lead to the notion of spreading models. Among other results, the facts a), b), c), just mentioned will be clearly obtained, so no previous knowledge of these properties is required, except the fact d), which will be obtained in section V.

For the sake of completeness, we start with some basic considerations about the Brunel–Sucheston spaces; most of them will be given without proof.

I. Basic properties of Brunel–Sucheston's spreading models.

PROPOSITION 1. (Extraction of good subsequences, according to Brunel–Sucheston):

Let $(x_n)_{n\in\mathsf{N}}$ be a bounded sequence in a Banach space E. There exists a subsequence $(e_n)_{n\in\mathsf{N}}$ of $(x_n)_{n\in\mathsf{N}}$ and a semi-norm L, defined on the set S of the finite sequences of scalars (complex or real), such that:

$$\forall \varepsilon > 0,\ \forall a \in S,\ \exists \nu \in \mathsf{N},\ \nu < n_1 < n_2 < \dots$$

imply

$$\big|\, \|\textstyle\sum a_i e_{n_i}\| - L(a)\big| < \varepsilon, \qquad \textit{where } a = (a_i) \in S\ .$$

The proof of this proposition, which depends on Ramsey's theorem, can be found in Brunel–Sucheston ([7] or [8]) for real scalars; the extension to the complex case is immediate.

If the sequence $(x_n)_{n\in\mathsf{N}}$ has no Cauchy subsequence (which we assume from now on), the semi-norm L is a norm. We can then define a norm on the vector space spanned by the e_n's by putting

$$|a_1e_1 + \dots + a_ke_k| = L(a)\ ,$$

if $a = (a_1, \dots, a_k)$.

We call F the completion of this space under the norm $|.|$. The norm $|.|$ is clearly invariant under spreading, which means that, for all $n_1 < n_2 < \dots < n_k$, all $a_1 \dots a_k$:

$$|a_1e_1 + \dots + a_ke_k| = |a_1e_{n_1} + \dots + a_ke_{n_k}|\ .$$

The space F will be called the spreading model of E built on the sequence $(x_n)_{n\in\mathsf{N}}$. The properties of F depend both on E and on the properties of the sequence $(x_n)_{n\in\mathsf{N}}$. It is an interesting question to describe on F some properties, either of E, or of $(x_n)_{n\in\mathsf{N}}$, and very little is known in this direction. In the following pages, we shall answer it in a special case.

Let us mention a few more things (due to Brunel–Sucheston) of the space F:

it is finitely representable in E, and the shift T defined by $T(\sum \alpha_k e_k)=\sum \alpha_k e_{k+1}$ is an isometry of F.

The sequence (e_n) will be called the fundamental sequence (in short f.s.) of F (it is not a basis in general). We shall now enter in a few technical facts concerning it.

LEMMA 1. (Brunel–Sucheston [9]). *The differences* $e_{2k-1}-e_{2k}$ *are unconditional in* F; *more precisely, if* $A \subset B$ *are finite subsets of the integers, one has, for any sequence of scalars* (a_i)

$$\left|\sum_{i\in A} a_i(e_{2i-1}-e_{2i})\right| \leqq \left|\sum_{i\in B} a_i(e_{2i-1}-e_{2i})\right| .$$

PROOF OF LEMMA 1. It is enough to show that for any n, for any i_0 between 1 and n,

$$\left|\sum_{\substack{i=1\\ i\neq i_0}}^{n} a_i(e_{2i-1}-e_{2i})\right| \leqq \left|\sum_{1}^{n} a_i(e_{2i-1}-e_{2i})\right| .$$

But, for $m \in \mathbf{N}$, one has, for all $k=0,\ldots,m-1$:

$$\left|\sum_{i=1}^{n} a_i(e_{2i-1}-e_{2i})\right| = \left|\sum_{i=1}^{i_0-1} a_i(e_{2i-1}-e_{2i})+a_{i_0}(e_{2i_0-1+k}-e_{2i_0+k}) + \sum_{i=i_0+1}^{n} a_i(e_{2i-1+m}-e_{2i+m})\right| .$$

Summing up these equalities and dividing by m, one obtains:

$$\begin{aligned}\left|\sum_{i=1}^{n} a_i(e_{2i-1}-e_{2i})\right| &\geqq \left|\sum_{i=1}^{i_0-1} a_i(e_{2i-1}-e_{2i})+a_{i_0}(e_{2i_0-1}-e_{2i_0+m-1})/m + \sum_{i=i_0+1}^{n} a_i(e_{2i-1+m}-e_{2i+m})\right| \\ &\geqq \left|\sum_{\substack{i=1\\ i\neq i_0}}^{n} a_i(e_{2i-1}-e_{2i})\right| -2|a_{i_0}|/m ,\end{aligned}$$

and this proves the lemma.

There is a case when the sequence $(e_n)_{n\in\mathbf{N}}$ itself is unconditional in F, namely when $(x_n)_{n\in\mathbf{N}}$ is weakly null in E:

LEMMA 2. *If* $x_n \xrightarrow[n\to\infty]{} 0$ *for* $\sigma(E,E^*)$, *the sequence* $(e_n)_{n\in\mathbf{N}}$ *is unconditional in* F.

PROOF OF LEMMA 2. We shall prove more precisely that if $A \subset B$ are finite subsets of the integers, we have, for all sequences of scalars (α_k):

$$\left|\sum_{k \in A} \alpha_k e_k\right| \leqq \left|\sum_{k \in B} \alpha_k e_k\right| .$$

Let us show for example that

$$|\alpha_1 e_1 + \alpha_3 e_3| \leqq |\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3| ,$$

the proof of the general statement being completely similar. Let $\varepsilon > 0$. We can find $v \in \mathbf{N}$ such that if $v < n_1 < n_2 < n_3$,

$$\text{①} \quad \begin{cases} \big|\,|a_1 e_1 + a_2 e_2 + a_3 e_3| - \|a_1 e_{n_1} + a_2 e_{n_2} + a_3 e_{n_3}\|\,\big| < \varepsilon \\ \big|\,|a_1 e_1 + a_3 e_3| - \|a_1 e_{n_1} + a_3 e_{n_3}\|\,\big| < \varepsilon . \end{cases}$$

Since $e_n \xrightarrow[n \to \infty]{} 0$ in E for $\sigma(E, E^*)$, there exist positive rational coefficients

$$\frac{p_1}{N}, \ldots, \frac{p_k}{N}, \quad \text{with } p_1 + \ldots + p_k = N, \text{ and}$$

$$\left\|\frac{1}{N}\left(p_1 e_{n_1+1} + p_2 e_{n_1+2} + \ldots + p_k e_{n_1+k}\right)\right\| < \frac{\varepsilon}{|a_2|} .$$

From ① it follows that for $j = 1 \ldots k$, one can write

$$|a_1 e_1 + a_2 e_2 + a_3 e_3| \geqq \|a_1 e_{n_1} + a_2 e_{n_1+j} + a_3 e_{n_1+N+1}\| - \varepsilon .$$

We repeat p_1 times the inequality with $j = 1, p_2$ with $j = 2, \ldots p_k$ with $j = k$, we sum up and divide by N. We obtain:

$$\begin{aligned} |a_1 e_1 + a_2 e_2 + a_3 e_3| &\geqq \|a_1 e_{n_1} + a_2(p_1 e_{n_1+1} + \ldots + p_k e_{n_1+k})/N + a_3 e_{n_1+N+1}\| - \varepsilon \\ &\geqq \|a_1 e_{n_1} + a_3 e_{n_1+N+1}\| - 2\varepsilon \geqq |a_1 e_1 + a_3 e_3| - 3\varepsilon , \end{aligned}$$

and this proves the lemma.

The space F has one more property (proved by Brunel–Sucheston [8]) which will be useful to us: it is so close to E that the convergence of the Cesaro means $n^{-1} \sum_1^n e_k$ in F leads to the same property for a subsequence of $(e_n)_{n \in \mathbf{N}}$ in E:

PROPOSITION 2 (Brunel–Sucheston [8]). *If the Cesaro means $n^{-1} \sum_1^n e_k$ converge in F, there is a subsequence $(e'_n)_{n \in \mathbf{N}}$ of $(e_n)_{n \in \mathbf{N}}$, such that, for all subsequences $(e''_n)_{n \in \mathbf{N}}$ of $(e'_n)_{n \in \mathbf{N}}$, the Cesaro means $n^{-1} \sum_1^n e''_k$ are norm convergent in E.*

The proof of this proposition can be found in [8, prop. 3]. We shall come back on it later, when we deal with the alternate sums $n^{-1}\sum_1^n(-1)^k e_k$, for which the same property will be true.

We shall now investigate the following question: when is the sequence $(e_n)_{n\in\mathbf{N}}$, in F, equivalent to the usual l_1-basis?

II. Spreading models isomorphic with l_1.

Let us consider the following two properties of a Banach space E:

$(\mathscr{P}_1)$ There exist $\delta>0$ and a bounded sequence $(x_n)_{n\in\mathbf{N}}$ in E such that, for all k, $\varepsilon_1,\dots,\varepsilon_k=\pm 1$, all $n_1<n_2<\dots<n_k$,

$$\left\|\frac{1}{k}\sum_1^k \varepsilon_i x_{n_i}\right\| \geqq \delta .$$

$(\mathscr{P}_2)$ There exist $\delta>0$ and a bounded sequence $(x_n)_{n\in\mathbf{N}}$ in E such that, for all k, if $k\leqq n_1<\dots<n_{2^k}$, for all scalars $c_1\dots c_{2^k}$,

$$\left\|\sum_1^{2^k} c_i x_{n_i}\right\| \geqq \delta \sum_1^{2^k}|c_i| .$$

Property $(\mathscr{P}_2)$ was first considered by H. P. Rosenthal [16], who proved that it holds if E does not have B.S.R. property.

Property $(\mathscr{P}_1)$ will be the key for the study of A.B.S. property, in connection with l_1-spreading models. In fact, one has:

THEOREM 1. *For a Banach space E, $(\mathscr{P}_1)$ and $(\mathscr{P}_2)$ are equivalent, and they are satisfied if and only if E has a spreading model the fundamental sequence of which is equivalent to the l_1-basis.*

Actually, we shall prove more: if the properties $(\mathscr{P}_1)$ and $(\mathscr{P}_2)$ are satisfied, one can sharpen the estimates occuring in them. More precisely, let us introduce:

$(\mathscr{P}'_1)$ For all $\eta>0$, there exists a bounded sequence $(x_n)_{n\in\mathbf{N}}$ in E such that, for all k, all $\varepsilon_1\dots\varepsilon_k=\pm 1$, all $n_1<\dots<n_k$,

$$1-\eta \leqq \left\|\frac{1}{k}\sum_1^k \varepsilon_i x_{n_i}\right\| \leqq 1+\eta .$$

$(\mathscr{P}'_2)$ For all $\eta>0$, there exists a bounded sequence $(x_n)_{n\in\mathbf{N}}$ in E such that for all k, if $k\leqq n_1<\dots<n_{2^k}$, for all scalars $c_1\dots c_{2^k}$,

$$(1-\eta)\sum_1^{2^k}|c_i| \leqq \left\|\sum_1^{2^k} c_i x_{n_i}\right\| \leqq (1+\eta)\sum_1^{2^k}|c_i| .$$

Then:

THEOREM 2. *For a Banach space E, the properties* $(\mathscr{P}_1)$, $(\mathscr{P}_2)$, $(\mathscr{P}'_1)$, $(\mathscr{P}'_2)$ *are all equivalent; they are satisfied if and only if E has a spreading model, the fundamental sequence of which is equivalent to the* l_1*-basis.*

PROOF OF THEOREM 2. 1°) Let us assume that $(\mathscr{P}_1)$ holds in E, for a sequence $(x_n)_{n\in\mathbf{N}}$ and a $\delta>0$. Then, obviously, the same property holds for $(e_n)_{n\in\mathbf{N}}$ in F, the spreading model built on the $(x_n)_{n\in\mathbf{N}}$. Therefore, we have for all k, all $\varepsilon_1 \dots \varepsilon_k = \pm 1$:

$$\left|\frac{1}{k}\sum_1^k \varepsilon_i e_i\right| \geqq \delta .$$

LEMMA 3. *If, for all k, all* $\varepsilon_1 \dots \varepsilon_k = \pm 1$,

$$\left|\frac{1}{k}\sum_1^k \varepsilon_i e_i\right| \geqq \delta ,$$

then, for some $\delta'>0$, *for all k, all scalars* $c_1 \dots c_k$:

$$\left|\sum_1^k c_i e_i\right| \geqq \delta' \sum_1^k |c_i| .$$

PROOF OF LEMMA 3. a) Let us first assume the scalars (c_i) to be real. Then it is enough to prove the lemma for $(c_i) \in \mathbf{Z}$. But we have, if we put $\varepsilon_i = \operatorname{sgn} c_i$, $p_i = |c_i|$:

$$|\varepsilon_1 p_1 e_1 + \dots + \varepsilon_k p_k e_k| \geqq |\varepsilon_1(e_1 + \dots + e_{p_1}) + \varepsilon_2(e_{p_1+1} + \dots + e_{p_1+p_2}) + \dots$$
$$+ \varepsilon_k(e_{p_1+\dots+p_{k-1}+1} + \dots + e_{p_1+\dots+p_k})| \geqq \delta(p_1 + \dots + p_k) ,$$

which, in this case, proves the lemma; the f.s. (e_n) of the spreading model F is therefore equivalent to the usual l_1-basis.

b) If F is a complex Banach space, the preceeding computation also holds for real scalars. Therefore, no subsequence of (e_n) can be weak Cauchy (a subsequence is weak Cauchy if $\lim \xi(e'_n)$ exists for all $\xi \in F^*$, but then this limit also exists for real valued $\xi \in F^*$, and this is not possible if (e_n) is, for real scalars, equivalent to the l_1-basis); a subsequence $(e'_n)_{n\in\mathbf{N}}$ of $(e_n)_{n\in\mathbf{N}}$ must be equivalent to the complex l_1-basis (L. Dor [10]), and therefore the sequence (e_n) itself, since it is I.S.

2°) Let us now assume that E has a spreading model F, built on some bounded sequence $(x_n)_{n\in\mathbf{N}}$, such that the f.s. $(e_n)_{n\in\mathbf{N}}$ is equivalent to the l_1-basis. We shall show now that, for all $\eta>0$, we can find a subsequence $(e'_n)_{n\in\mathbf{N}}$ of a sequence of blocks on the $(e_n)_{n\in\mathbf{N}}$ which satisfies $(\mathscr{P}'_2)$.

For the $(e_n)_{n\in\mathbf{N}}$, we have, in F, the estimates

$$m \sum |c_i| \leqq |\sum c_i e_i| \leqq M \sum |c_i| .$$

for some constant m, $M>0$. We shall start by improving these estimates, using a computation due to R. C. James [13]. Here, it is specially easy, because the norm of F is I.S. on the sequence $(e_n)_{n\in\mathbf{N}}$.

Let us put

$$K = \inf\{|\sum \alpha_i e_i|, \sum |\alpha_i|=1\} ,$$

then $K \geqq m$. Let $\eta>0$; we can find

$$f_1 = \sum_{i=1}^{n_0} \alpha_i^0 e_i, \quad \text{with } \sum_1^{n_0} |\alpha_i^0| = 1 ,$$

such that

$$K \leqq |f_1| \leqq K(1+\eta/4) .$$

Since the norm is I.S., we also have:

$$K = \inf\left\{\left|\sum_{i>n_0} \alpha_i e_i\right|, \sum |\alpha_i|=1\right\} .$$

Therefore, if we put, with the same sequence of scalars $\alpha_1^0 \ldots . \alpha_{n_0}^0$,

$$f_k = \sum_{i=1}^{n} \alpha_i^0 e_{(k-1)n_0+i} ,$$

we have, for all $k \in \mathbf{N}$, $K \leqq |f_k| \leqq K(1+\eta/4)$.

If (c_i) is a finite sequence of scalars, with $\sum |c_i|=1$, we have:

$$K \leqq |\sum c_i f_i| \leqq K(1+\eta/4)$$

and if we put $f'_k = K^{-1} f_k$, we obtain, for all finite sequences of scalars (c_i):

$$\sum |c_i| \leqq |\sum c_i f'_i| \leqq (1+\eta/4) \sum |c_i| .$$

Let $k \in \mathbf{N}$ be given. We chose an $\eta/8$-net in the unit sphere of $l_1^{(2^k)}$, which we call $(c_1^{(l)},\ldots,c_{2^k}^{(l)})_{l=1\ldots L}$. There exists an integer ν_k such that, if

$$\nu_k \leqq n_1 < n_2 < \ldots < n_{2^k} ,$$

we have, for all $l=1,\ldots,L$:

$$\left|\left\|\sum_{i=1}^{2^k} c_i^{(l)} f'_{n_i}\right\| - \left\|\sum_{i=1}^{2^k} c_i^{(l)} f'_i\right\|\right| < \eta/8$$

(because the f'_{n_i} always use the same coefficients (α_i^0), and because the sum $\sum_1^{2^k} c_i^{(l)} f'_{n_i}$ "starts" on the integer $(n_1-1)n_0+1 \geqq \nu_k$). Therefore, since $\sum_{i=1}^{2^k} |c_i^{(l)}| =1$, for all l:

$$1-\eta/8 \leqq \left\|\sum_{i=1}^{2^k} c_i^{(l)} f'_{n_i}\right\| \leqq 1+3\eta/8 .$$

From this, we deduce that for all finite sequences of scalars $c_1 \dots c_{2^k}$:

$$1-3\eta/8 \leqq \left\|\sum_1^{2^k} c_i f'_{n_i}\right\| \leqq 1+5\eta/8 .$$

Now, if we chose $e'_k = f'_{\nu_k}$ for all $k \in \mathbf{N}$, the sequence $(e'_k)_{k\in\mathbf{N}}$ will satisfy $(\mathscr{P}'_2)$: we shall have, for all k, all sequences $c_1 \dots c_{2^k}$ of scalars, if $k \leqq n_1 < \dots < n_{2^k}$,

$$(1-3\eta/8)\sum_1^{2^k} |c_i| \leqq \left\|\sum_{i=1}^{2^k} c_i e'_{n_i}\right\| \leqq (1+5\eta/8)\sum_1^{2^k} |c_i| .$$

REMARK. If we wish only to obtain $(\mathscr{P}_2)$ instead of $(\mathscr{P}'_2)$, that is, if we do not try to improve the estimates, we do not need to take blocks on the x_n's: we obtain $(\mathscr{P}_2)$, by this method, on a subsequence of the x_n's. Let us observe, however, that if $(x_n)_{n\in\mathbf{N}}$ is weakly null in E, so is $(e'_n)_{n\in\mathbf{N}}$.

3°) Let us now show that $(\mathscr{P}'_2)$ implies $(\mathscr{P}'_1)$. Let $\eta>0$, $k \in \mathbf{N}$, $n_1 < \dots < n_k$, $\varepsilon_1 \dots \varepsilon_k = \pm 1$, be given. Let $(x_j)_{j\in\mathbf{N}}$ be the sequence satisfying $(\mathscr{P}'_2)$ in E with η replaced by $\eta' = \eta/2$.

Let us denote by $[x]$ the integer part of x, $x \in \mathbf{R}^+$, and let us call $k' = [\log_2 k]$. We have

$$\left\|k^{-1}\sum_1^k \varepsilon_i x_{n_i}\right\| \geqq \left\|k^{-1}\sum_{k'+1}^k \varepsilon_i x_{n_i}\right\| - k^{-1}\left\|\sum_1^{k'} \varepsilon_i x_{n_i}\right\| .$$

For the sum $\sum_{k'+1}^k \varepsilon_i x_{n_i}$, we obtain by $(\mathscr{P}'_2)$:

$$\left\|\sum_{k'+1}^k \varepsilon_i x_{n_i}\right\| \geqq (1-\eta')(k-k')$$

and

$$\left\|\sum_1^{k'} \varepsilon_i x_{n_i}\right\| \leqq k'(1+\eta') .$$

Therefore:

$$\left\|\frac{1}{k}\sum_1^k \varepsilon_i x_{n_i}\right\| \geqq (1-\eta')\frac{k-k'}{k} - (1+\eta')\frac{k'}{k} .$$

$$\geqq 1-\eta$$

if k is greater than some k_0 (depending on η).

Let us put

$$y_1 = \frac{1}{k_0}\sum_{i=1}^{k_0} x_i, \dots, y_j = \frac{1}{k_0}\sum_{i=1}^{k_0} x_{(j-1)k_0+i}, \dots .$$

We then have, by the preceeding computation, for all k, all $\varepsilon_1 \dots \varepsilon_k = \pm 1$, all $n_1 < \dots < n_k$:

$$\left\| \frac{1}{k} \sum_1^k \varepsilon_i y_{n_i} \right\| = \left\| \frac{1}{k} \sum_{i=1}^k \varepsilon_i \left(\frac{1}{k_0} \sum_{i=1}^{k_0} x_{(j-1)k_0+i} \right) \right\| \geqq 1 - \eta ,$$

and since $\|y_i\| \leqq 1+\eta$, property $(\mathscr{P}'_1)$ holds, for the sequence $(y_n)_{n \in \mathbf{N}}$.

Obviously, $(\mathscr{P}'_1)$ implies $(\mathscr{P}_1)$ and $(\mathscr{P}'_2)$ implies $(\mathscr{P}_2)$. If $(\mathscr{P}_2)$ holds for some $\delta > 0$, it can be proved as in 3°) that $(\mathscr{P}_1)$ holds with $\delta/2$. Therefore, theorem 2 is proved.

We shall now apply these tools in order to obtain Rosenthal's result [16]: property $(\mathscr{P}_2)$ holds is E does not have B.S.R. property.

PROPOSITION 1. (H. P. Rosenthal [16]). *If there exists in E a weakly null sequence such that no subsequence has norm-converging Cesaro averages, E has property* $(\mathscr{P}_2)$.

PROOF OF PROPOSITION 1. Let (x_n) be such a sequence; it follows from Lemma I.2 that (e_n) is an unconditional basis of the spreading model F built on the sequence (x_n). From proposition I.2. follows that the Cesaro means $n^{-1} \sum_1^n e_k$ cannot converge in F.

LEMMA 1. *If* $n^{-1} \sum_1^n e_k$ *does not converge in F, there exists a* $\delta > 0$ *such that for all n* $|n^{-1} \sum_1^n e_k| > \delta$.

PROOF OF LEMMA 1. There exists an increasing sequence of integers n_k and a $\delta_1 > 0$ such that, for all k:

$$\left| n_k^{-1} \sum_1^{n_k} e_j \right| > \delta_1 .$$

We shall show that this implies for all n

$$\left| n^{-1} \sum_j^n e_j \right| > \delta_1/2 .$$

Let $n \in \mathbf{N}$. We choose j great enough to have $(n \sup_i |e_i|)/n_j < \delta_1/2$. Dividing $N = n_j$ by n, we obtain

$$N = nk + r, \quad r < n .$$

But:

$$\delta_1 \leqq \frac{1}{N} |e_1 + \dots + e_N| \leqq \frac{1}{N} |e_1 + \dots + e_{nk}| + \frac{1}{N} |e_{nk+1} + \dots + e_{nk+r}|$$

which implies

$$\frac{1}{N}|e_1+\ldots+e_{nk}| \geqq \delta_1/2 .$$

Therefore

$$\frac{1}{nk}|e_1+\ldots+e_{nk}| \geqq \delta_1/2 .$$

But since

$$|e_1+\ldots+e_n| = |e_{n+1}+\ldots+e_{2n}| = \ldots = |e_{n(k-1)}+\ldots+e_{nk}| ,$$

the lemma follows.

Since (e_n) is monotonic-unconditional, we obtain from lemma 1:

$$\left|\frac{1}{n}\sum_1^n \varepsilon_i e_i\right| \geqq \frac{1}{2}\left|\frac{1}{n}\sum_1^n e_i\right| \geqq \frac{\delta}{2},$$

for all n and all $\varepsilon_1 \ldots \varepsilon_n = \pm 1$. It now follows as in the proof of theorem 2, 1°), that (e_n) is equivalent, in F, to the l_1-basis. Theorem 2 now implies that the properties $(\mathscr{P}_2)$ and $(\mathscr{P}'_2)$ are satisfied, which proves proposition 1.

Remark. We obtained the estimates $(\mathscr{P}'_2)$, which slightly improves Rosenthal's result, who gave only $(\mathscr{P}_2)$. In fact, to obtain estimates as good as one wants is of great help for proving factorization results, as we shall see in section IV.

Let us now say a few words about properties $(\mathscr{P}_1)$ or $(\mathscr{P}_2)$. Obviously, if they hold in E, the space cannot have Banach–Saks property: if (x_n) satisfies $(\mathscr{P}_1)$, one can write, for any subsequence (x'_n) of (x_n):

$$\left\|\frac{1}{n}\sum_1^n x'_k - \frac{1}{2n}\sum_1^{2n} x'_k\right\| = \left\|\frac{1}{2n}\left(\sum_1^n x'_k - \sum_{n+1}^{2n} x'_k\right)\right\| \geqq \delta$$

and (x'_k) cannot have norm-convergent Cesaro averages. But the space c_0 does not have Banach–Saks property, and, as will be seen in the next paragraph, does not possess $(\mathscr{P}_1)$ either: $(\mathscr{P}_1)$, therefore, is not a characterization of Non B.S. But of course, for reflexive spaces, both coincide:

Proposition 2. *For reflexive spaces, $(\mathscr{P}_1)$ and Non* B.S. *are equivalent.*

Proof of proposition 2. For reflexive spaces, Non-B.S. and Non-B.S.R. are equivalent, and the latter implies $(\mathscr{P}_1)$.

Also, if $(\mathscr{P}_1)$ is satisfied, so is $(\mathscr{P}_2)$, and E must contain $l_1^{(n)}$ uniformly. But c_0 contains $l_1^{(n)}$ uniformly and does not possess $(\mathscr{P}_1)$: $(\mathscr{P}_1)$ is not equivalent to the fact of containing $l_1^{(n)}$ uniformly. It is of course satisfied if E contains l_1, but conversely, Baernstein space [1] is reflexive, does not have B.S. property, therefore does not have B.S.R. property (since they are equivalent for reflexive spaces), so has $(\mathscr{P}_1)$ (Proposition 1 and Theorem 2), but does not contain l_1. Therefore, $(\mathscr{P}_1)$ is strictly intermediate between containing l_1 and containing $l_1^{(n)}$ uniformly.

We have seen that $(\mathscr{P}_1)$ does not describe Non-B.S.; it does not describe Non-B.S.R. either: Proposition 1 says that Non-B.S.R. implies $(\mathscr{P}_1)$, but l_1 has obviously both $(\mathscr{P}_1)$ and B.S.R. There is, however, a class of spaces for which $(\mathscr{P}_1)$ and Non-B.S.R. are equivalent:

PROPOSITION 3. *For Banach spaces non-containing l_1, the properties $(\mathscr{P}_1)$ and Non-*B.S.R. *are equivalent.*

PROOF OF PROPOSITION 3. In any Banach space, Non-B.S.R. implies $(\mathscr{P}_1)$. Conversely, assume $(\mathscr{P}_1)$ to be satisfied for a bounded sequence $(x_n)_{n\in\mathbf{N}}$. If E does not contain l_1, the sequence $(x_n)_{n\in\mathbf{N}}$ contains, according to a result of H. P. Rosenthal [15], a subsequence $(x'_n)_{n\in\mathbf{N}}$ which is weak-Cauchy: this means that for all $\xi \in E^*$, the limit $\lim_{n\to\infty} \xi(x'_n)$ exists. If we take the differences $y_n = \frac{1}{2}(x'_{2n-1} - x'_{2n})$, we obtain a weakly null sequence, which satisfies (with the same δ) the estimates $(\mathscr{P}_1)$. Therefore, as already observed, non subsequence of $(y_n)_{n\in\mathbf{N}}$ can have norm-convergent Cesaro averages, and E does not have B.S.R. property.

In the next paragraph, we shall investigate a new version of the Banach–Saks property, the A.B.S. property, which will be intermediate between B.S. and B.S.R. It will be found that the equivalent formulations occuring in theorem 2 provide a complete characterization of this new B.S. property.

III. The Alternate signs Banach–Saks property.

Let us make an attempt to find properties weaker than the usual Banach–Saks property. Les us consider:

(1) For every bounded sequence $(x_n)_{n\in\mathbf{N}}$ in E, there is a subsequence $(x'_n)_{n\in\mathbf{N}}$ and a sequence $(\varepsilon_n)_{n\in\mathbf{N}}$ of signs such that the averages $n^{-1}\sum_1^n \varepsilon_k x'_k$ are norm-convergent.

(2) For every bounded sequence $(x_n)_{n\in\mathbf{N}}$ in E, there is a subsequence $(x'_n)_{n\in\mathbf{N}}$ and a sequence $(\varepsilon_n)_{n\in\mathbf{N}}$ of signs such that the averages $n^{-1}\sum_1^n \varepsilon_k x'_k$ are norm-convergent to zero.

(3) For every bounded sequence $(x_n)_{n\in\mathbf{N}}$ in E, there is a subsequence $(x'_n)_{n\in\mathbf{N}}$ such that the alternate signs averages $n^{-1}\sum_1^n (-1)^k x'_k$ are norm-convergent.

(4) For every bounded sequence $(x_n)_{n\in\mathbf{N}}$ in E, there is a subsequence $(x'_n)_{n\in\mathbf{N}}$ such that the alternate signs averages $n^{-1}\sum_1^n (-1)^k x'_k$ are norm-convergent to zero.

Then, the main theorem of this paragraph is:

Theorem 1. *For a Banach space E, properties* (1), (2), (3), (4), *are all equivalent. They are satisfied if and only if the equivalent formulations occuring in theorem* II.2 *are not.*

We shall say that E has Alternate signs Banach–Saks property if (1), (2), (3), (4), hold. This property has already been studied by A. Brunel and L. Sucheston [9], who proved that a B. convex space has it, but that B. convexity is not necessary, since c_0 also has it. Our result is of course much stronger, since it provides a complete description of the A.B.S. property.

Proof of theorem 1. Obviously, (4) $\Rightarrow$ (2) $\Rightarrow$ (1); (4) $\Rightarrow$ (3) $\Rightarrow$ (1). Obviously also, if $(\mathscr{P}_1)$ holds none of (1), (2), (3), (4), can hold, as we have already observed.

Assume that (4) does not hold. Then we can find a bounded sequence $(x_n)_{n\in\mathbf{N}}$ in E such that, for all subsequence $(x'_n)_{n\in\mathbf{N}}$, the sums $n^{-1}\sum_1^n (-1)^k x'_k$ do not converge to zero. Of course, we can restrict ourselves to the good subsequence $(e_n)_{n\in\mathbf{N}}$ of $(x_n)_{n\in\mathbf{N}}$, built by prop. I.1., and we can assume $\|x_n\|=1\ \forall n$, therefore $\|e_n\|=1\ \forall n$.

Proposition 1. *If the sequence $n^{-1}\sum_1^n (-1)^k e_k$ converges in F, there is a subsequence $(e'_n)_{n\in\mathbf{N}}$ of the sequence $(e_n)_{n\in\mathbf{N}}$ such that, for all subsequences $(e''_n)_{n\in\mathbf{N}}$, $n^{-1}\sum_1^n (-1)^k e''_k$ converges to zero in E.*

Proof of Proposition 1. This proposition is quite analogous to Proposition 3 of Brunel–Sucheston [8], and our proof will follow the same lines. The only difference is that here we deal with alternate signs, whereas in [9] all signs are $+$.

Let us assume that $n^{-1}\sum_1^n (-1)^k e_k \xrightarrow[n\to\infty]{} y$ in F.

Lemma 1. *If $y=\lim_{n\to\infty} n^{-1}\sum_1^n (-1)^k e_k$ in F, then $y=0$.*

Proof of lemma 1. Let us put $s_n=n^{-1}\sum_1^n (-1)^k e_k$. If $s_n \xrightarrow[n\to\infty]{} y$ in F, $s_n-s_{2n}\to 0$. But:

$$s_n - s_{2n} = \frac{1}{n}\sum_1^n (-1)^k e_k - \frac{1}{2n}\sum_1^{2n} (-1)^k e_k$$
$$= \frac{1}{2n}\sum_1^n (-1)^k e_k - \sum_{n+1}^{2n} (-1)^k e_k$$

and therefore:

$$|s_n - s_{2n}| = \left|\frac{1}{2n}\left(\sum_1^n (-1)^k e_k - \sum_{n+1}^{2n} (-1)^k e_{k+1}\right)\right|$$
$$= \left|\frac{1}{2n}\left(\sum_1^{2n+1} (-1)^k e_k - (-1)^{n+1} e_{n+1}\right)\right|$$

hence:

$$|s_n - s_{2n}| \geqq \frac{2n+1}{2n}|s_{2n+1}| - \frac{1}{2n}$$

from which follows $s_{2n+1} \xrightarrow[n\to\infty]{} 0$, and $y=0$, which proves the lemma.

Let us come back to the proof of Proposition 1. For all $\varepsilon>0$, there exists an integer $P(\varepsilon)$ such that, if $p \geqq P(\varepsilon)$,

$$\left|p^{-1}\sum_1^p (-1)^k e_k\right| < \varepsilon .$$

From the construction of Brunel–Sucheston's spreading model (prop. I.1), it follows that, when $p \geqq P(\varepsilon)$, there exists an integer N (depending on ε, p), such that if

$$N \leqq n_1 < n_2 < \dots n_p ,$$

then

$$\text{①} \qquad \left\|p^{-1}\sum_1^p (-1)^k e_{n_k}\right\| < \varepsilon .$$

Let us choose by induction an increasing sequence of integers $(P_n)_{n\in\mathbf{N}}$, with:
$\begin{cases} P_n \geqq P(1/2^n) \\ P_n \geqq nP_{n-1} \end{cases}$. Let us put $\nu(n) = N(2^{-n}, P_n)$, $r_n = \sum_{j=1}^n (\nu(j) + P_j)$. For the r_n's, we have:

$$\begin{cases} r_{n+1} \geqq r_n + P_n \\ r_n \geqq \nu(n) . \end{cases}$$

From ① follows that, if $r_n \leqq m_1 < m_2 < \ldots m_{P_n}$, then

② $$\left\| \frac{1}{P_n} \sum_1^{P_n} (-1)^j e_{m_j} \right\| \leqq 1/2^n .$$

Let us consider the terms (e_i) of indexes

$$r_1+1,\ r_1+2, \ldots, r_1+P_1\ ,\ r_2+1, \ldots, r_2+P_2, \ldots, r_n+1, \ldots, r_n+P_n, \ldots$$

written in this order, and let us call them $y_1, y_2 \ldots$. Let (z_j) be a subsequence of the (y_i). Let $n \in \mathbf{N}$ and let k be the integer defined by

$$P_1 + \ldots + P_k \leqq n \leqq P_1 + \ldots + P_{k+1} .$$

Set $m = n - \sum_1^k P_i$. We write the Euclidean division:

$$m = d_k P_k + Q_k, \quad Q_k < P_k$$

and again

$$Q_k = d'_k P_{k-1} + Q'_k, \quad Q'_k < P_{k-1} .$$

We can now write:

$$\sum_{j=1}^{n} (-1)^j z_j = [(-1) z_1 + \ldots + (-1)^{P_1} z_{P_1}] +$$
$$[(-1)^{P_1+1} z_{P_1+1} + \ldots + (-1)^{P_1+P_2} z_{P_1+P_2}] +$$
$$+ \ldots + [(-1)^{P_1+\ldots+P_{k-1}+1} z_{P_1+\ldots P_{k+1}+1} + \ldots + (-1)^{P_1+\ldots P_k} z_{P_1+\ldots P_k}]$$
$$+ (-1)^{P_1+\ldots+P_k+1} z_{P_1+\ldots+P_k+1} + \ldots + (-1)^n z_n .$$

For the terms between brackets, the condition ② implies:

$$\|(-1) z_1 + \ldots + (-1)^{P_1} z_{P_1}\| \leqq P_1 \cdot 2^{-1}$$
$$\|(-1)^{P_1+1} z_{P_1+1} + \ldots + (-1)^{P_1+P_2} z_{P_1+P_2}\| \leqq P_2 \cdot 2^{-2}$$
$$\|(-1)^{P_1+\ldots+P_{k-1}+1} z_{P_1+\ldots+P_{k-1}+1} + \ldots + (-1)^{P_1+\ldots+P_k} z_{P_1+\ldots+P_k}\|$$
$$\leqq P_k 2^{-k} .$$

We have now to consider the m terms of the sum

$$(-1) z_{P_1+\ldots+P_k+1} + \ldots + (-1)^m z_{P_1+\ldots+P_k+m} .$$

For the first $d_k P_k$, by grouping them into d_k blocks of P_k terms, we have, by using ②, a majoration by $d_k P_k 2^{-k}$. There remains Q_k terms. For the first $d'_k P_{k-1}$, we obtain again $d'_k P_{k-1} 2^{-k+1}$. For the last Q'_k terms, we use the triangular inequality.

Finally, dividing by n, we obtain:

$$\left\|\frac{1}{n}\sum_1^n z_j\right\| \leqq \frac{\sum_{j=1}^k 2^{-j}P_j}{\sum_1^k P_j}+2^{-k}+2^{1-k}+\frac{Q'_k}{P_k}$$

because $d_kP_k/n \leqq 1$ (since $d_kP_k \leqq m \leqq n$), and

$$\frac{Q_{k-1}}{n} \leqq \frac{Q_{k-1}}{P_k} \qquad (\text{since } n \geqq P_k)\,.$$

But if $n \to +\infty$, $k \to +\infty$, and all these terms go to zero (since $Q'_k/P_k < P_{k-1}/P_k < 1/k$); the proof of Proposition 1 follows.

Let us come back to the proof of theorem 1. If (4) does not hold, it follows clearly from Proposition 1 that $n^{-1}\sum_1^n(-1)^k e_k$ does not converge in F. Exactly as in Lemma II.1, it follows that we can find a $\delta>0$ such that, for all n:

$$\left|n^{-1}\sum_1^n(-1)^k e_k\right| > \delta\,.$$

LEMMA 2. *If, for all* n, $|n^{-1}\sum_1^n(-1)^k e_k|>\delta$, *then, for all choices of signs* $\varepsilon_\varepsilon \ldots \varepsilon_n = \pm 1$,

$$\left|n^{-1}\sum_1^n \varepsilon_k e_k\right| > \delta/4\,.$$

PROOF OF LEMMA 2. Let $n \in \mathbf{N}$, $\varepsilon_1 \ldots \varepsilon_n = \pm 1$; we can write:

$$|\varepsilon_1 e_1+\ldots+\varepsilon_n e_n| = |\varepsilon_1 e_1+\varepsilon_2 e_3+\ldots+\varepsilon_n e_{2n-1}|$$

and also

$$= |-\varepsilon_1 e_2-\ldots-\varepsilon_n e_{2n}|\,.$$

Therefore:

$$|\varepsilon_1 e_1+\ldots+\varepsilon_n e_n| \geqq |\tfrac{1}{2}[\varepsilon_1(e_1-e_2)+\varepsilon_2(e_3-e_4)+\ldots+\varepsilon_n(e_{2n-1}-e_{2n})]|\,.$$

But from Lemma I.1, follows that:

$$|\varepsilon_1(e_1-e_2)+\ldots+\varepsilon_n(e_{2n-1}-e_{2n})| \geqq |\tfrac{1}{2}[e_1-e_2+\ldots+e_{2n-1}-e_{2n}]|\,,$$

which proves the Lemma.

But then Lemma II.3 implies that $(e_n)_{n\in\mathbf{N}}$ is equivalent, in F, with the l_1-basis; the other properties follow from Theorem II.2. Our theorem is proved.

In comparison with the equivalent formulations occuring in theorem 1, let us mention the following result obtained by B. Maurey and G. Pisier [14]: a Banach E is B convex if and only if, for any bounded sequence $(x_n)_{n\in\mathbf{N}}$, there exists a sequence of signs $(\varepsilon_n)_{n\in\mathbf{N}}$ such that $n^{-1}\sum_1^n \varepsilon_k x_k \xrightarrow[n\to\infty]{} 0$ in E (no extraction of subsequence is needed).

It is clear that A.B.S. property is weaker than B.S.; it is stronger that B.S.R., since Non B.S.R. implies $(\mathscr{P}_1)$.

c_0 is an example of space having A.B.S. (Brunel–Sucheston [9]) without having B.S.; l_1 is an example of space having B.S.R. without A.B.S.

It is also clear that B. convexity implies A.B.S.: as we already observed, properties $(\mathscr{P}_1)$ and $(\mathscr{P}_2)$ imply that E contains $l_1^{(n)}$ uniformly.

The previous results allow us to give easily a geometric description of the negation of Banach–Saks property:

PROPOSITION 2. *A Banach space E does not have Banach–Saks property if and only if the following property holds*:

$(\mathscr{P}_3)$ *There exists a bounded sequence $(x_n)_{n\in\mathbf{N}}$ in E and a number $\delta>0$ such that, for all subsequences $(x'_n)_{n\in\mathbf{N}}$ of $(x_n)_{n\in\mathbf{N}}$, all $m\in\mathbf{N}$, all k with $1\leqq k\leqq m$, one has*:

$$\left\|\frac{1}{m}\left(\sum_1^k x'_j-\sum_{k+1}^m x'_j\right)\right\| \geqq \delta .$$

PROOF OF PROPOSITION 2. a) If this property holds for a bounded sequence (x_n), no subsequence of it can have norm convergent Cesaro averages, and E cannot have Banach–Saks property.

b) Assume now that E does not have Banach–Saks property. Then two cases are possible:

α) E is reflexive. Then, E does not have B.S.R. property either, then has property $(\mathscr{P}_1)$, which obviously implies $(\mathscr{P}_3)$.

β) E is not reflexive. Then, by a result of R. C. James [12], there exists a sequence of points $(x_n)_{n\in\mathbf{N}}$ of norm 1, satisfying for all $k\in\mathbf{N}$,

③ $$\operatorname{dist}(\operatorname{conv}(x_1,\dots,x_k), \operatorname{span}(x_{k+1},\dots)) \geqq \tfrac{9}{10} .$$

We shall see that this condition implies $(\mathscr{P}_3)$.

For $m,k\in\mathbf{N}$, with $k\leqq m$, if $k<[\frac{m}{3}]$, one has, for any subsequence $(x'_j)_{j\in\mathbf{N}}$ of $(x_j)_{j\in\mathbf{N}}$:

$$\left\|\frac{1}{m}\left(\sum_1^k x'_j-\sum_{k+1}^m x'_j\right)\right\| \geqq \frac{1}{m}\left(\left\|\sum_{[m/3]+1}^m x'_j\right\|-\left\|\sum_1^k x'_j\right\|-\left\|\sum_{k+1}^{[m/3]} x'_j\right\|\right)$$

$$\geqq \frac{2}{3}\cdot\frac{9}{10}-\frac{1}{3}=\frac{4}{15},$$

and if $k>[m/3]$, since by ③ we can write:

$$\left\|\frac{1}{k}\sum_{1}^{k} x'_j-\frac{1}{k}\sum_{k+1}^{m} x'_j\right\| \geqq \frac{9}{10},$$

we obtain

$$\left\|\frac{1}{m}\left(\sum_{1}^{k} x'_j-\sum_{k+1}^{m} x'_j\right)\right\| \geqq \frac{9}{10}\frac{k}{m} \geqq \frac{3}{10},$$

from which $(\mathscr{P}_3)$ follows.

Let us now say a word about the super-properties associated with the several Banach–Saks properties. For the usual Banach–Saks property, the super-property is obviously super-reflexivity (since B.S. is weaker than super-reflexivity and stronger than reflexivity).

For A.B.S. property, we have:

PROPOSITION 3. *The super-property* A.B.S. *is* B-*convexity.*

PROOF OF PROPOSITION 3. If E is B convex, it has A.B.S., therefore super A.B.S. If E is not, l_1 is finitely representable in E, and l_1 does not have A.B.S.

For super-B.S.R., no complete description is known: it is only clear that l_1 and the B-convex spaces have this property.

Let us now turn to the Banach–Saks properties for operators; we shall try to prove for them some factorization theorems.

IV. Factoring Banach–Saks properties.

Let now A, A_1 be Banach spaces, and T a continuous operator from A into A_1. We shall say that:

— T has B.S. property if any bounded sequence $(x_n)_{n\in\mathrm{N}}$ contains a subsequence $(x'_n)_{n\in\mathrm{N}}$ such that the averages $n^{-1}\sum_1^n Tx'_k$ converge in A_1.

— T has A.B.S. property if any bounded sequence $(x_n)_{n\in\mathrm{N}}$ in A contains a subsequence $(x'_n)_{n\in\mathrm{N}}$ such that the averages $n^{-1}\sum_1^n (-1)^k Tx'_k$ converge in A_1.

— T has B.S.R. property if any weakly null sequence $(x_n)_{n\in\mathrm{N}}$ in A contains a subsequence $(x'_n)_{n\in\mathrm{N}}$ such that the averages $n^{-1}\sum_1^n T(x'_k)$ converge in A_1.

Finally, we shall say that T factors through a Banach space Y if there exist operators $U\colon A\to Y$, $V\colon Y\to A_1$, with $V\circ U=T$.

Our aim, in this paragraph, will be to establish the following theorems (the first one already appeared in [4]).

THEOREM 1. *Any operator which possesses Banach–Saks property factors through a space which has the same property.*

THEOREM 2. *For an operator, the corresponding formulations of* (1), (2), (3), (4) *of section* III *are all equivalent* (*we call them* A.B.S.); *any operator satisfying them factors through a space with* A.B.S.

For B.S.R. property, the result is not so satisfactory:

THEOREM 3. *Any injection, starting from a space which does not contain* l_1, *possessing* B.S.R. *property, factors through a space with* B.S.R.

Let us call A_0 the space A_1 equipped with the gauge of $T(B_A)$ (this space is isometric with $A/\ker T$). There is a continuous injection i between A_0 and A_1, and any space Y intermediate between A_0 and A_1 is a factorization space for T. The spaces possessing the required properties will be a special class of intermediate spaces: the Lions–Peetre Interpolation spaces.

We refer to [5] for a detailed study of these spaces. Let us recall briefly that the norm is defined by:

① $$\|x\|_{(A_0,A_1)_{\theta,p}} = \inf_{\int_{-\infty}^{+\infty} x(t)\,dt = x} \max\left(\|e^{\xi_0 t}x(t)\|_{L^p(A_0)}, \|e^{\xi_1 t}x(t)\|_{L^p(A_1)}\right)$$

with $\xi_0<0$, $\xi_1>0$, $1<p<\infty$, $\xi_0/(\xi_0-\xi_1)=\theta$ and that the following formulas hold:

② $$\|x\|_{(A_0,A_1)_{\theta,p}} = \inf_{\int_{-\infty}^{+\infty} x(t)\,dt = x} \|e^{\xi_0 t}x(t)\|_{L^p(A_0)}^{1-\theta} \cdot \|e^{\xi_0 t}x(t)\|_{L^p(A_1)}^{\theta}$$

and, for some C, all $x \in A_0$

③ $$\|x\|_{(A_0,A_1)_{\theta,p}} \leqq C\|x\|_{A_0}^{1-\theta} \cdot \|x\|_{A_1}^{\theta}.$$

The proof of the three theorems will follow from the next proposition:

PROPOSITION 1. *If the spaces* $(A_0, A_1)_{\theta,p}$ *have property* $(\mathscr{P}_1)$, *there exists in* A_0 *a bounded sequence* $(e_n)_{n\in\mathbf{N}}$ *which satisfies property* $(\mathscr{P}_1)$ *in* A_1 (*for some* $\delta>0$) (*we identify a point of* A_0 *and its image in* A_1 *by the canonical embedding*).

If $(e_n)_{n\in\mathbf{N}}$ is bounded in A_0 and gives $(\mathscr{P}_1)$ in A_1, it gives automatically $(\mathscr{P}_1)$ in A_0 also: we shall then say that A_0 and A_1 have $(\mathscr{P}_1)$ homothetically.

Our proof of proposition 1 follows the same lines as in [4], but will be much simpler, since we obtained in section II, for Property $(\mathscr{P}_1)$, estimates as good as we want.

Let $\eta>0$ be small enough for, if we set

$$\eta' = 1-\min\left(\left[\frac{1-\eta}{(1+2\eta)^{\theta}}\right]^{1/1-\theta}, \left[\frac{1-\eta}{(1+2\eta)^{1-\theta}}\right]^{1/\theta}\right)$$

we have $\eta'<(\frac{1}{20})^p$. For all n, we choose $e_n(t)$ with $\int_{-\infty}^{+\infty} e_n(t)\,dt=e_n$, and

$$④ \quad \max\left(\|e^{\xi_0 t}e_n(t)\|_{L^p(A_0)}, \|e^{\xi_1 t}e_n(t)\|_{L^p(A_1)}\right) \leqq 1+2\eta \,.$$

If $(A_0, A_1)_{\theta,p}$ has property $(\mathscr{P}_1)$, it has $(\mathscr{P}'_1)$, (by theorem II.2), from which follows that, by (2):

$$1-\eta \leqq \left\|\frac{1}{k}\sum_1^k \varepsilon_i e_{n_i}\right\|_{(A_0, A_1)_{\theta,p}}$$

$$\leqq \left\|e^{\xi_0 t}\frac{1}{k}\sum_1^k \varepsilon_i e_{n_i}(t)\right\|_{L^p(A_0)}^{1-\theta} \cdot \left\|e^{\xi_1 t}\frac{1}{k}\sum_1^k \varepsilon_i e_{n_i}(t)\right\|_{L^p(A_1)}^{\theta}$$

which gives:

$$⑤ \quad \begin{cases} \left\|e^{\xi_0 t}\dfrac{1}{k}\displaystyle\sum_1^k \varepsilon_i e_{n_i}(t)\right\|_{L^p(A_0)} \geqq 1-\eta' \\ \left\|e^{\xi_1 t}\dfrac{1}{k}\displaystyle\sum_1^k \varepsilon_i e_{n_i}(t)\right\|_{L^p(A_1)} \geqq 1-\eta' \,. \end{cases}$$

LEMMA 1. *There exists a number $M>0$ and an integer i_0 such that, for all $i \geqq i_0$, one has simultaneously*:

$$⑥ \quad \begin{cases} \left(\displaystyle\int_{-M}^{+M} \left\|e^{\xi_0 t}e_i(t)\right\|_{A_0}^p dt\right)^{1/p} \geqq 1-2\eta' \\ \left(\displaystyle\int_{-M}^{+M} \left\|e^{\xi_1 t}e_i(t)\right\|_{A_1}^p dt\right)^{1/p} \geqq 1-2\eta' \,. \end{cases}$$

This lemma means of course that the $(e_i(t))_{i\in\mathbb{N}}$ take almost their whole mass on a fixed compact set, both in A_0 and in A_1.

PROOF OF LEMMA 1. It is clearly enough to prove separately, for A_0 and A_1, the existence of M and i_0. Let us prove it for A_0. If the conclusion was false, one could find a sequence of (M_k) strictly increasing to infinity, a sequence of (i_k), strictly increasing, such that:

$$\left(\int_{-M_k}^{M_k} \|e^{\xi_0 t} e_{i_k}(t)\|_{A_0}^p \, dt\right)^{1/p} < 1-2\eta'$$

$$\left(\int_{|t|>M_{k+1}} \|e^{\xi_0 t} e_{i_k}(t)\|_{A_0}^p \, dt\right)^{1/p} < \eta'/2 \, .$$

We put

$$\begin{aligned} e'_{i_k}(t) &= e_{i_k}(t) \qquad \text{if } M_k \leqq |t| < M_{k+1} \\ &= 0 \qquad \text{if not ,} \end{aligned}$$

and $e''_{i_k}(t) = e_{i_k}(t) - e'_{i_k}(t)$.
The $e'_{i_k}(t)$ are disjointly supported, and for the e''_{i_k}, we have

$$\|e^{\xi_0 t} e''_{i_k}(t)\|_{L^p(A_0)} \leqq 1-2\eta' + \frac{\eta'}{2} = 1 - \frac{3\eta'}{2} \, .$$

Then we obtain, for all n:

$$\begin{aligned} 1-\eta' \leqq \left\| \frac{e^{\xi_0 t}}{n} \sum_1^n e_{i_k}(t) \right\|_{L^p(A_0)} &\leqq \left\| \frac{e^{\xi_0 t}}{n} \sum_1^n e'_{i_k}(t) \right\|_{L^p(A_0)} \\ &+ \left\| \frac{e^{\xi_0 t}}{n} \sum_1^n e''_{i_k}(t) \right\|_{L^p(A_0)} \leqq (1+2\eta) \frac{1}{n^{1-1/p}} + 1 - \frac{3\eta'}{2} \end{aligned}$$

and this cannot happen if n is large enough; this contradiction proves the lemma.

We eliminate the first i_0 terms of the sequence $(e_i(t))$; the conclusion of the lemma follows, after renumerating, with $i_0 = 1$.

Let us put

$$\begin{aligned} f_k(t) &= e_k(t) \qquad \text{if } |t| \leqq M \\ &= 0 \qquad \text{if not ,} \end{aligned}$$

and $g_k(t) = e_k(t) - f_k(t)$; let us put finally:

$$f_k = \int_{-\infty}^{+\infty} f_k(t) \, dt ; \qquad g_k = \int_{-\infty}^{+\infty} g_k(t) \, dt \, .$$

LEMMA 2. *The points f_k are in A_0, and their norms are bounded in A_0.*

PROOF OF LEMMA 2. We have:

$$\|f_k\|_{A_0} = \left\| \int_{-M}^{+M} e_k(t) \, dt \right\| \leqq \int_{-M}^{+M} e^{-\xi_0 t} \|e^{\xi_0 t} e_k(t)\|_{A_0} \, dt$$

and with $q=p/p-1$:

$$\leqq \left(\int_{-M}^{+M} e^{-\xi_0 tq}\, dt\right)^{1/q} \cdot \left(\int_{-M}^{+M} \|e^{\xi_0 t} e_k(t)\|_{A_0}^p\, dt\right)^{1/p}$$

$$\leqq (1-2\eta)\left(\int_{-M}^{+M} e^{-\xi_0 qt}\, dt\right)^{1/q},$$

which proves the lemma.

We shall now see that, in $(A_0, A_1)_{\theta, p}$, the differences $e_k - f_k$ have a small norm:

LEMMA 3. *For all $k \in \mathbf{N}$, one has*:

$$\|e_k - f_k\|_{(A_0, A_1)_{\theta, p}} \leqq 8(\eta')^{1/p}.$$

PROOF OF LEMMA 3. We have $e_k - f_k = \int_{|t|>M} e_k(t)\, dt$, and therefore:

$$\|e_k - f_k\|_{(A_0, A_1)_{\theta, p}} \leqq \max\left(\|e^{\xi_0 t} g_k(t)\|_{L^p(A_0)}, \|e^{\xi_1 t} g_k(t)\|_{L^p(A_1)}\right)$$

which, by estimates ④ and ⑥, can be shown to be at most equal to $8(\eta')^{1/p}$.

It follows from Lemma 3 that, in $(A_0, A_1)_{\theta, p}$ the f_k's give property $(\mathscr{P}_1)$: for all $k \in \mathbf{N}$, all $\varepsilon_1 \ldots \varepsilon_k$, all $n_1 < \ldots < n_k$, one has:

$$\left\|\frac{1}{k}\sum_1^k \varepsilon_i f_{n_i}\right\|_A \geqq \left\|\frac{1}{k}\sum_1^k \varepsilon_i e_{n_i}\right\|_A - \frac{1}{k}\sum_1^k \left\|e_{n_i} - f_{n_i}\right\|_A$$

$$\geqq 1 - \eta - (8\eta')^{1/p} \geqq 1/2,$$

by the choices of η and η'. Since the points (f_k) are in A_0, formula ③ and Lemma 2 give:

$$\frac{1}{2} \leqq \left\|\frac{1}{k}\sum_1^k \varepsilon_i f_{n_i}\right\|_A \leqq C\left\|\frac{1}{k}\sum_1^k \varepsilon_i f_{n_i}\right\|_{A_0}^{1-\theta} \cdot \left\|\frac{1}{k}\sum_1^k \varepsilon_i f_{n_i}\right\|_{A_1}^{\theta}$$

and since $\|f_i\|_{A_0} \leqq C_1$, we obtain, for some $\delta' > 0$:

$$\left\|\frac{1}{k}\sum_1^k \varepsilon_i f_{n_i}\right\|_{A_1} \geqq \delta',$$

which proves Proposition 1.

We shall now deduce from Proposition 1 the proof of the three theorems we have mentioned:

PROOF OF THEOREM 1. Let us assume that T, from A into A_1, has B.S. One checks immediately that i, from A_0 into A_1, also has B.S. Then i is weakly compact, and, by [2], $(A_0, A_1)_{\theta, p}$ is reflexive.

But A_0 and A_1 cannot have $(\mathscr{P}_1)$ homothetically, since i has B.S. From Proposition 1 follows that $(A_0, A_1)_{\theta, p}$ does not have $(\mathscr{P}_1)$ either, and it must have B.S., by section III.

Proof of theorem 2. Let us assume that T has A.B.S.; then i has A.B.S. also, and A_0, A_1 cannot have $(\mathscr{P}_1)$ homothetically. It follows from Proposition 1 that $(A_0, A_1)_{\theta, p}$ does not have $(\mathscr{P}_1)$, and has A.B.S. by theorem III.2.

Proof of tjeorem 3. If $(A_0, A_1)_{\theta, p}$ does not have B.S.R., it has $(\mathscr{P}_1)$, and A_0, A_1 have $(\mathscr{P}_1)$ homothetically, on a sequence $(e_n)_{n \in \mathbf{N}}$ bounded in A_0. Since A_0 does not contain l_1, by a result of H. P. Rosenthal [15], some subsequence of $(e_n)_{n \in \mathbf{N}}$ is weak-Cauchy; the consecutive differences on this subsequence are weakly null on A_0 and satisfy $(\mathscr{P}_1)$ in A_1: this contradicts the fact that i has B.S.R.

We have assumed, in theorem 3, that T was an injection, because it is not clear that i has B.S.R. if T has it.

Let us finally mention that it follows easily from theorems 1 and 2 that uniformly convexifying operators (see [2] for definition) factor through a space with Banach–Saks property, and that Type Rademacher operators (see [3] for definition) factor through a space with Alternate Signs Banach–Saks property.

V. Construction of a Baernstein space.

In this paragraph, we shall show how the tools introduced and the results obtained allow us to build a reflexive space without Banach–Saks property. This has already been observed by W. J. Davis, T. Figiel, W. B. Johnson, A. Pelizynski in [18].

Our starting point will be a space close to the one introduced by J. Schreier [17], and built along the same lines. We shall call it S. Let us give its construction.

If $A = \{n_1, n_2, \ldots, n_k\}$, with $n_1 < n_2 < \ldots < n_k$, we call A admissible if $k \leqq n_1$. We denote by $\mathscr{N}$ the set of all admissible (finite) subsets of $\mathbf{N}$. The space S is the closure of the finite sequences of real scalars under the norm

$$\|x\|_S = \sup_{A \in \mathscr{N}} \sum_{i \in A} |x(i)|$$

(where, as usually, we denote by $x(i)$ the ith term of the sequence x). The following facts follow clearly from the definition:

a) There is a canonical injection of norm one from l_1 into S; the points of the canonical basis $(e_n)_{n\in\mathbf{N}}$ of l_1 are of norm one in S.

b) One also has

$$\|x\|_S = \sup\left\{\sum_{i\in A} x(i)\xi(i)\ ;\ A \text{ admissible set, } \xi(i) = \pm 1\right\}$$

and therefore S is isometric to a subspace of a space $\mathscr{C}(K)$, where K is a countable compact set; it follows that S cannot contain l_1.

c) The sequence $(e_n)_{n\in\mathbf{N}}$ satisfies property $(\mathscr{P}_1)$ in S with $\delta = \frac{1}{2}$. Since it also satisfies $(\mathscr{P}_1)$ in l_1, S and l_1 have $(\mathscr{P}_1)$ homothetically.

d) For any subsequence $(e'_n)_{n\in\mathbf{N}}$ of $(e_n)_{n\in\mathbf{N}}$, it is easily checked that:

$$\left\|\frac{1}{n}\sum_1^n \frac{1}{2^k}\sum_{i=2^k}^{2^{k+1}-1} e'_i\right\|_S = \frac{1}{n},$$

from which follows that the sequence $(e_n)_{n\in\mathbf{N}}$ must be weakly convergent to zero. By the Krein–Smulian theorem, the closed convex hull of the e_n's is weakly compact in S, and, therefore, the injection from l_1 into S is weakly compact.

Let us now take the interpolation space $(l_1, S)_{\theta,p}$ $(0<\theta<1,\ 1<p<\infty)$. From [2] follows that this space is reflexive, and from section III, theorem 1, follows that it cannot have Banach–Saks property.

VI. Reiteration of spreading models.

We shall now consider the following question, asked by H. P. Rosenthal: given a space E, a spreading model F_1 of E, built on some bounded sequence $(x_n)_{n\in\mathbf{N}}$ of E, a spreading model F_2 of F_1, built on some bounded sequence $(y_n)_{n\in\mathbf{N}}$ of F_1, can F_2 be represented as a spreading model of E? More precisely, does there exist another bounded sequence $(x'_n)_{n\in\mathbf{N}}$ of E such that the model built on $(x'_n)_{n\in\mathbf{N}}$ is isometric with F_2? We shall show that the answer to this question is negative.

Let us consider a space with property $(\mathscr{P}_1)$. By section II theorem 1 it has a spreading model F_1 isomorphic to l_1; let (e_n) be the basis of this spreading model. By James's argument, we can construct blocks

$$f_n = \sum_{n_k+1}^{n_{k+1}} \alpha_i e_i \quad \text{with } \alpha_i \geqq 0,\ \sum_{n_k+1}^{n_{k+1}} \alpha_i = 1,$$

such that $\|f_n\|_{F_1} \leqq 1 + 1/n$, $\forall\, n$, and, for any n, for any finite sequence of scalars (c_i),

$$\left\|\sum_{i\geqq n} c_i f_i\right\|_{F_1} \geqq \left(1-\frac{1}{n}\right)\sum_{i\geqq n} |c_i|.$$

It follows from these two conditions that the spreading model F_2 built on the sequence (f_n) in F_1 must be isometric with l_1. Therefore, to prove our claim, we need only to produce a space possessing property $(\mathscr{P}_1)$ and having no spreading model isometric with l_1. The following example of such a space is due to B. Maurey.

Let us call E the space S equipped with the norm

$$\|x\|_E = \|x\|_{l_\infty} + \|x\|_S .$$

The norm $\|.\|_E$ is equivalent with $\|.\|_S$; E must therefore have property $(\mathscr{P}_1)$.

We shall show that E has no spreading model isometric with l_1. Let us assume, on the contrary, that the spreading model F, built on a bounded sequence $(x_n)_{n\in\mathbf{N}}$, is isometric with l_1. On this sequence $(x_n)_{n\in\mathbf{N}}$, we can make the following assumptions, by passing to subsequences if necessary:

— $\|x_n\|_E = 1 \quad \forall n$
— $(x_n)_{n\in\mathbf{N}}$ is a good sequence, according to Brunel–Sucheston
— $(x_n)_{n\in\mathbf{N}}$ is weak Cauchy (since E does not contain l_1, by fact b) in the preceeding paragraph).

We can also assume that each of the x_n's is finitely supported: if not, we take x'_n, finitely supported, with $\|x_n - x'_n\| \leqq 1/n$.

Now, let us put $y_n = \frac{1}{2}(x_{2n} - x_{2n-1})$. The sequence $(y_n)_{n\in\mathbf{N}}$ is weakly convergent to zero, each of the y_n's is finitely supported: $y_n(i) = 0$ for $i > i(n)$, and the spreading model built on the y_n's is still isometric with l_1:

$$\text{①}\quad \begin{cases} \forall \varepsilon > 0,\ \forall (c_i) \text{ finite sequence of scalars, } \exists \nu \text{ such that if } \nu \leqq n_1 < n_2 < \dots \\ \qquad\qquad\qquad \left| \left\| \sum c_i y_{n_i} \right\|_E - \sum |c_i| \right| < \varepsilon . \end{cases}$$

By, again, eventually passing to a subsequence of the y_n's, we can assume, from ①, that

$$1 - 2^{-n} \leqq \|y_n\|_E \leqq 1 + 2^{-n} .$$

Since the coordinate functionals are continuous, and since $(y_n)_{n\in\mathbf{N}}$ is weakly null, we can find a strictly increasing sequence $(\alpha_n)_{n\in\mathbf{N}}$ of integers, and subsequence $(y'_n)_{n\in\mathbf{N}}$ of the $(y_n)_{n\in\mathbf{N}}$ with:

$$\text{②}\quad \begin{cases} \sum_{i \leqq \alpha_{n-1}} |y'_n(i)| \leqq 2^{-n} \\ y'_n(i) = 0 \quad \text{for } i > \alpha_n . \end{cases}$$

LEMMA 1. *The quantities $\|y'_n\|_{l_\infty}$ tend to zero when $n \to \infty$.*

PROOF OF LEMMA 1. Assume on the contrary that this is not the case. Then we can find a subsequence of the $(y'_n)_{n \in \mathbf{N}}$, still denoted by $(y'_n)_{n \in \mathbf{N}}$, and a $\delta > 0$ such that

③ $$\frac{9\delta}{10} \leqq \|y'_n\|_{l_\infty} \leqq \delta .$$

It follows from ① that there exists $\nu \in \mathbf{N}$ such that, if $\nu \leqq m_1 < m_2$:

④ $$2 - \frac{\delta}{10} \leqq \|y'_{m_1} + y'_{m_2}\|_E \leqq 2 + \frac{\delta}{10} .$$

But y'_{m_1} and y'_{m_2} are almost disjointly supported: we have, by ②:

$$\|y'_{m_1} + y'_{m_2}\|_{l_\infty} \leqq \max\left(\frac{1}{2^{m_2}} + \|y'_{m_1}\|_{l_\infty}, \|y'_{m_2}\|_{l_\infty}\right)$$

and if ν has been chosen large enough for $2^{-\nu} \leqq \delta/10$, we have

$$\|y'_{m_1} + y'_{m_2}\|_{l_\infty} \leqq \frac{11\delta}{10} .$$

But then

$$\begin{aligned} \|y'_{m_1} + y'_{m_2}\|_E &\leqq \frac{11\delta}{10} + \|y'_{m_1} + y'_{m_2}\|_S \\ &\leqq \|y'_{m_1}\|_E + \|y'_{m_2}\|_E - \frac{7\delta}{10}, \quad \text{by ③} \\ &\leqq 2 - \frac{\delta}{2} . \end{aligned}$$

But this, compared with ④, is impossible: this contradiction proves the lemma.

LEMMA 2. *For all $y \in E$ with $\|y\|_E = 1$, we have $\|y + y'_n\|_S \to 1$, when $n \to \infty$.*

PROOF OF LEMMA 2. Let $\varepsilon > 0$ be given; let $y \in E$; choose y' finitely supported with $\|y - y'\| < \varepsilon/2$. Let j_0 be such that $y'(j') = 0$ if $j > j_0$. Choose m_0 such that

$$\sum_{j \leqq j_0} |y_m(j)| \leqq 2^{-m} \qquad \text{for } m \geqq m_0 ,$$

and $m'_0 \geqq m_0$ such that, by lemma 1,

$$\|y'_m\|_{l_\infty} \leqq \frac{\varepsilon}{2j_0} \qquad \text{if } m \geqq m'_0 .$$

Let A be an admissible set. Then:

— either A begins before j_0, and then, has at most j_0 elements. Then we have:

$$\sum_{i\in A} |(y'+y'_m)(i)| \leqq \sum_{i\in A} |y'(i)|+\sum_{i\in A} |y'_m(i)|$$

$$\leqq 1+j_0\|y'_m\|_{l_\infty} \leqq 1+\varepsilon/2$$

— or A begins after j_0, and then:

$$\sum_{i\in A} |(y'+y'_m)(i)| = \sum_{i\in A} |y'_m(i)| \leqq 1+1/2m \leqq 1+\frac{\varepsilon}{2}$$

for m large enough.

But also, we can find a set A for which

$$\sum_{i\in A} |y'_m(i)| \geqq 1-\varepsilon/2 ,$$

from which follow that, for m large enough:

$$1-\frac{\varepsilon}{2} \leqq \|y'+y'_m\|_S \leqq 1+\varepsilon/2$$

and therefore $| \|y+y'_m\|_S-1|<\varepsilon$, which proves our lemma.

It is now clear that ① and lemma 2 are in contradiction: for v large enough, we have, if $v\leqq m_1<m_2$:

$$2-\tfrac{1}{10} \leqq \|y'_{m_1}+y'_{m_2}\|_E \leqq 2+\tfrac{1}{10}$$

and, by lemma 1:

$$\|y'_{m_1}\|_{l_\infty} \leqq 1/10, \ \|y'_{m_2}\|_{l_\infty} \leqq 1/10 ,$$

which implies:

$$2-\tfrac{3}{10} \leqq \|y'_{m_1}+y'_{m_2}\|_S \leqq 2+\tfrac{1}{10} ,$$

contradicting lemma 2. This achieves the construction of our example.

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