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A NOTE ON REFLECTION

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Through the recent development in the theory of recursion in normal, higher type functionals, the importance of Grilliot's selection theorem for subindividuals has become evident. After the Harrington–MacQueen-proof of this selection-property [1] and the reflection-properties obtained from it in Harrington [0], the theory has been tremendously enriched.

The development of degree-theory for normal higher-type functionals was one of the applications of these selection- and reflection principles.

Unfortunately, in order to use these principles one had to add restrictive assumptions such as the continuum hypothesis, so there was a need for improvements of these principles.

In this paper we will show that some genuine subsets of the individuals may share some of the reflection and selection-properties of the set of subindividuals. We actually show that for some well-behaved recursive well-orderings of the individuals, we may uniformly "search" through proper initial segments.

These results were first presented in my lecture at the GRT II-conference in Oslo June -77, and there is an application in Normann [6].

Our proof is based on the notion of set-recursion (Normann [6], [7], Moschovakis [5]), and it will be an advantage to know the original version of Grilliot's selection theorem as proved in Harrington–MacQueen [1] or Moldestad [3]. We have based our notation on the exposition in Moldestad [3]. We repeat the complete argument with the adjustments needed for the more general result, and to transform the argument to the context of set recursion, or actually E-recursion.

Throughout this note we will let I be a set of individuals. We will assume that I has recursive pairing and coding of countable sequence ($I = \mathrm{tp}\,(k)$ for $k \geqq 1$ or $I = H(\varkappa)$ for some cardinal $\varkappa$ whose cofinallity is not ω). We will let $<$ be a well-ordering of I. By "recursive" we will always mean E-recursive in the parameters $I, <$.

For standard notation in higher type recursion theory we refer to Moldestad [3].

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DEFINITION. A subset A of I is *reflecting* if for all $b \in I$, if A is recursive in b, then $\operatorname{Sup}\{K_0^{a,b} \; ; \; a \in A\}$ is K_0^b-reflecting.

DEFINITION. $<$ is *recursively regular* if there is no function recursive in some individual mapping an initial segment of $<$ onto a $<$-cofinal subset of I.

THEOREM. *The following two statements are equivalent:*
a) $<$ *is recursively regular*
b) *All proper initial segments of* $<$ *are reflecting.*

PROOF. b) $\Rightarrow$ a). Assume $<$ is not recursively regular. Let f, a be recursive in an individual c such that f maps $\{b \; ; \; b<a\}$ cofinally into $<$. By an argument of Moschovakis [4] (see also MacQueen [2] or Moldestad [3]) the semirecursive subsets of I will not be closed under existential quantification over I. But we may write $\exists b<a \; \exists d<f(b)$ for $\exists d \in I$, so the semirecursive subsets of I cannot be closed under $<$-bounded quantification, so the initial segments cannot all be reflecting.

In order to prove a) $\Rightarrow$ b) we need two lemmas.

LEMMA 1. *Assume that* $<$ *is recursively regular. Let* $A \subseteq I$ *be bounded and recursive in some* $c \in I$. *Then* $A \in L_{\|<\|}[I, <]$. ($\|<\|$ *is the order-type of* $<$. *We let* $x \in L_\alpha[x]$, *so we do not treat* $I, <$ *as relations above*).

PROOF. Let A be bounded by b_0. Since A is recursive in c, we will have that $A \in M_c^\gamma$ for some γ. (We here use notation from Normann [6] or [7]). In particular then, $A \in L_\gamma[I, <] \in M_c^{\gamma+\omega}$.

The wellordering $<$ is inducing a well ordering on $L_\gamma[I, <]$, which we use to define recursive Skolem-functions on $L_\gamma[I, <]$. Using these, let K_0 be the least substructure of $L_\gamma[I, <]$ that contains A, each $c<b_0$ and that is closed under the Skolem-functions.

K_0 is recursive in c and indicable over $\omega \times \{c \; ; \; c<b_0\}$. By the regularity assumption, $K_0 \cap I$ is bounded by some minimal element b_1.

Inductively, define K_i and b_{i+1} in the same way, and let $K = \bigcup_{i\in\omega} K_i$. Then $K \cap I$ is a bounded initial segment of I and K is indicable over $K \cap I$ recursively in c (as $L_\gamma[I, <]$ is indicable over I). This shows that the ordertype of the ordinals in K is less than $\|<\|$. Moreover, K is by construction an elementary substructure of $L_\gamma[I, <]$.

Let K' be the Mostowski collapse of K. By the collapsing function, A will be mapped on A, so $A \in K'$ and $K \cap I = K' \cap I$. Let γ_0 be the supremum of the ordinals in K'. Then $\gamma_0 < \|<\|$ and $K' = L_{\gamma_0}[K' \cap I, <\restriction K']$. But then

$$A \in L_{\gamma_0}[I, <] \subseteq L_{\|<\|}[I, <]$$

and lemma 1 is proved.

We will make use of the following corollary:

Let $A \subseteqq (\{c : c<a\} \cup \omega)^{\omega}$ (where x^{ω} is all *finite* sequences from x) be recursive in some individual. Then $A \in L_{\|<\|}[I, <]$.

In the next lemma we will repeat the Harrington–MacQueen argument with some modifications. An essential feature of their proof is that one need to have control over the possible arguments of a computation, e.g. they should be restricted to the individuals. This is not the fact for set recursion. In order to get around this difficulty, we cheet. We prove the theorem for recursion in the functional ^{I}E (existential, total quantification over I) and the essential relations. To avoid confution, we use superscripts 1 and 2 to indicate computations norms etc. in ^{I}E-recursion respectively set recursion.

The two theories are sufficiently equivalent to justify this trick.

The proof of the lemma is essentially the same as the Harrington–MacQueen-proof for the general Grilliot selection, say for ^{4}E. At some point in their proof they use that quantification over the power-set of subindividuals, is computable in ^{4}E. At the analogue point we will have to quantify over the subsets of the individuals with low cardinality. We show that we only need to consider such subsets that are computable in ^{3}E and a real. By recursive regularity and the corollary to Lemma 1 quantification over the family of such sets will be computable in ^{3}E.

Elsewhere we repeat the original argument as given in Moldestad [3].

LEMMA 2. *Let $<$ be recursively regular. There is an index e such that*

$$\{e\}^2(e', \vec{a})\downarrow$$

if and only if

$$\exists b < a_1 \ \{e'\}^1(b, \vec{a})\downarrow$$

and then

$$\|\langle e, e', \vec{a}\rangle\|^2 \geqq \min\{\|\langle e', c, \vec{a}\rangle\|^1 \ ; c<a_1\}$$

PROOF. Let X consist of those elements of $L_{\|<\|}[I, <]$ that are on the form

$$\alpha = \langle e_x, \vec{a}_x\rangle_{x<a}, \quad a \in I, \ \vec{a} \in I^n, \ n \in \omega \, .$$

We let α_x be $\langle e_x, \vec{a}_x\rangle$, and

$$\|\alpha\| = \min\{\|\alpha_x\|^1 \ ; x<a\} \, .$$

Let α, β range over X. It is sufficient to prove

SUBLEMMA. *There is an index m such that*
i) $\|\beta\|<\infty \Rightarrow \{m\}^2(\beta)\downarrow$ and $\|\beta\| \leqq \|\langle m,\beta\rangle\|^2$
ii) $\{m\}^2(\beta)\downarrow \Rightarrow \|\beta\|<\infty$

PROOF. To find m we use the recursion theorem for set recursion. i) is first proved by induction on $\|\beta\|$ and then ii) is proved by induction on $\|\langle m,\beta\rangle\|^2$.

Given $\mu \in$ On, assume as an induction hypothesis that $\|\beta\|<\mu \Rightarrow \{m\}^2(\beta)\downarrow$ and $\|\beta\| \leqq \|\langle m,\beta\rangle\|^2$. Let α be given and assume that $\|\alpha\|=\mu$.

Let S be the relation obtained by the following proposition:

PROPOSITION. *In ^{I}E-recursion there is a semirecursive set S such that if τ is a computation then $S(\tau,\sigma)$ if and only if σ is an immediate subcomputation of τ.*

(The method needed for proving this proposition goes back to Moschovakis [4]. For generalizations in various directions, see MacQueen [2] and Moldestad [3]). Define the relation R by

$$R(x,y,w) \Leftrightarrow S(x,y) \text{ if } x \text{ is not on the form of a substitution } \{e\}^1(a)\approx\{e_2\}^1(\{e_1\}^1(a),a)$$

$$R(x,y,w) \Leftrightarrow (y=\langle e_1,a\rangle \text{ or } w \text{ is a computation and } \|\langle e_1,a\rangle\|^1 \leqq \|w\|^1 \text{ and } y=\langle e_2,\{e_1\}^1(a),a\rangle) \text{ otherwise.}$$

Let a_0 be the bound of the index-set for α. For an ordinal σ, let T_σ be the relation defined by

$$T_\sigma = \{\beta \in X : \forall x<a_0\ R(\alpha_x,\beta_x,w)\} \quad \text{where } \|w\|^1 = \sigma .$$

Then

$$\beta \in T_\sigma \vee \|\beta\| < \|\alpha\| \quad \text{and} \quad \sigma<\tau \vee T_\sigma \subseteqq T_\tau .$$

Now, let W be the set of prewellorderings of $\{c : c<a_0\}^\omega$ that are in $L_{\|<\|}[I,<]$. W is uniformly E-recursive in $I,<,\alpha$.

For $\delta \in W$, let $O(\delta)$ be the length of the prewellordering δ. Let $\lambda=\sup\{O(\delta)\,;\,\delta \in W\}$.

Let C be the set of ^{I}E-computations.

There is an index m_1 such that $\{m_1\}^2(m,\alpha,w)\downarrow$ if

i) $w \in C$ or w is an E-computation in $I,<$.
ii) $\{m\}^2(\beta)\downarrow$ for all $\beta \in T_{\|w\|^1}\cdot(T_{\|w\|^2})$

and if $w \in C$, then

$$\|\langle m_1,m,\alpha,w\rangle\|^2 > \|\langle m,\beta\rangle\|^2 \quad \text{for all } \beta \in T_{\|w\|^1}(T_{\|w\|^2}) .$$

By the recursion theorem one can find an index m_2 such that $\{m_2\}^2(m, \alpha, \gamma)\downarrow$ if $\gamma \in W$ and for all $\gamma' \in W$

$$O(\gamma') < O(\gamma) \Rightarrow \{m_2\}^2(m, \alpha, \gamma')\downarrow \text{ and } \{m_1\}^2(m, \alpha, \langle m_2, m, \alpha, \gamma' \rangle)\downarrow$$

Moreover, m_2 may be chosen such that

$$\|\langle m_2, m, \alpha, \gamma \rangle\|^2 > \|\langle m_2, m, \alpha, \gamma' \rangle\|^2, \|\langle m_1, m, \alpha, \langle m_2, m, \alpha, \gamma' \rangle\rangle\|^2$$

for all γ' such that $O(\gamma') < O(\gamma)$.

Then, if $O(\gamma') < O(\gamma)$ and $\beta \in T_{\|\langle m_1, m, \alpha, \gamma' \rangle\|^2}$ it follows that $\|\beta\| < \|\alpha\|$ and by the induction hypothesis

$$\|\langle m_2, m, \alpha, \gamma \rangle\|^2 > \|\langle m_1, m, \alpha, \langle m_2, m, \alpha, \gamma \rangle\rangle\|^2 > \|\langle m, \beta \rangle\|^2 \geqq \|\beta\|$$

There is an index m_3 such that

$$\{m_3\}^2(m, \alpha)\downarrow \quad \text{if } \forall \gamma \in W \{m_2\}^2(m, \alpha, \gamma)\downarrow$$

and

$$\|\langle m_3, m, \alpha \rangle\|^2 > \|\langle m_2, m, \alpha, \gamma \rangle\|^2 \quad \text{for all } \gamma \in W.$$

For $\tau < \lambda$, let

$$\sigma(\tau) = \inf\{\|\langle m_2, m, \alpha, \gamma \rangle\|^2 \ ; \ \gamma \in W \text{ and } O(\gamma) = \tau\}.$$

Then $\{\sigma(\tau) \ ; \ \tau < \lambda\}$ is a strictly increasing sequence of ordinals bounded above by $\|\langle m_3, m, \alpha \rangle\|^2$

CLAIM 1. *There is an ordinal $\tau' < \lambda$ such that $T_{\sigma(\tau')} = T_{\sigma(\tau)}$ when $\tau' \leqq \tau \leqq \lambda$.*

PROOF. Assume that this is not the case. We are going to construct an element of W of length λ, and there by obtain a contradiction.

Suppose $\forall \tau' < \lambda \, \exists \tau \, (\tau' < \tau < \lambda)$ and $T_{\sigma(\tau')} \neq T_{\sigma(\tau)}$. Take $\tau' < \lambda$. Let τ be minimal such that $\tau' < \tau$ and $T_{\sigma(\tau')} \not\supseteq T_{\sigma(\tau)}$.

We may effectively in τ' and τ choose w' and w such that $\|w\|^1 = \tau$, $\|w'\|^1 = \tau'$ and $w, w' \in C$. If $\beta \in T_{\sigma(\tau)} \setminus T_{\sigma(\tau')}$, then

$$\forall x < a_0 \ R(\alpha_x, \beta_x, w) \quad \text{and} \quad \neg \forall x < a_0 \ R(\alpha_x, \beta_x, w').$$

If $\neg R(\alpha_x, \beta_x, w')$, then α_x is a substitution

$$\{e\}^1(a) \cong \{e'\}^1(\{e_1\}^1(a)a), \qquad \beta_x = \langle e', \{e_1\}^1(a), a \rangle$$

and $\|w'\|^1 < \|\langle e_1, a \rangle\|^1 \leqq \|w\|^1$. Hence $R(\alpha_x, \beta_x, w'')$ for all w'' such that $\|w''\|^1 \geqq \|w\|^1$.

Let

$$P(\tau') = \{x < a_0 \ ; \ \exists \beta \in T_{\sigma(\tau)} \setminus T_{\sigma(\tau')} \neg R(\alpha_x, \beta_x, w').$$

$P(\tau')$ is independent of the choice of w, w', $P(\tau')$ is nonempty, $P(\tau')=P(v)$ for $\tau' \leqq v < \tau$, and for $\tau \leqq v$ will $P(\tau')$ and $P(v)$ be disjoint.

For each $v < \lambda$, select $\gamma_v \in W$ such that $O(\gamma_v)=v$. Define

$$s_1 * \langle t_1 \rangle \prec s_2 * \langle t_2 \rangle \text{ if for some minimal } \tau_1, \tau_2 ,$$
$$t_1 \in P(\tau_1) \wedge t_2 \in P(\tau_2) \wedge s_1 \in \text{Field}\,(\gamma_{\tau_1}) \wedge s_2 \in \text{Field}\,(\gamma_{\tau_2})$$
$$\wedge\ (\tau_1 < \tau_2 \vee (\tau_1 = \tau_2 = \tau \wedge s_1 <_{\gamma_\tau} s_2))$$

where $*$ is the concatenation of sequences.

$\prec$ will be a prewellordering on $\{c\ ;\ c < a_0\}^{\omega}$ and will be recursive in I, $<$, α, and thus an element of $L_{\|<\|}[I, <]$ by lemma 1. So $\prec \in W$. But $\|\prec\| = \lambda$ which is absurd. This ends the proof of claim 1.

Let $\sigma = \text{Sup}\,\{\sigma(\tau) : \tau < \lambda\}$

CLAIM 2. $\sigma \geqq \|\alpha\|$ (Hence $\|\langle m_3, m, \alpha\rangle\|^2 > \|\alpha\|$)

PROOF. Suppose $\sigma < \|\alpha\|$. Let $x < a_0$. If α_x is a substitution

$$\{e\}^1(a) \approx \{e'\}^1(\{e_1\}^1(a), a)$$

and $\sigma(\tau') < \|\langle e_1, a\rangle\|^1 < \sigma$, choose $\beta \in T_{\sigma(\tau')}$ (where τ' comes from claim 1).

Let $\beta'_y = \beta_y$ if $y \neq x$ and let $\beta'_x = \langle e', \{e_1\}^1(a), a\rangle$. Then $\beta' \in X$ and $\beta' \in T_\sigma \setminus T_{\sigma(\tau')}$. This contradicts claim 1, so either $\|\langle e_1, a\rangle\|^1 \leqq \sigma(\tau')$ or $\|\langle e_1, a\rangle\|^1 \geqq \sigma$.

Now, let β be defined in the following way: Let $x < a_0$. If α_x is not a substitution, let β_x be such that $S(\alpha_x, \beta_x)$ and $\|\beta_x\|^1 \geqq \sigma$. By the assumption it is always possible to find such β_x, and by the recursive wellordering we may select β_x from x in a recursive way.

If α_x is the substitution $\{e\}^1(a) \approx \{e'\}^1(\{e_1\}^1(a), a)$, let

$$\beta_x = \begin{cases} \langle e', \{e_1\}^1(a), a\rangle & \text{if } \|\langle e_1, a\rangle^1 \leqq \sigma(\tau') \\ \langle e_1, a\rangle & \text{otherwise} \end{cases}$$

Then $\|\beta\| \geqq \sigma$, and $\beta \in X$ by lemma 1. By construction, $\beta \in T_{\sigma(\tau')}$. Choose τ such that $\tau' < \tau < \lambda$. Choose $\gamma', \gamma \in W$ such that $O(\gamma') = \tau'$, $O(\gamma) = \tau$,

$$\sigma(\tau') = \|\langle m_2, m, \alpha, \gamma'\rangle\|^2 \quad \text{and} \quad \sigma(\tau) = \|\langle m_2, m, \alpha, \gamma\rangle\|^2 .$$

By the construction of m_2,

$$\|\langle m_2, m, \alpha, \gamma\rangle\|^2 > \|\beta'\| \quad \text{for all } \beta' \in T_{\|\langle m_2, m, \alpha, \gamma'\rangle\|^2} .$$

Hence $\|\langle m_2, m, \alpha, \gamma\rangle\|^2 > \|\beta\|$ since $\beta \in T_{\sigma(\tau')} = T_{\|\langle m_2, m, \alpha, \gamma'\rangle\|^2}$, contradicting the fact that $\|\langle m_2, m, \alpha, \gamma\rangle\|^2 = \sigma(\tau) < \sigma \leqq \|\beta\|$. Thus claim 2 is proved.

By the second recursion theorem for set recursion there is an index m such that

$$\{m_3\}^2(m,\alpha) \cong \{m\}^2(\alpha) \qquad \|\langle m,\alpha\rangle\|^2 > \|\langle m_3,m,\alpha\rangle\|^2$$

for all $\alpha \in X$. This m satisfies part i) of the sublemma. Part ii of the sublemma is proved by induction on $\|\langle m; \beta\rangle\|^2$.

As an induction hypothesis, suppose that $\{m\}^2(\alpha)\downarrow$ and that ii) is satisfied for all β such that $\|\langle m,\beta\rangle\|^2 < \|\langle m,\alpha\rangle\|^2$.

Since $\{m\}^2(\alpha)\downarrow$,

$$\{m_3\}^2(m,\alpha)\downarrow \quad \text{and} \quad \|\langle m,\alpha\rangle\|^2 > \|\langle m_3,m,\alpha\rangle\|^2 .$$

Also $\{m_2\}^2(m,\alpha,\gamma)\downarrow$ for all $\gamma \in W$. Let the ordinals $\{\sigma(\tau) : \tau<\lambda\}$ and σ be defined as before. Choose $\tau'<\lambda$ as before. As in claim 2, if α_x is a substitution

$$\{e\}^1(a) \cong \{e'\}^1(\{e_1\}^1(a),a)$$

then either

$$\|\langle e_1,a\rangle\|^1 \leqq \sigma(\tau') \quad \text{or} \quad \|\langle e_1,a\rangle\|^1 \geqq \sigma .$$

We will prove that $\|\alpha\| \leqq \sigma$, so assume in order to obtain a contradiction that $\|\alpha\| > \sigma$. Construct β as in claim 2. By construction $\beta \in T_{\sigma(\tau')}$ and $\|\beta\| \geqq \sigma$. Choose $\gamma' \in W$ such that $\sigma(\tau') = \|\langle m_2,m,\alpha,\gamma'\rangle\|^2$. Choose $\gamma \in W$ such that $O(\gamma') < O(\gamma)$. By the construction of m_2:

$$\|\langle m_2,m,\alpha,\gamma\rangle\|^2 > \|\langle m_1,m,\alpha,\langle m_2,m,\alpha,\gamma'\rangle\rangle\|^2$$

By construction of m_1:

$$\|\langle m_1,m,\alpha,\langle m_2,m,\alpha,\gamma'\rangle\rangle\| > \|\langle m,\beta\rangle\|^1 ,$$

since $\beta \in T_{\|\langle m_2,m,\alpha,\gamma'\rangle\|^2} = T_{\sigma(\tau')}$.

By the induction hypothesis and part i) of the sublemma, $\|\beta\| \leqq \|\langle m,\beta\rangle\|^1$. Hence $\|\beta\| < \|\langle \mathrm{m}_2,\mathrm{m},\alpha,\gamma\rangle\|^2$ for all $\gamma \in W$ such that $O(\gamma') < O(\gamma)$. By the definition of $\sigma(\tau)$, $\|\beta\| < \sigma(\tau)$ when $\tau' < \tau < \lambda$, so $\|\beta\| < \sigma$. This contradicts the assumption, and the sublemma and lemma 2 is established.

From lemma 1 and lemma 2 it is now trivial to prove theorem 1.

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