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HARMONIC MAPS FROM SURFACES TO CERTAIN KAEHLER MANIFOLDS

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Introduction.

In this paper we study some relations between harmonic maps of negatively curved manifolds and bounds on the homology invariants of such maps. The classical example we have in mind is Kneser's beautiful inequality on the degree of a map between surfaces of genus greater than one [8]. This inequality has recently been generalized by M. Gromov to maps between compact manifolds of constant negative curvature.

Since the volume-decreasing properties of holomorphic maps between hyperbolic complex manifolds give similar inequalities for holomorphic maps, cf. [9], it is natural to ask whether harmonic maps between negatively curved manifolds enjoy similar volume-decreasing properties. If this were true one would obtain very easily many inequalites of Kneser type, since in the presence of negative curvature every smooth map can be deformed to a harmonic one [4].

Sufficient conditions for a harmonic mapping to be volume-decreasing have been given by Chern and Goldberg in [2], but there is no discussion of their necessity. In section 3 we construct examples of harmonic maps between closed surfaces of constant negative curvature, and hence of the Poincaré disk, which are not area-decreasing. Thus there is no general area-decreasing (or volume-decreasing) Schwarz lemma, and Kneser inequalities cannot be derived in the same fashion as for holomorphic maps.

But in [5] Eells and Wood showed that Kneser's classical inequality can still be derived from special properties of harmonic maps. In a similar spirit we derive in section 4 an inequality for the degree of a map from a Riemann surface to a compact Kaehler manifold of constant negative holomorphic sectional curvature. By degree we mean essentially the intersection number with a hyperplane section, i.e., the integral of the Kaehler class. This inequality gives a lower bound on the genus of a surface representing a homology class of given degree, and it is interesting to compare it with a conjecture of Thom in

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the dual positively curved situation. Thom asks if the minimal genus of a smoothly embedded surface of a given degree in the complex projective plane is that of a non-singular algebraic curve of the same degree.

We remark that this inequality on the degree can be derived from inequalities on the first Chern class of flat bundles with structure group being the group of biholomorphic maps of the unit ball in $\mathbf{C}^n$, and that the inequalities on flat bundles follow from the methods of [3] and [13]; cf. Lusztig's remarks on Kneser's inequality quoted in [5]. The essential use of harmonic mappings is not in the derivation of the inequality, but rather in deciding when equality can hold in Theorem (4.14). We prove that a map has maximal degree if and only if it can be deformed to a holomorphic totally geodesic immersion. The proof of this fact is a generalization of the proof that a harmonic degree-one map of a negatively curved surface is a diffeomorphism [11], [12]. The point here is that if an inequality is proved by deforming the map to a good canonical map, then if equality holds the structure of this map must be particularly simple. It is interesting to note that Kneser's proof [8] was in this spirit, including his discussion of equality.

Our results rely on formulas for the Laplacian of the (1,0) and (0,1) energy which are stated in section 1 and proved in section 2. A more general formula of this type was first proved by Lichnerowicz [6]. For maps between surfaces there is a proof in [12], and a different proof of the formula used here appears in [14], where applications are given to the dual situation of maps to projective space. Since notational conventions in this subject vary so widely, we have found it preferable to include our own derivation.

I am very grateful to R. Schoen and S. Yau for making a preprint of [12] available to me. The discussion of equality in (4.3) is based totally on their work. I also thank J. Eells for his comments on the manuscript, and the Universities of Warwick and Copenhagen for their hospitality while most of this work was in progress.

1. The basic formulas.

Let S be a Riemann surface with Hermitian metric g and Kaehler form ω_S, and let X be an n-dimensional Kaehler manifold with Hermitian metric h and Kaehler form ω_X. In local coordinates

$$g = g dz d\bar{z}, \qquad h = h_{\alpha\bar{\beta}} dw^\alpha d\bar{w}^\beta ,$$

$$\omega_S = \frac{i}{2} g dz \wedge d\bar{z}, \qquad \omega_X = \frac{i}{2} h_{\alpha\bar{\beta}} dw^\alpha \wedge d\bar{w}^\beta .$$

Note that $\omega_S = dA$, the area form of g.

Let $f: S \to X$ be a C^∞ map. We write df for the real differential of f and

$$\partial^{1,0}f: T^{1,0}S \to T^{1,0}X, \qquad \partial^{0,1}f: T^{1,0}S \to T^{0,1}X,$$

for the corresponding components of the complexification of df. In local coordinates $\partial^{1,0}f$, $\partial^{0,1}f$ correspond to $w_z^\alpha, w_{\bar z}^\alpha$ respectively.

Recall [4] that f is called *harmonic* if it is an extremal of the energy functional

$$E(f) = \int_S e(f)\,dA$$

where the energy density $e(f)$ is defined by

$$e(f) = \tfrac{1}{2}\|df\|^2 = \tfrac{1}{2}\operatorname{tr} df(df)^t .$$

In local coordinates

$$e(f) = \frac{1}{g}(h_{\alpha\beta}w_z^\alpha \bar{w}_{\bar z}^\beta + h_{\alpha\beta}w_{\bar z}^\alpha \bar{w}_z^\beta) = e'(f)+e''(f)$$

where $e'(f)=\|\partial^{1,0}f\|^2$, $e''(f)=\|\partial^{0,1}f\|^2$. Thus f is holomorphic (anti-holomorphic) if and only if $e''(f)=0$ $(e'(f)=0)$. Observe that

$$e'(f)-e''(f) = \frac{1}{g}h_{\alpha\beta}(w_z^\alpha \bar{w}_{\bar z}^\beta - w_{\bar z}^\alpha \bar{w}_z^\beta)$$

which can be written as

$$(1.1) \qquad (e'(f)-e''(f))\omega_S = f^*\omega_X .$$

The differential equation [4, p. 109] for an extremal can be written in the Kaehler context as

$$(1.2) \qquad w_{z\bar z}^\beta + \theta_{\alpha\gamma}^\beta w_z^\alpha w_{\bar z}^\gamma = 0$$

where $\theta = \theta_{\alpha\gamma}^\beta dw^\gamma$ is the connection form of the Hermitian connection on $T^{1,0}X$. Recall that

$$(1.3) \qquad \theta = (\partial N)N^{-1} \quad \text{where } N_\alpha^\beta = h_{\alpha\beta} .$$

Also if

$$(1.4) \qquad K_{\alpha\gamma\bar\delta}^\beta = (\theta_{\alpha\gamma}^\beta)_{\bar w^\delta} ,$$

then $\Omega_\alpha^\beta = K_{\alpha\gamma\bar\delta}^\beta dw^\gamma \wedge d\bar{w}^\delta$ is the curvature form of the Hermitian connection.

In the following theorem we use the notation

$$\beta'(f) = \nabla_1\partial^{1,0}f, \qquad \beta''(f) = \nabla_2\partial^{0,1}f,$$

where ∇_1, ∇_2 denotes the covariant derivatives on $(T_S^{1,0})^*\otimes f^*T^{1,0}X$, $(T_S^{1,0})^*\otimes f^*T^{0,1}X$ respectively. We also write

$$(1.5)\qquad \begin{aligned} q'(f) &= \frac{1}{g^2} K_{\alpha\bar\beta\gamma\bar\delta}(w_z^\gamma \bar w_z^\delta - w_{\bar z}^\gamma \bar w_{\bar z}^\delta) w_z^\alpha \bar w_{\bar z}^\beta \\ q''(f) &= \frac{1}{g^2} K_{\alpha\bar\beta\gamma\bar\delta}(w_z^\gamma \bar w_{\bar z}^\delta - w_{\bar z}^\gamma \bar w_z^\delta) w_{\bar z}^\alpha \bar w_z^\beta . \end{aligned}$$

In terms of the curvature form Ω we could equivalently write, at each $x \in S$,

$$q'(f)\omega_S = \langle f^*\Omega(\partial^{1,0}f(v)), \partial^{1,0}f(v)\rangle$$
$$q''(f)\omega_S = \langle f^*\Omega(\partial^{0,1}f(v)), \partial^{0,1}f(v)\rangle$$

where v is a unit vector in $T_x^{1,0}S$ and $\langle\cdot,\cdot\rangle$ is the Hermitian inner product f^*h.

THEOREM 1.6. *If $f: S \to X$ is harmonic, then*

$$\tfrac{1}{4}\Delta e'(f) = \|\beta'(f)\|^2 + q'(f) + \tfrac{1}{2}K_S e'(f)$$
$$\tfrac{1}{4}\Delta e''(f) = \|\beta''(f)\|^2 - q''(f) + \tfrac{1}{2}K_S e''(f),$$

where Δ is the Laplace operator of f and K_S is the Gaussian curvature of S.

COROLLARY 1.7. *If $f: S \to X$ is harmonic, then at every point of S where $e' \neq 0$ (respectively $e'' \neq 0$),*

$$\tfrac{1}{4}\Delta \log e'(f) = \alpha'(f) + \frac{1}{e'(f)} q'(f) + \tfrac{1}{2}K_S$$

$$\tfrac{1}{4}\Delta \log e''(f) = \alpha''(f) - \frac{1}{e''(f)} q''(f) + \tfrac{1}{2}K_S,$$

where $\alpha'(f), \alpha''(f) \geqq 0$.

COROLLARY 1.8. *If X is a Riemann surface and $f: S \to X$ is harmonic, then*

$$\tfrac{1}{4}\Delta e'(f) = \|\beta'(f)\|^2 - \tfrac{1}{2}K_X(e'(f) - e''(f))e'(f) + \tfrac{1}{2}K_S e'(f)$$
$$\tfrac{1}{4}\Delta e''(f) = \|\beta''(f)\|^2 + \tfrac{1}{2}K_X(e'(f) - e''(f))e''(f) + \tfrac{1}{2}K_S e''(f).$$

Note that the sum of the two formulas of Theorem 1.6 gives the formula for Δe of Eells and Sampson [4, p. 123]. The difference of the two formulas of Corollary 1.8 gives a formula for Δu, $u = (e' - e'')^2$. This u has the same meaning as in the paper of Chern and Goldberg [2], and this gives the two dimensional case of their formula (57).

2. Proof of the formulas.

We compute at each $x \in S$ the value of $\tfrac{1}{4}\Delta e'$ by using holomorphic normal coordinates centered at x and $f(x)$. Recall that in such coordinates we have

$$g = 1, \quad g_z = 0, \quad g_{z\bar{z}} = -\tfrac{1}{2}K_S \quad \text{at } x, \tag{2.1}$$

and

$$h_{\alpha\bar{\beta}} = \delta_{\alpha\beta}, \quad h_{\alpha\bar{\beta}w^\gamma} = \theta^\beta_{\alpha\gamma} = 0, \quad h_{\alpha\bar{\beta}w^\gamma\bar{w}^\delta} = \theta^\beta_{\alpha\gamma\bar{w}^\delta} = K_{\alpha\bar{\beta}\gamma\bar{\delta}} \tag{2.2}$$

at $f(x)$, where the last two equations follow from (1.3) and (1.4). We obtain

$$\begin{aligned}\tfrac{1}{4}\Delta e' &= \frac{1}{g}e'_{z\bar{z}} = \frac{1}{g}\left(\frac{1}{g}h_{\alpha\bar{\beta}}w^\alpha_z\bar{w}^\beta_{\bar{z}}\right)_{z\bar{z}} \\ &= h_{\alpha\bar{\beta}z\bar{z}}w^\alpha_z\bar{w}^\beta_{\bar{z}} + w^\alpha_{zz\bar{z}}\bar{w}^\alpha_{\bar{z}} + w^\alpha_{zz}\bar{w}^\alpha_{\bar{z}\bar{z}} + w^\alpha_z\bar{w}^\beta_{\bar{z}z\bar{z}} + \tfrac{1}{2}K_S w^\alpha_z\bar{w}^\alpha_{\bar{z}}\end{aligned} \tag{2.3}$$

at x, where in the last expression we have simplified the formula by using the relations (2.1), (2.2) and

$$w_{z\bar{z}} = \bar{w}_{\bar{z}z} = 0 \quad \text{at } x \tag{2.4}$$

which results from (2.2) and the harmonic equation (1.2). Using the chain rule and simplifying by (2.2) we get

$$h_{\alpha\bar{\beta}z\bar{z}} = h_{\alpha\bar{\beta}w^\gamma\bar{w}^\delta}w^\gamma_z\bar{w}^\delta_{\bar{z}} + h_{\alpha\bar{\beta}\bar{w}^\gamma w^\delta}\bar{w}^\gamma_z w^\delta_{\bar{z}}\,. \tag{2.5}$$

Differentiating the harmonic equation (1.2) we obtain in the same way

$$w^\beta_{z\bar{z}z} = -\theta^\beta_{\alpha\gamma\bar{w}^\mu}w^\alpha_z w^\gamma_{\bar{z}}\bar{w}^\mu_z \qquad \bar{w}^\beta_{\bar{z}z\bar{z}} = -\overline{(\theta^\beta_{\alpha\gamma\bar{w}^\mu})}\bar{w}^\alpha_{\bar{z}}\bar{w}^\gamma_z w^\mu_{\bar{z}} \tag{2.6}$$

at x, where the second equation is just the conjugate of the first. If we use the symmetry relation

$$\overline{(\theta^\beta_{\alpha\gamma\bar{w}^\mu})} = \theta^\alpha_{\beta\mu\bar{w}^\gamma}\,,$$

which results from $h_{\alpha\bar{\beta}} = \overline{h_{\beta\bar{\alpha}}}$, rewrite (2.5) and (2.6) in terms of $K_{\alpha\bar{\beta}\gamma\bar{\delta}}$ (last equation (2.2)), and substitute the result in (2.3) we get

$$\tfrac{1}{4}\Delta e' = w^\alpha_{zz}\bar{w}^\alpha_{\bar{z}\bar{z}} + K_{\alpha\bar{\beta}\gamma\bar{\delta}}(w^\gamma_z\bar{w}^\delta_{\bar{z}} - w^\gamma_{\bar{z}}\bar{w}^\delta_z)w^\alpha_z\bar{w}^\beta_{\bar{z}} + \tfrac{1}{2}K_S w^\alpha_z\bar{w}^\alpha_{\bar{z}} \tag{2.7}$$

at x. By (2.4), the first term is $\|\beta'(f)\|^2(x)$. The first formula of Theorem 1.6 follows, because the remaining terms of (2.7) clearly agree with the corresponding terms in (1.6). The second equation is proved in the same way.

To prove Corollary 1.7 we use

$$\tfrac{1}{4}\Delta \log e' = \frac{1}{g}(\log e')_{z\bar{z}} = \tfrac{1}{4}\Delta e'/e' - \frac{1}{g}e'_z e'_{\bar{z}}\,.$$

From this we obtain the first formula with

$$\alpha'(f) = \frac{\beta'(f)\|^2}{e'} - \frac{1}{g}\frac{e'_z e'_{\bar{z}}}{(e')^2}\,.$$

In holomorphic normal coordinates we have, because of (2.4),

$$\|\beta'(f)\|^2 = w^\alpha_{zz}\bar{w}^\alpha_{\bar{z}\bar{z}}, \qquad e'_z = w^\alpha_{zz}\bar{w}^\alpha_{\bar{z}}, \qquad e'_{\bar{z}} = w^\alpha_z \bar{w}^\alpha_{\bar{z}\bar{z}} \ .$$

Schwarz's inequality gives $|e'_z e'_{\bar{z}}| \leqq w^\alpha_{zz}\bar{w}^\alpha_{\bar{z}\bar{z}} e'$, hence $\alpha'(f) \geqq 0$. The same reasoning gives the second formula, with $\alpha''(f) \geqq 0$.

To prove Corollary 1.8 observe that when X has complex dimension one there is only one $K_{\alpha\bar{\beta}\gamma\bar{\delta}}$ and this is equal to $-\frac{1}{2}K_X$ (cf. (2.1) and (2.2)).

3. Harmonic maps to surfaces.

Our counterexamples to the Schwarz lemma are based on the following more general observation:

THEOREM 3.1. *Let S and X be compact Riemann surfaces with metrics g and h respectively, and let f: $S \to X$ be a harmonic map that preserves area (i.e., $f^*\omega_X = \pm\omega_S$) and Gaussian curvature (i.e., $f^*K_X = K_S$). Then either*

(1) *f is a holomorphic or anti-holomorphic local isometry, or*
(2) *(S, g) and (X, h) are both flat tori.*

PROOF. Assume first that f preserves orientation, i.e., $f^*\omega_X = \omega_S$, which by (1.1) is equivalent to $e'(f) - e''(f) \equiv 1$. Since we also have $f^*K_X = K_S$, Corollary 1.8 gives

$$\tfrac{1}{4}\Delta e'(f) = \|\beta'(f)\|^2 \geqq 0 \, ,$$

thus e' is subharmonic, hence constant, and $e'' = e' - 1$ is also constant. Thus if z and w are holomorphic normal coordinates centered at $p \in S$ and $f(p) \in X$, we have

$$0 = e''(f)_{\bar{z}} = (w_{\bar{z}}\bar{w}_z)_{\bar{z}} = w_{\bar{z}\bar{z}}\bar{w}_z \ .$$

Since at p we have $\beta''(f) = w_{\bar{z}\bar{z}}$ and $e''(f) = |\bar{w}_z|^2$, we conclude that

$$\text{either } \beta''(f) = 0 \text{ or } e''(f) = 0 \, . \tag{3.2}$$

Now if f: $S \to X$ is harmonic where X is any Kaehler manifold, it is known that $(h_{\alpha\bar{\beta}} \circ f) w^\alpha_z \bar{w}^\beta_z (dz)^2$ is a *holomorphic* quadratic differential on S. This is checked by computing its derivative with respect to $\bar{z}$ using (1.2), and implies

LEMMA 3.3. *Let f: $S \to X$ be harmonic. Then each of the functions $e'(f)$, $e''(f)$ is either identically zero or has only isolated zeros.*

From (3.2) and this lemma we conclude that either $e'(f) \equiv 0$ or $\beta''(f) \equiv 0$. In the first case f would be anti-holomorphic, which contradicts the assumption that f is orientation-preserving, thus $\beta''(f) \equiv 0$. If we now use in the second equation of Corollary 1.8 that $e'(f) - e''(f) \equiv 1$, $\Delta e''(f) = \|\beta''(f)\| \equiv 0$ and $f^*K_X = K_S = K$, say, we conclude that $Ke''(f) \equiv 0$. From Lemma 3.3 we see again that either $K \equiv 0$ (and both surfaces are flat tori) or $e''(f) \equiv 0$, and f is holomorphic, hence also locally isometric since it is both conformal and area-preserving.

Finally, if f reverses orientation, i.e., $f^*\omega_X = -\omega_S$, we argue in the same way starting from the second equation (1.8) to conclude that either both surfaces are flat tori or f is anti-holomorphic.

COROLLARY 3.4. *Let S be a compact Riemann surface with metric of constant negative curvature, and let $f\colon S \to S$ be harmonic and area-preserving. Then f is a holomorphic or anti-holomorphic isometry.*

If $f\colon S \to S$, we say that f is *area-decreasing* if $|e'(f) - e''(f)| \leqq 1$ at all points of S. By (1.1), this is equivalent to $|f^*\omega_S| \leqq |\omega_S|$, thus, strictly speaking, we should say that f does not increase area.

COROLLARY 3.5. *Let S be a compact Riemann surface with metric of constant negative curvature, and let $\varphi\colon S \to S$ be a smooth map of degree one that is not homotopic to a holomorphic map. Then the harmonic map f homotopic to φ is not area-decreasing.*

PROOF. If f were area-decreasing we would have $|e'(f) - e''(f)| \leqq 1$ and

$$\int (e'(f) - e''(f))\omega_S = \int \omega_S\,,$$

hence $e'(f) - e''(f) \equiv 1$, i.e., f is area-preserving. By the previous corollary f would be holomorphic, contrary to the hypothesis.

Observe that there are plenty of maps φ that satisfy the hypothesis of Corollary 3.5, for example, any diffeomorphism which is not of finite order in homology, e.g., a "Dehn twist". A harmonic map homotopic to φ exists by the main theorem of Eells and Sampson [4], and moreover it is also a diffeomorphism, cf. [11], [12]. Note, on the other hand, that all these diffeomorphisms are homotopic to area-preserving diffeomorphisms of S. This follows, say, by composing φ with the diffeomorphism constructed in [10] which brings $\varphi^*\omega_S$ to ω_S.

COROLLARY 3.6. *There exist harmonic diffeomorphisms of the unit disk in $\mathbf{C}$ with the Poincaré metric that are not area-decreasing.*

PROOF. Lift one of the maps f of the previous Corollary to the universal cover of S.

Finally we remark that a flat torus has plenty of harmonic diffeomorphisms that are area-preserving but neither holomorphic nor anti-holomorphic.

4. Maps to quotients of the ball.

Let S be a compact surface with metric g of constant curvature -1 and fundamental group Γ. We think of S as D/Γ, where D is the unit disk with the Poincaré metric and Γ is a Fuchsian group. Let B^n be the unit ball in $\mathbf{C}^n$ with metric

$$h = 4\frac{(1-\sum w^\alpha \bar{w}^\alpha)(\sum dw^\alpha d\bar{w}^\alpha)+(\sum \bar{w}^\alpha dw^\alpha)(\sum w^\alpha d\bar{w}^\alpha)}{(1-\sum w^\alpha \bar{w}^\alpha)^2} \tag{4.1}$$

of constant holomorphic sectional curvature -1 and invariant under the group $G^n = \mathrm{SU}(1,n)$/center of biholomorphic maps of B^n. Recall that $\mathrm{SU}(1,n)$ is the subgroup of $\mathrm{SL}(n+1,\mathbf{C})$ that preserves the form $|z_0|^2-|z_1|^2-\ldots-|z_n|^2$ and acts on B^n via the natural identification of B^n with the set of lines through the origin in $\mathbf{C}^{n+1}$ on which the form is positive. Finally let Γ' be a discrete subgroup of G^n which acts freely on B^n. The quotient manifold $X = B^n/\Gamma'$ is then a Kaehler manifold of constant holomorphic sectional curvature -1. Our goal is to prove the following theorem:

THEOREM 4.2. *Let S and X be as above, and let $f\colon S \to X$ be a harmonic map. Then*

$$\left|\int_S f^*\omega_X\right| \leqq \int_S \omega_S . \tag{4.3}$$

Moreover equality holds if and only if there is a metric of constant curvature -1 on the surface S with respect to which f is a totally geodesic holomorphic or anti-holomorphic immersion.

We recall that a map f is totally geodesic if and only if it maps geodesics to geodesics. This is equivalent to $\beta(f)\equiv 0$, where β is the second fundamental form of f (cf. [4, pp. 123, 131]). In our situation $\beta(f)\equiv 0$, if and only if $\beta'(f) \equiv \beta''(f)\equiv 0$, with β', β'' as in section 1.

Note that if f is a holomorphic totally geodesic immersion with respect to some metric of constant negative curvature on S, then $f^*\omega_X$ is the Kaehler form of this metric, thus Gauss–Bonnet gives

$$\int_S f^*\omega_X = -2\pi\chi(S) = \int_{S'} \omega_S ,$$

and equality does hold in (4.3). Similarly, if f is an anti-holomorphic totally geodesic immersion, then

$$\int_S f^*\omega_X = -\int_S \omega_S$$

and equality again holds in (4.3).

LEMMA 4.4. *Let $f: D \to B^n$ be harmonic. Then*

$$q'(f) = \tfrac{1}{2}e'(f)(e'(f)-e''(f))+p(f)$$

$$q''(f) = \tfrac{1}{2}e''(f)(e'(f)-e''(f))-p(f)$$

where $p(f)\geqq 0$ and $p(f)(x)=0$ if and only if $d_xf(T_xD)$ is contained in a complex one-dimensional subspace of $T_{f(x)}B^n$.

PROOF. Let $x \in D$. Since $q'(f)$ is invariant under G^n, to compute $q'(f)(x)$ we may assume $f(x)=0 \in B^n$. At 0 one finds by direct computation from (4.1) that

$$K_{\alpha\beta\gamma\bar\delta} = 4\{\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}\} = \tfrac{1}{4}\{h_{\alpha\bar\beta}h_{\gamma\bar\delta}+h_{\alpha\bar\delta}h_{\bar\beta\gamma}\}\,. \tag{4.4}$$

Direct calculation from (1.5) gives

$$\begin{aligned} q'(f)(x) &= \frac{4}{g^2}\{2(w_z^\alpha\bar w_{\bar z}^\alpha)^2-(w_{\bar z}^\gamma\bar w_z^\gamma)(w_z^\alpha\bar w_{\bar z}^\alpha)-(w_z^\alpha\bar w_z^\alpha)(w_{\bar z}^\gamma\bar w_{\bar z}^\gamma)\} \\ &= \tfrac{1}{4}\{2(e')^2-e'e''-\frac{1}{g^2}\langle w_z, w_{\bar z}\rangle\langle w_{\bar z}, w_z\rangle\} \\ &= \tfrac{1}{2}\{(e')^2-e'e''\}+\frac{1}{4g^2}\{|w_z|^2|w_{\bar z}|^2-|\langle w_z, w_{\bar z}\rangle|^2\} \end{aligned}$$

from which the first formula follows, letting $p(f)$ be the second term in this expression, and from Schwarz's inequality we see that $p(f)\geqq 0$.

For the last assertion, let $V=d_xf(T_xD)\subset T_{f(x)}B^n$, and let W be the smallest J-invariant subspace of $T_{f(x)}B^n$ containing V. Then

$$W\otimes\mathbf{C} = \pi^{1,0}(V\otimes\mathbf{C})\oplus\pi^{0,1}(V\otimes\mathbf{C}) = \pi^{1,0}(V\otimes\mathbf{C})\oplus\overline{\pi^{1,0}(V\otimes\mathbf{C})}\,,$$

where $\pi^{1,0},\pi^{0,1}$ are the projections of $TB^n\otimes\mathbf{C}$ onto $T^{1,0}B^n$, $T^{0,1}B^n$ respectively. Clearly $\dim_{\mathbf{R}} W\leqq 2$ if and only if $\dim_{\mathbf{C}}\pi^{1,0}(V\otimes\mathbf{C})\leqq 1$. Since the latter space is spanned by $w_z^\alpha, w_{\bar z}^\alpha$, the equality part of Schwarz's inequality gives that $\dim_{\mathbf{C}}\pi^{1,0}(V\otimes\mathbf{C})\leqq 1$ if and only if $p(f)(x)=0$, and this completes the proof of the lemma.

We remark that a more geometric expression for $p(f)$ is the following. Let u, v be an orthonormal basis for T_xD, and let α be the angle between the planes $df(T_xD)$, $Jdf(T_xD)$ in $T_{f(x)}B^n$, and let A be the area of the parallelogram determined by $df(u)$ and $df(v)$. Then $16p(f)(x) = A^2 \sin^2 \alpha$, from which the asserted properties of $p(f)$ follow easily.

The lemma gives the following simplification of formulas (1.7):

COROLLARY 4.5. *If $f\colon D \to B^n$ is harmonic then*

$$\tfrac{1}{4}\Delta \log e'(f)\ \tfrac{1}{4}\Delta \log e'(f) = \alpha'(f) + \frac{p(f)}{e'(f)} + \tfrac{1}{2}(e' - e'' - 1)$$

$$\tfrac{1}{4}\Delta \log e''(f) = \alpha''(f) + \frac{p(f)}{e''(f)} - \tfrac{1}{2}(e' - e'' + 1)$$

where $\alpha', \alpha'', p \geqq 0$ and p is as in (4.4).

LEMMA 4.6. *Let p be an isolated zero of e', and let z be a holomorphic coordinate centered at p. Then there exists a positive integer m_p, a positive constant c, and a function $\varrho(z)$ which is C^∞ for $z \neq 0$, $\varrho \in O(|z|)$ and $\partial\varrho/\varrho x$, $\partial\varrho/\partial y \in O(1)$, so that*

$$e'(z) = |z|^{2m_p}(c + \varrho(z))$$

in a neighborhood of p. A similar local representation holds near an isolated zero of e''.

PROOF. We use equation (1.2) exactly as in the proof of equation (5) in [5] to show that the leading term in the Taylor expansion of w_z is holomorphic, i.e.,

$$w_z = Az^m + O(|z|^{m+1})$$

for some positive integer m and some non-zero vector $A = (a^1, \ldots, a^m)$. Thus

$$e'(z) = \frac{h_{\alpha\bar{\beta}}}{g} w_z^\alpha \bar{w}_z^\beta = \frac{h_{\alpha\bar{\beta}}(0)}{g(0)} a^\alpha \bar{a}^\beta |z|^{2m} + O(|z|^{2m+1}).$$

If we let $m_p = m$, $c = 1/g(0) h_{\alpha\bar{\beta}}(0) a^\alpha \bar{a}^\beta$ and let ϱ be the remainder term divided by $|z|^{2m}$ we obtain the desired representation. The bounds on ϱ follow from the standard properties of the remainder in Taylor's formula. The representation for e'' is obtained in the same way.

We now continue the proof of the theorem. To prove the inequality we consider two cases: (1) f is holomorphic or anti-holomorphic, (2) f is neither holomorphic nor anti-holomorphic. Case (1) follows from well-known area-decreasing properties of holomorphic maps, but for completeness we repeat the standard argument: Suppose f is holomorphic. Then $e'' \equiv 0$, $f^*\omega_B = e'\omega_S$ and the first equation of (1.6) together with (4.4) gives

$$\tfrac{1}{4}\Delta e' = \|\beta'(f)\|^2 + \tfrac{1}{2}(e')^2 - \tfrac{1}{2}e' \tag{4.7}$$

because $p(f) \equiv 0$. Let x be a point of S where e' attains its maximum. At x we have $\Delta e' \leqq 0$, hence $e'(e'-1) \leqq 0$ at x. But we must then have $e' \leqq 1$ at all points of S, hence $|\int f^*\omega_B| = \int e'\omega_S \leqq \int \omega_S$. If f is anti-holomorphic we get $e' = 0$, $e'' \leqq 1$.

Consider now case (2). We can apply the formulas of Corollary 4.5 at all points of S except the zeros of e', respectively e'', which by Lemma 3.3 are isolated. Let S_ε be the surface with boundary obtained by removing a disk of radius ε about each of the zeros $x_1, \ldots, x_k$ of e', and let $|z|^{2m_i'}(c_i + \varrho_i(z))$ be the local expression for e' near x_i given by Lemma 4.6. Green's Theorem and the bounds on ϱ_i easily give

$$\lim_{\varepsilon \to 0} \int_{S_\varepsilon} \Delta \log e' \omega_S = -2\pi \sum_i 2m_i' .$$

From Corollary 4.5 we have

$$\int_{S_\varepsilon} \left(\tfrac{1}{4}\Delta \log e' - \alpha'(f) - \frac{p(f)}{e'(f)} \right) \omega_S = \tfrac{1}{2} \int_{S_\varepsilon} (e' - e'' - 1)\omega_S ,$$

and letting $\varepsilon \to 0$

$$-\pi \sum_i m_i' - \int_S (\alpha' + p/e')\omega_S = \tfrac{1}{2} \int_S (e' - e'')\omega_S - \tfrac{1}{2} \int_S \omega_S . \tag{4.8}$$

Since m_i', α', $p \geqq 0$ it follows that

$$\int_S f^*\omega_X = \int_S (e' - e'')\omega_S \leqq \int_S \omega_S , \tag{4.9}$$

which is one half of the asserted inequality. Arguing on the same way with the zeros $y_1, \ldots, y_l$ of e'' and the corresponding exponents $2m_j''$ given by Lemma 4.6, we get

$$-\pi \sum_j m_j'' - \int_S \left(\alpha'' + \frac{p}{e''} \right) \omega_S = -\tfrac{1}{2} \int_S (e' - e'')\omega_S - \tfrac{1}{2} \int_S \omega_S ,$$

from which we get the second half of the inequality, namely

$$\int_S f^*\omega_X \geqq -\int_S \omega_S . \tag{4.10}$$

Now we examine when equality can hold in (4.3). First, if f is holomorphic, we must have $e' \equiv 1$, hence from (4.7) that $\beta'(f) \equiv 0$. Since f is holomorphic $\beta''(f) \equiv 0$, hence the second fundamental form $\beta(f) \equiv 0$ and f is totally geodesic, and an immersion since $e' \equiv 1$, thus $f^*\omega_X$ is never zero. The case that f is anti-holomorphic is handled similarly.

Now suppose that f is neither holomorphic nor anti-holomorphic and equality holds in (4.3). We must then have equality in (4.9) or (4.10). Suppose that equality holds in (4.9). Then the left hand side of (4.8) must be identically zero. Since the $m_i' > 0$, this sum must be empty, i.e., $e' > 0$, and we must also have $\alpha' \equiv p \equiv 0$. We summarize the situation as follows:

LEMMA. *If* $\int_S f^*\omega_X = \int_S \omega_S$, *then*

(4.11) $$e'(x) > 0 \quad \text{for all } x \in S\,,$$

(4.12) *for each* $x \in S$, $d_x f(T_x S)$ *is contained in a complex one-dimensional subspace of* $T_{f(x)}X$,

(4.13) $$\tfrac{1}{4}\Delta \log e'/e'' = -\alpha'' + e' - e''\,.$$

Note that the second assertion follows from $p \equiv 0$ and Lemma 4.4, and the third is obtained from the difference of the two equations of Corollary 4.5 once we set $\alpha' = p = 0$.

It now follows as in Theorem 3.1 of [12] that $e' - e'' \geqq 0$. Namely, from (4.13) we see that $\log e'/e''$ is superharmonic on the set where $e' - e'' < 0$, and from (4.11) it follows that $\log e'/e''$ is never $-\infty$. If the set where $e' - e'' < 0$ were non-empty, $\log e'/e''$ would take its minimum on the boundary, which is impossible since it is identically zero on the boundary.

We must then have $e' - e'' \geqq 0$, hence $\log e'/e'' \geqq 0$. Arguing as in Proposition 2.2 of [12] we see that $e' - e'' > 0$: Namely, if $e' - e'' = 0$ at some point x of S, since $e' > 0$ we must also have $e''(x) > 0$. Taking a small disk about x where $e'' > 0$, from (4.13) we get

$$\tfrac{1}{4}\Delta \log \frac{e'}{e''} \leqq e' - e'' = e''\left(\frac{e'}{e''} - 1\right) \leqq C \log e'/e''\,.$$

But Lemma 6′ of [7] asserts that a function $g \geqq 0$ which satisfies an inequality of the form $\Delta g \leqq Cg$ on some disk D centered at x must also satisfy

$$\iint_D g\, dA \leqq C' g(x)\,.$$

Applying this to $g = \log e'/e''$, we see that if $g(x) = 0$, then $g \equiv 0$ in a neighborhood of x, thus the zero set of g, which is the same as that of $e' - e''$, would be open in S. If $e'-e''$ vanished at some x, then it would vanish identically, i.e., $f^*\omega_X \equiv 0$, which is impossible when equality, holds in (4.9).

It follows that $e'-e''>0$ everywhere on S, i.e., $f^*\omega_X$ is never zero, and in particular f is an immersion. By (4.12) we have that for each $x \in S$, $d_x f$ gives an isomorphism between T_xS and a complex line in $T_{f(x)}X$. These isomorphisms give a complex structure on S with respect to which f is holomorphic. If we give S a metric of constant curvature -1 which is compatible with this complete structure, from the above discussion of equality in (4.3) for holomorphic maps it follows that f is totally geodesic.

Finally the case that equality holds in (4.10) is handled in the totally analogous way, thus completing the proof of Theorem 4.2.

THEOREM 4.14. *Let S be a compact orientable surface of genus $g>1$, let X be a compact Kaehler manifold of constant holomorphic sectional curvature -1, and let $\varphi\colon S \to X$ be a smooth map. Then*

$$\left|\int_S \varphi^*\omega_X\right| \leqq 4\pi(g-1)\,.$$

Equality holds if and only if there is a metric of constant curvature -1 on S and a map f homotopic to φ which is a totally geodesic holomorphic or anti-holomorphic immersion.

PROOF. X must be of the form B^n/Γ' for some discrete $\Gamma' \subset G^n$ which acts freely on B^n, and S can be represented as D/Γ. Since X is compact and negatively curved, by the main theorem of Eells and Sampson [4] φ is homotopic to a harmonic map f. The theorem follows by applying Theorem 4.2 to f and rewriting the right hand side of (4.3) by the Gauss–Bonnet theorem.

We remark that $\omega_X = -4\pi/n+1c_1(X)$, where $c_1(X)$ is the first Chern class of X and $n=\dim_{\mathbf{C}} X$. Thus the inequality in (4.14) can be equivalently written as

$$\left|\int_S \varphi^* c_1(X)\right| \leqq (n+1)(g-1)\,.$$

Finally we should give an example where the hypotheses of Theorem 4.14 are satisfied. Represent B^n not as the unit ball in $\mathbf{C}^n$, but rather as the ball of radius $\sqrt[4]{2}$, i.e., as the set of lines through the origin in $\mathbf{C}^{n+1}$ where the form $\sqrt{2}|z_0|^2-|z_1|^2-\ldots-|z_n|^2>0$. The group G^n of automorphisms of B^n is then represented as the subgroup of SL $(n+1, \mathbf{C})$ that preserves this form, modulo its center. Let I be the ring of integers in the field $\mathbf{Q}(i,\sqrt{2})$ and let $\Gamma'=\Gamma'(n)$ be the image in G^n of the subgroup of SL $(n+1, I)$ consisting of all matrices that preserve the above hermitian form and which are congruent to the identity matrix modulo 3. Then from the arguments at the end of Borel's paper [1] it follows that $X = B^n/\Gamma'(n)$ is a compact manifold of constant negative holomorphic curvature. Taking D/Γ to be $B^1/\Gamma'(1)$, then the map $f\colon D/\Gamma \to B^n/\Gamma'(n)$ induced by the map $(z_0, z_1) \to (z_0, z_1, 0, \ldots, 0)$ is a holomorphic totally geodesic immersion. (f is in fact an embedding, since some reflection shows that it is generically injective and its image is a component of the fixed point set of the isometry of X induced by the map

$$(z_0, z_1, z_2, \ldots, z_n) \to (z_0, z_1, -z_2, \ldots, -z_n) \text{ of } \mathbf{C}^{n+1}) .$$

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