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ON THE ASYMPTOTIC BEHAVIOUR OF NONLINEAR CONTRACTION SEMIGROUPS

GUSTAF GRIPENBERG

1. Introduction and statement of results.

The purpose of this paper is to study the asymptotic behaviour of nonlinear contraction semigroups. This question has been investigated in [1–3, 5, 7, 9–12] from various points of view. Here we shall only consider the strong convergence of semigroups.

Let X be a real Banach space with norm $|\cdot|$ and let X^* be its dual (with norm $|\cdot|^*$). The duality mapping $F\colon X \to X^*$ is defined by

$$F(x) = \{x^* \in X^* \mid (x,x^*) = |x|^2 = (|x^*|^*)^2\},$$

where (x,x^*) denotes the value of x^* at x. Recall that $S\colon C \to C \subset X$ is a contraction semigroup if

$$S(t+s)x = S(t)S(s)x, \quad |S(t)x - S(t)y| \leqq |x-y|$$

and

$$\lim_{t\to 0+} S(t)x = S(0)x = x, \quad t,s \geqq 0,\ x,y \in C.$$

A subset $A \subset X \times X$ is said to be accretive if for every $[x_i,y_i] \in A$, $i=1,2$, there exists $z \in F(x_1-x_2)$ such that $(y_1-y_2,z) \geqq 0$. We use the notation

$$R(I+\lambda A) = \{x+\lambda y \mid [x,y] \in A\},\ D(A) = \{x \mid \exists y \text{ such that } [x,y] \in A\}$$

and

$$A^{-1}(y) = \{x \mid [x,y] \in A\}.$$

For more information on accretive sets and the generation of semigroups in Banach spaces, (especially the existence of the limit in (1.5) below) see [3], [6].

Our first result is

THEOREM 1. *Assume that*

(1.1) *X is a uniformly convex real Banach space,*

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(1.2) *$A \subset X \times X$ is closed and accretive,*
(1.3) *there exists $\lambda_0 > 0$ such that $R(I+\lambda A) \supset \mathrm{cl}\,(D(A))$, $0<\lambda<\lambda_0$,*
(1.4) *there exists $x_0 \in A^{-1}(0)$ and a continuous function k_0: $(0,\infty)\times(0,\infty) \to [0,\infty)$ such that if $[x_i, y_i] \in A$, $x_i \neq x_0$, $i=1,2$ then $(y_1+y_2, z_3) \geqq -k_0(|x_1-x_0|, |x_2-x_0|)((y_1,z_1)+(y_2,z_2))$ for some $z_i \in F(x_i-x_0)$, $i=1,2$ and $z_3 \in F(x_1+x_2-2x_0)$,*
(1.5) $S(t)x = \lim_{n\to\infty} (I+n^{-1}tA)^{-n}x$, $x \in \mathrm{cl}\,(D(A))$,
(1.6) $\lim_{t\to\infty} |S(t+h)x - S(t)x| = 0$ *for every $h>0$ and $x \in \mathrm{cl}\,(D(A))$.*

Then

(1.7) $\lim_{t\to\infty} S(t)x$ *exists for all $x \in \mathrm{cl}\,(D(A))$.*

If A is an odd mapping, i.e. $[x,y] \in A$ iff $[-x, -y] \in A$ and $[0,0] \in A$, then (1.4) is a consequence of (1.2). In this case Theorem 1 has been established in [2, Th. 4.1]. Here we have the following sufficient conditions for (1.4) (for simplicity we take $x_0=0$).

PROPOSITION 1. *Assume that* (1.1) *holds and that*

(1.8) *X^* is strictly convex*
(1.9) *$[x,y] \in A$ iff $y=y_1+y_2$, $[x,y_i] \in A_i$, $i=1,2$ where A_1 is odd and accretive,*
(1.10) *A_2 is accretive, $0 \in \mathrm{int}\,(D(A_2))$ and $[0,0] \in A_2$,*
(1.11) *for every $\delta>0$ there exists $\varepsilon>0$ such that if $[x_i,y_i] \in A_2$, $i=1,2$, $z_1 \in F(x_1)$, $|x_1| \geqq \delta$, $|x_2| \leqq \varepsilon$, then $(y_1, z_1) \geqq |y_2|(|x_1|+|x_2|)$.*

Then (1.4) *holds.*

Observe that (1.11) will certainly be satisfied if (1.10) holds and $0 \in \mathrm{int}\,(A_2^{-1}(0))$. In the second example related to Theorem 1 we consider the case when A is the subdifferential of a convex function.

PROPOSITION 2. *Assume that* (1.5) *holds and that*

(1.12) *X is a real Hilbert space with scalar product $(\cdot,\cdot)$,*
(1.13) *φ: $X \to [0,\infty]$, $\varphi \not\equiv +\infty$ is lower semicontinuous and convex and $\varphi(x_0) = \min_{x\in X} \varphi(x)$ where $x_0 \in X$,*
(1.14) *$[x,y] \in A$ iff $\varphi(x)<\infty$ and $y \in \{w \mid (w, z-x) \leqq \varphi(z)-\varphi(x),\ z \in X\}$,*
(1.15) *there exists a continuous function k_1: $(0,\infty) \to (0,1]$ such that $\varphi(x) \geqq \varphi(x_0 - k_1(|x-x_0|)(x-x_0))$, $x \in X$, $x \neq x_0$.*

Then (1.7) *holds.*

Note that according to a result in [1] the assumptions (1.5) and (1.12)–(1.14) do not imply (1.7).

Is is rather easy to see that if one can take the function k_0 to be a constant and X is a Hilbert space, then (1.4) is equivalent to [4, line (4)] (with $U = S(t)$, $t > 0$). In [2, Th. 4.3] it is shown that if $A = a(I-T)$, $a > 0$ and T is a nonexpansive mapping, then (1.6) holds, and obviously (1.6) is a necessary condition for the conclusion of Theorem 1. To see that some assumption like (1.1) is essential, consider the following simple example: Let

$$X = \{f \in C([0,\infty)) \cap L^\infty([0,\infty)) \mid \lim_{\tau \to \infty} |f(\tau+h)-f(\tau)| = 0 \quad \text{for all } h>0\}$$

with sup-norm and let

$$(S(t)f)(\tau) = f(t+\tau), \quad t \geqq 0,\ \tau \geqq 0 .$$

Obviously S is a linear contraction semigroup on X which satisfies (1.6) but $\lim_{t\to\infty} S(t)f$ does not exist for all $f \in X$.

It follows from [9, Th. 4] that if (1.1)–(1.3), (1.5) hold, X^* is uniformly convex and $\operatorname{int}(A^{-1}(0)) \neq \emptyset$ then (1.7) holds. In the Hilbert space case this result has been established in [3, Th. 3.13] and [10, Coroll. 3.6]. In the next theorem we extend this result in the case when X is uniformly convex, replacing the assumption $\operatorname{int}(A^{-1}(0)) \neq \emptyset$ by a weaker one.

THEOREM 2. *Assume that* (1.1)–(1.3) *and* (1.5) *hold and that*

(1.16) *there exists a real topological vector space V and a linear injection j: $V \to X$, such that,*

(1.17) $D(A) \subset j(V)$,

(1.18) $\operatorname{int}(j^{-1}(A^{-1}(0))) \neq \emptyset$,

and either

(1.19) *there exists $d > 0$, $x_0 \in A^{-1}(0)$ and a bounded set B in V such that if $[x,y] \in A$, $x \notin j(B)$, then $(y,z) \geqq d$ for some $z \in F(x-x_0)$,*

or

(1.20) *there exists a sequence $\{B_n\}_{n=1}^{\infty}$ of bounded sets in V such that if*

$[x,y] \in A$ and $|x|+|y| \leqq n$, then $x \in j(B_n)$.

Then (1.7) *holds.*

Note that if $V = X$, then (1.20) is trivially satisfied and so the assumption $\operatorname{int}(A^{-1}(0)) \neq \emptyset$ is a special case of this theorem. As another example assume that (1.12) holds and that V is a reflexive Banach space, the injection j: $V \to X$

is continuous and $\psi : V \to [0, \infty)$ is convex, lower semicontinuous and satisfies

$$\lim_{|x|_V \to \infty} \psi(x) = +\infty .$$

Moreover, assume that $\operatorname{int}(\{x \in V \mid \psi(x)=0\}) \neq \emptyset$ and take A to be the subdifferential (see (1.14)) of the function ψ_X where $\psi_X(x)=\psi(j^{-1}(x))$ on $j(V)$ and $\psi_X(x)=+\infty$ on $X \setminus j(V)$. Then one can show that the assumptions of Theorem 2 (with (1.19)) are satisfied.

In the next theorem X is a Hilbert space and we consider a combination of the assumption $\operatorname{int}(A^{-1}(0)) \neq \emptyset$ and the condition that $(I+A)^{-1}$ is compact, which has been used in [7, 9, 10].

THEOREM 3. *Assume that* (1.5) *and* (1.12) *hold and that*

(1.21) $A \subset X \times X$ *is maximal accretive,*

(1.22) *there exists* $x_0 \in A^{-1}(0)$ *such that* $x \in A^{-1}(0)$ *whenever* $[x,y] \in A$ *and* $(y, x-x_0)=0$,

(1.23) *there exists a closed subspace E of X such that* $(A^{-1}(0)-x_0) \supset U$ *where U is an open neighborhood of* 0 *in E,*

(1.24) $P(I+A)^{-1}$ *is compact, where P is the orthogonal projection onto the orthogonal complement* $E^{\perp}$ *of E.*

Then (1.7) *holds.*

Observe that assumption (1.22) was introduced in [5] and termed "firm positivity". A related result is to be found in [10, Th. 3.7] where it is assumed that the closed affine space spanned by $A^{-1}(0)$ has codimension 1 and that for some sequence $\{t_n\}_{n=1}^{\infty}$ of positive numbers tending to $+\infty$, $\lim_{n\to\infty} S(t_n)x$ exists.

The following result answers a question raised in [1, P. II, Chap. 4].

PROPOSITION 3. *Assume that* (1.5) *and* (1.12)–(1.15) *hold with* $x_0=0$, $k_1(r)=1$, $r>0$ *and that* $[x_1,y_1] \in A$. *Then it does not follow that the semigroup* S^1 *generated (in the sense of* (1.5)) *by* $-A^1$, *where*

$$A^1 = \{[x,y] \mid [x+x_1, y+y_1] \in A\}$$

satisfies (1.7).

It is easy to see that A^1 in the proposition above is the subdifferential of the convex function

$$\psi(x) = \varphi(x+x_1) - (y_1, x+x_1) .$$

Observe that the approach taken in this paper differs from that in [9, 11] since there the following convergence condition plays a central role: $A^{-1}(0)$ is nonempty and $[x_n, y_n] \in A$, $|x_n| \leqq C$, $|y_n| \leqq C$ and $\lim_{n\to\infty} (y_n, z_n) = 0$ imply that

$$\liminf_{n\to\infty} |x_n - Px_n| = 0$$

where $z_n \in F(x_n - Px_n)$ and $P\colon X \to A^{-1}(0)$ is the nearest point mapping.

Finally we remark that the convergence of continuous contraction semigroups studied here is closely related to the convergence of discrete semigroups of the form $T^k x$, $x \in C$ where k is a nonnegative integer and T is a nonexpansive operator on C.

2. Proof of Theorem 1.

First we establish the following easy

LEMMA 2.1. *Assume that* (1.1)–(1.3) *and* (1.5) *hold and that* $\lim_{t\to\infty} S(t)x$ *exists for all* $x \in D(A)$. *Then* $\lim_{t\to\infty} S(t)x$ *exists for all* $x \in \mathrm{cl}\,(D(A))$.

PROOF. This is a direct consequence of the fact that it follows from the accretivity of A that S is a contraction semigroup, i.e.

$$|S(t)x - S(t)y| \leqq |x-y|, \qquad t \geqq 0,\ x, y \in \mathrm{cl}\,(D(A))\,,$$

cf. [6].

We may without loss of generality assume that $x_0 = 0$ in (1.4), otherwise we perform a translation. Let $x \in D(A)$ be arbitrary and put $u(t) = S(t)x$. It follows from [6, Prop. 2.3, Th. 3.4] that

$$u \text{ is Lipschitz-continuous on } [0, \infty)\,, \tag{2.1}$$

hence differentiable a.e. (since X is reflexive by (1.1)) and satisfies

$$[u(t), -u'(t)] \in A \qquad \text{a.e. } t \geqq 0\,. \tag{2.2}$$

Since $|u(t)|$ is also differentiable a.e. we have by (1.2), (2.2) and [8, Lemma 3.1] that for any $v \in A^{-1}(0)$

$$d/dt|u(t)-v|^2 = 2(u'(t), z(t)) \leqq 0 \qquad \text{a.e. } t \geqq 0 \tag{2.3}$$

where $z(t) \in F(u(t)-v)$.

Assume that $\lim_{t\to\infty} u(t)$ does not exist. Then there exists by (2.3) (since $0 \in A^{-1}(0)$) a constant $c_1 > 0$ such that

$$|u(t)| \geqq c_1 = \lim_{s\to\infty} |u(s)|, \qquad t \geqq 0\,. \tag{2.4}$$

We also conclude that there exist sequences $\{r_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$ of real numbers and a constant c_2 such that

$$|u(r_n)-u(s_n)| \geqq c_2 > 0, \quad n \geqq 1 . \tag{2.5}$$

Invoking (1.4), (2.2) and the fact that $|u(t)|$ is contained in a compact subset of $(0, \infty)$ (see (2.3) and (2.4)), we deduce the existence of a constant $c_3 > 0$ such that

$$-\left(u'(r_n+t)+u'(s_n+t), z_{3,n}(t)\right)$$
$$\geqq c_3\left(\left(u'(r_n+t), z_{1,n}(t)\right)+\left(u'(s_n+t), z_{2,n}(t)\right)\right), \quad n \geqq 1, \text{ a.e. } t \geqq 0$$

where $z_{1,n}(t) \in F(u(r_n+t))$, $z_{2,n}(t) \in F(u(s_n+t))$ and $z_{3,n}(t) \in F(u(r_n+t)+u(s_n+t))$. Integrate this inequality over $(0, t)$ for some $t > 0$, and use [8, Lemma 3.1]. This yields

$$|u(r_n+t)+u(s_n+t)|^2 \leqq |u(r_n)+u(s_n)|^2+c_3(|u(r_n)|^2-|u(r_n+t)|^2 + |u(s_n)|^2-|u(s_n+t)|^2), \quad n \geqq 1,\ t \geqq 0 . \tag{2.6}$$

Assume that $r_n < s_n$ for all n. Then $|u(s_n)| \leqq |u(r_n)|$ by (2.3) (since $0 \in A^{-1}(0)$) and by (1.1) and (2.5) there exists a constant $c_4 \in (0, 1)$ such that

$$|u(r_n)+u(s_n)|^2 \leqq 4c_4|u(r_n)|^2, \quad n \geqq 1 . \tag{2.7}$$

On the other hand we have by the triangle inequality

$$|u(r_n+t)+u(s_n+t)|^2 \geqq (2|u(r_n+t)|-|u(s_n+t)-u(r_n+t)|)^2, \quad n \geqq 1 . \tag{2.8}$$

Insert (2.7) and (2.8) into (2.6) and first let $t \to \infty$ and then $n \to \infty$. This yields by (1.6) and (2.4)

$$4c_1^2 \leqq 4c_4c_1^2$$

and since $c_1 > 0$, $c_4 < 1$ we have a contradiction. Consequently $\lim_{t\to\infty} u(t)$ exists and as $x \in D(A)$ was arbitrary, the assertion of Theorem 1 follows from Lemma 2.1.

3. Proofs of Propositions 1 and 2.

It is well-known that if (1.1) and (1.8) hold, then F is a bijection and F^{-1} (the inverse of F) is uniformly continuous on bounded sets of X^*. Obviously we have only to show that (1.4) holds with A replaced by A_2, (the same fact for A_1 is trivial).

Let $[x_i, y_i] \in A_2$, $x_i \neq 0$, $i = 1, 2$. Define

$$\delta = \min\{|x_1|, |x_2|\}, \quad \gamma = \max\{|x_1|, |x_2|\} . \tag{3.1}$$

Choose ε so small that the condition in (1.11) is satisfied and so that $\{x \mid |x| < \varepsilon\} \subset D(A_2)$, (this is possible by (1.10)). Since F^{-1} is uniformly continuous we deduce that there exists a constant c_1 depending on δ, γ so that for some $x_3 \in X$, $|x_3| < \varepsilon$

$$F(x_1 - x_3) = F(x_1) + c_1 F(x_1 + x_2) . \tag{3.2}$$

By our choice of ε there exists y_3 so that $[x_3, y_3] \in A_2$ and then we have by the accretivity of A_2 and (1.11) that

$$(y_1, F(x_1)) + (y_3, F(x_1 - x_3)) + (y_1 - y_3, F(x_1 - x_3)) \geqq 0 .$$

Using (3.2) we see that this inequality is equivalent to

$$(y_1, F(x_1 + x_2)) \geqq -(2/c_1)(y_1, F(x_1)) . \tag{3.3}$$

In the same way we deduce that

$$(y_2, F(x_1 + x_2)) \geqq -(2/c_1)(y_2, F(x_2)) \tag{3.4}$$

and adding (3.3) and (3.4) we get (1.4) when we note that we may obviously choose the constant c_1 to depend continuously on $|x_1|$ and $|x_2|$. This completes the proof of Proposition 1.

To prove Proposition 2 we note that all assumptions in Theorem 1 except (1.4) follow from (1.12)–(1.14) (cf. [3, p. 25, p. 89]). To show that (1.4) holds we let $[x_i, y_i] \in A$, $i = 1, 2$. By the definition of the subdifferential we obtain

$$(y_1, x_1 - x_0 + c_2(x_2 - x_0)) \geqq \varphi(x_1) - \varphi(x_0 - c_2(x_2 - x_0))$$

and

$$(y_2, x_2 - x_0 + c_2(x_1 - x_0)) \geqq \varphi(x_2) - \varphi(x_0 - c_2(x_1 - x_0))$$

where $c_2 = \min\{k_1(|x_1 - x_0|), k_1(|x_2 - x_0|)\}$. Adding these inequalities and using (1.13) and (1.15) we conclude that (1.4) holds with

$$k_0(s, t) = (\min\{k_1(s), k_1(t)\})^{-1} - 1 .$$

Now we can apply Theorem 1 and the proof of Proposition 2 is completed.

4. Proof of Theorem 2.

Let $x \in D(A)$ be arbitrary, define $u(t) = S(t)x$, $t \geqq 0$ and assume that $\lim_{t \to \infty} u(t)$ does not exist. We may clearly assume that $0 \in \operatorname{int}(j^{-1}(A^{-1}(0)))$. Again it follows from [6, Prop. 2.3, Th. 3.4] that (2.1)–(2.4) hold. From (2.3) we deduce that

(4.1) $$\int_0^\infty |(u'(t), z(t))|\, dt < \infty, \qquad z(t) \in F(u(t)-v),\ t \geqq 0,\ v \in A^{-1}(0)\,.$$

Now it follows from either (1.19), (2.2) and (4.1) or from (1.20) and (2.1)–(2.3) that there exists a bounded set B in V such that if

(4.2) $$E = \{t \mid t \geqq 0,\ u(t) \in j(B)\}$$

then

(4.3) $$\lim_{t\to\infty} m([t,\infty)\setminus E) = 0$$

where m is Lebesgue mesure. As we assume that $\lim_{t\to\infty} u(t)$ does not exist, it follows from (2.1) and (4.3) that for some sequences $\{r_n\}_{n=1}^\infty$ and $\{s_n\}_{n=1}^\infty$ of real numbers tending to $+\infty$, such that $r_n, s_n \in E$, $n \geqq 1$, the inequality (2.5) holds.

From the definition of the set E and the fact that $0 \in \operatorname{int}(j^{-1}(A^{-1}(0)))$ we conclude that there exists a constant $c_3 > 0$ such that $c_3(u(r_n)-u(s_n)) \in A^{-1}(0)$, $n \geqq 1$. This yields by (2.3)

(4.4) $$|u(s_n)+c_3(u(s_n)-u(r_n))| \leqq |u(r_n)+c_3(u(s_n)-u(r_n))|, \qquad n \geqq 1$$

if we assume that $r_n < s_n$ for all n. In the same way we also obtain

(4.5) $$|u(s_n)| \leqq |u(r_n)|, \qquad n \geqq 1\,.$$

Fix n. We introduce the notation

(4.6)
$$v_n = u(r_n), \quad w_n = u(s_n), \quad x_n = v_n + c_3(w_n - v_n), \quad y_n = w_n + c_3(w_n - v_n)\,.$$

From the triangle inequality we have

(4.7) $$|y_n| \geqq (1+c_3)|w_n| - c_3|v_n|\,.$$

From (4.4) we conclude that, see [6, p. 74])

$$(w_n - v_n, z) \leqq 0, \qquad z \in F(x_n)\,,$$

and consequently, adding and subtracting one term,

(4.8) $$|x_n|^2 \leqq (v_n, z), \qquad z \in F(x_n)\,.$$

We may safely assume that $c_3 \leqq 1$ and hence it follows from (4.5) and (4.6) that $|x_n| \leqq |v_n|$. This fact combined with (1.1), (2.4) and (2.5) gives the existence of a constant $c_4 \in (0,1)$, such that

(4.9) $$|v_n + x_n| \leqq 2c_4|v_n|\,.$$

Now we get

$$(4.10)\qquad (v_n, z) \leqq \lim_{\lambda \to 0+} (2\lambda)^{-1}(|x_n+\lambda v_n|^2-|x_n|^2) \leqq (2c_4|v_n|-|x_n|)|x_n|, \qquad z \in F(x_n)\,,$$

where the first inequality follows from [6, p. 74] and the second from the triangle inequality and (4.9). Combining (4.4), (4.6)–(4.8) and (4.10) we get

$$(1+c_3)|u(s_n)|-c_3|u(r_n)| \leqq c_4|u(r_n)|\,.$$

If we let $n \to \infty$ in this inequality it follows from (2.4) that $c_1 \leqq c_1 c_4$ and since $c_1>0$ and $c_4<1$ we have a contradiction. This completes the proof of Theorem 2, as $x \in D(A)$ was arbitrary and we can apply Lemma 2.1.

5. Proof of Theorem 3.

Let $x \in D(A)$ be arbitrary and define $u(t)=S(t)x$, $t \geqq 0$. Without loss of generality we may assume that $x_0=0$ in (1.22) and (1.23) and that $E \neq \{0\}$, (cf. [5, p. 22]). First we are going to establish

LEMMA 4.1. *If the assumptions of Theorem* 3 *hold, then* $u(t)$ *converges weakly in* X *as* $t \to \infty$.

PROOF. By [6, Prop. 2.3, Th. 3.4] we conclude that (2.1) and (2.2) hold and that moreover

$$(5.1)\qquad u(t) \in D(A), \quad t \geqq 0\,,$$

since (1.3) follows from (1.12) and (1.21), see [3, Prop. 2.2]. We want to apply [5, Th. 1] and hence we must establish the following complement to [5, Th. 3]

(5.2) if $u_n \rightharpoonup u$ (weakly) as $n \to \infty$, $[u_n, y_n] \in A$, $n \geqq 1$, $\{y_n\}_{n=1}^{\infty}$ is bounded and $\lim_{n\to\infty}(y_n, u_n)=0$, then $u \in A^{-1}(0)$.

Let $\{u_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be such that the assumptions in (5.2) hold. By (1.23) there exists $r>0$ such that

$$(5.3)\qquad v \in A^{-1}(0) \quad \text{if } v \in E \text{ and } |v| \leqq r\,.$$

Put $u=q+s$, $u_n=q_n+s_n$ and $y_n=w_n+z_n$ where q, q_n, $w_n \in E$ and s, s_n, $z_n \in E^{\perp}$. Since $s_n=P(I+A)^{-1}(u_n+y_n)$ and $\{u_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are bounded, it follows from (1.24) and the weak convergence of u_n that

$$(5.4)\qquad s_n \to s \quad \text{as } n \to \infty\,.$$

Suppose that $w_n \neq 0$ for all n. Then it is a consequence of (5.3) and the accretivity of A that if $v_n=rw_n|w_n|^{-1}$, then

$$(y_n, u_n) = (y_n, u_n - v_n) + r|w_n| \geqq r|w_n|$$

and since $\lim_{n\to\infty} (y_n, u_n) = 0$ we get

$$w_n \to 0 \quad \text{as } n \to \infty . \tag{5.5}$$

Without loss of generality we may assume that $y_n \rightharpoonup y$ (weakly) as $n \to \infty$ and we proceed to show that $[u, y] \in A$. Let $[u_0, y_0] \in A$ be arbitrary. Then $(y_n - y_0, u_n - u_0) \geqq 0$ and using (5.4) and (5.5) one easily sees that $(y - y_0, u - u_0) \geqq 0$ and the desired conclusion follows from the maximal accretivity of A. In the same way we deduce that

$$(y, u) = \lim_{n\to\infty} (y_n, u_n) = 0$$

and by (1.22) we conclude that the assertion of (5.2) holds. Now we have only to combine (2.1), (2.2), (5.1), (5.2) with [5, Th. 1] and the proof of Lemma 5.1 is completed.

Put $u(t) = q(t) + s(t)$ where $q(t) \in E$, $s(t) \in E^{\perp}$ for all $t \geqq 0$. Now

$$s(t) = P(I + A)^{-1}(u(t) - u'(t)) \quad \text{a.e. } t \geqq 0$$

by (2.2) and so it follows from (1.24), (2.1), Lemma 5.1 and the obvious fact that $|u(t)|$ is bounded, that

$$\lim_{t\to\infty} s(t) \text{ exists} . \tag{5.6}$$

The proof will be completed if we show that $q(t)$ also converges.

Observe that (2.3) holds. As an easy consequence we have for $t_1 > t_0 > 0$

$$(u(t_1) - u(t_0), u(t_0) - v) \leqq 0 \quad \text{for every } v \in A^{-1}(0) . \tag{5.7}$$

Fix $t_1 > t_0 > 0$ and let

$$q = q(t_1) - q(t_0), \qquad s = s(t_1) - s(t_0) .$$

The relations (5.3) and (5.7) (with $v = -rq|q|^{-1}$) yield

$$(q + s, u(t_0)) \leqq -r|q|$$

since $(q, s) = 0$ and so it follows from this inequality that

$$|u(t_1)|^2 \leqq -r|q| + |q|^2 + |s|^2 + (u(t_1), u(t_0))$$

and as moreover $|u(t_1)| \leqq |u(t_0)|$ we conclude that

$$|q|(r - |q|) \leqq |u(t_0)|^2 - |u(t_1)|^2 + |s|^2 . \tag{5.8}$$

Since $|u(t)|^2$ and $s(t)$ converge by (2.3) and (5.6) we can deduce from (4.8) and the definitions of q and s that $q(t)$ converges as $t \to \infty$. This completes the proof of Theorem 3, since $x \in D(A)$ was arbitrary and we can apply Lemma 2.1.

6. Proof of Proposition 3.

It follows from a result in [1, P. II, Chap. 4] that there exists a Lipschitz continuous function $B: X \to X$, (X is the real Hilbert space of square summable sequences), such that $B(0)=0$ and B is the subdifferential (in the sense of (1.14)) of a convex, continuous function $\psi: X \to [0, \infty)$, $\psi(0)=0$, but the semigroup generated by $-B$ does not converge for all $x \in X$. Let u_0 be such an element in X and choose $r > |u_0|$. Since B is Lipschitz continuous there exists a constant c such that

$$(6.1) \qquad |\psi(x)-\psi(y)| \leqq c|x-y|, \qquad |x|, |y| \leqq r .$$

Choose $z \in X$ so that

$$(6.2) \qquad |z| \geqq (r+1)(c+1) .$$

Define

$$(6.3) \qquad C = \{u \in X \mid |z-u| \leqq r\}, \qquad D = \operatorname{cl}\operatorname{co}(C \cup -C)$$

and

$$(6.4) \qquad \varphi(u) = \begin{cases} \psi(u-z)+(z,u) & \text{if } u \in C \\ \psi(-u-z)-(z,u) & \text{if } u \in -C . \end{cases}$$

Let $x \in C$, $y \in -C$, $\alpha, \beta \geqq 0$, $\alpha+\beta=1$ be such that $\alpha x+\beta y \in C \cup -C$, assume for example that $\alpha x+\beta y \in C$. Then we have by (6.1)–(6.4), the convexity and nonnegativity of ψ and the fact that $\psi(0)=0$

$$\begin{aligned}(6.5) \quad & \alpha\varphi(x)+\beta\varphi(y)-\varphi(\alpha x+\beta y) \\ & \geqq \alpha\psi(x-z)+\beta\psi(0)+\alpha(z,x)-\beta(z,y)-\psi(\alpha x+\beta y-z)-(z,\alpha x+\beta y) \\ & \geqq \psi(\alpha x+\beta z-z)-\psi(\alpha x+\beta y-z)-2(z,\beta(y+z))+2\beta|z|^2 \\ & \geqq -c|\beta z-\beta y|-2\beta|z|r+2\beta|z|^2 \\ & \geqq 2\beta(|z|^2-c|z|-cr-|z|r) \geqq 0 .\end{aligned}$$

Since C is closed and convex, every element $u \in D$ can be written in the form $u=\alpha x+\beta y$, $x \in C$, $y \in -C$, $\alpha, \beta \geqq 0$, $\alpha+\beta=1$. Define φ on D by

$$(6.6) \quad \varphi(u) = \inf\{\alpha\varphi(x)+\beta\varphi(y) \mid u=\alpha x+\beta y,\ \alpha, \beta \geqq 0,\ \alpha+\beta=1,\ x \in C,\ y \in -C\} .$$

Since ψ is convex and (6.5) holds, this definition agrees with (6.4) on $C \cup -C$.

Next we show that φ is convex on D. Let $u_1, u_2 \in D$, $\alpha, \beta \geqq 0$, $\alpha+\beta=1$ and let $\varepsilon > 0$ be arbitrary. Then there exists $x_i \in C$, $y_i \in -C$, $\alpha_i, \beta_i \geqq 0$, $\alpha_i+\beta_i=1$, $i=1,2$ such that

$$\alpha_i\varphi(x_i)+\beta_i\varphi(y_i) \leqq \varphi(u_i)+\varepsilon, \quad u_i = \alpha_i x_i+\beta_i y_i,\ i = 1,2\ . \tag{6.7}$$

From (6.6), (6.7) and the convexity of φ on the convex sets C and $-C$ we have

$$\begin{aligned}\varphi(\alpha u_1+\beta u_2) &\leqq \alpha\alpha_1\varphi(x_1)+\beta\alpha_2\varphi(x_2)+\alpha\beta_1\varphi(y_1)+\beta\beta_2\varphi(y_2)\\ &\leqq \alpha\varphi(u_1)+\beta\varphi(u_2)+\varepsilon\ .\end{aligned}$$

As $\varepsilon > 0$ was arbitrary, the desired conclusion follows. Finally let $\varphi(u) = +\infty$ if $u \notin D$. It is clear from the definition that φ is even and since we have shown that φ is convex we have only to check that φ is lower semicontinuous. Let $u_n \in D$, $u_n \to u$ as $n \to \infty$. Then there exist for all n $x_n \in C$, $y_n \in -C$, $\alpha_n, \beta_n \geqq 0$, $\alpha_n+\beta_n=1$ so that

$$\alpha_n\varphi(x_n)+\beta_n\varphi y_n) \leqq \varphi(u_n)+n^{-1}, \quad u_n = \alpha_n x_n+\beta_n y_n,\ n \geqq 1\ . \tag{6.8}$$

Taking subsequences if necessary we may assume that $\alpha_n \to \alpha$, $\beta_n \to \beta$, $x_n \rightharpoonup x$, $y_n \rightharpoonup y$ (weakly) as $n \to \infty$ and using the fact that C and $-C$ are convex and φ is weakly lower semicontinuous on C and $-C$ we conclude from (6.8) that

$$\begin{aligned}\varphi(u) \leqq \varphi(\alpha x+\beta y) &\leqq \alpha\varphi(x)+\beta\varphi(y)\\ &\leqq \liminf_{n\to\infty}(\alpha_n\varphi(x_n)+\beta_n\varphi(y_n)) \leqq \liminf_{n\to\infty}\varphi(u_n)\ .\end{aligned}$$

Hence φ is lower semicontinuous.

Define A by (1.14) and choose $\{x_1, y_1] = [z, z]$. Now it is easy to see, using the convexity of φ, that $\{y \mid [x, y] \in A\}$ only depends on the values of φ in a neighborhood of x and so

$$[x,y] \in A^1 \text{ iff } Bx=y \text{ provided } |x| \leqq r_1 < r \tag{6.9}$$

where $r_1 > |u_0|$. Since clearly $[0,0] \in A^1$ it follows from (2.1)–(2.3) and (6.9) that S^1 equals the semigroup generated by $-B$ on the set $|x| < r_1$ and this semigroup did not converge for all x, $|x| < r_1$. This completes the proof of Proposition 3.

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INSTITUTE OF MATHEMATICS
HELSINKI UNIVERSITY OF TECHNOLOGY
SF-02150 ESPOO 15
FINLAND

AND

MATHEMATICS DEPARTMENT
UNIVERSITY OF WISCONSIN
MADISON, WISCONSIN 53706
U.S.A.