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WHITNEY (b)-REGULARITY IS WEAKER THAN KUO'S RATIO TEST FOR REAL ALGEBRAIC STRATIFICATIONS

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We give examples of real algebraic hypersurfaces such that the full partition by dimension gives a stratification which is Whitney (b)-regular, but which fails to satisfy Kuo's ratio test (r), and hence also fails to satisfy the (w)-regularity of Verdier. Such a hypersurface can be a C^1 submanifold, so that the stratification is C^1 trivial, showing that (r) and (w) are not invariant under C^1 changes of coordinates, although they are C^2 invariant. We show that (w)-regularity is characterised by the possibility of extending rugose vector fields defined on some strata to rugose vector fields tangent to the remaining strata.

1. On regularity.

Let X be a C^1 submanifold of $\mathbf{R}^n$, and a subanalytic set (defined in [2]). Let Y be an analytic submanifold of $\mathbf{R}^n$ such that $0 \in Y \subset \bar{X} \setminus X$. Verdier [8] defines X to be (w)-*regular* over Y at 0 if,

(w) There is a constant $C > 0$ and a neighborhood U of 0 in $\mathbf{R}^n$ such that if $x \in U \cap X$ and $y \in U \cap Y$, then $d(T_yY, T_xX) \leqq C|x-y|$.

Here $d(.,.)$ is defined as follows.

DEFINITION. Let A, B, be vector subspaces of $\mathbf{R}^n$.

$$d(A,B) = \sup_{\substack{a \in A \\ |a|=1}} |a - \pi_B(a)| ,$$

where π_B is orthogonal projection onto B.

This is not symmetric in A and B. Clearly $d(A,B)=0$ if and only if $A \subseteqq B$. It is clear from the definition of (w) that it is a C^2 invariant, or more precisely

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that it is invariant under a C^1 diffeomorphism with Lipschitz derivative. We shall see below that it is not a C^1 invariant.

Kuo's ratio test.

We suppose that Y is linear (apply a local analytic isomorphism at 0 to $\mathbf{R}^n$). Let π_Y denote orthogonal projection onto Y.

Reformulate (w) by the condition that $d(T_yY, T_xX)/|x-y|$ is bounded near 0. Then in particular $d(T_0Y, T_xX)/|x-\pi_Y(x)|$ is bounded for x near 0 (recall Y is linear). Then it is clear that if X is (w)-regular over Y at 0, then $(X, Y)_0$ satisfies the ratio test of Kuo [3]:

(r) Given any vector $v \in T_0Y$,

$$\lim_{\substack{x\to 0\\ x\in X}} \frac{|\pi_x(v)|\cdot|x|}{|x-\pi_Y(x)|} = 0 .$$

Here π_x denotes orthogonal projection onto the normal space to X at x, so that for unit vectors v, $|\pi_x(v)| = d(\langle v\rangle, T_xX)$. In [3] Kuo proved that (r) implies Whitney (b)-regularity (defined in [9]) and that (b) implies (r) when Y is 1-dimensional. In [6] a fairly complicated semialgebraic example was given with Y 2-dimensional showing that (b) is weaker than (r). We give a simple algebraic example below.

First observe that if (b) (respectively (w)) holds for a pair of strata (X, Y) at 0 in $\mathbf{R}^n$, then (b) (respectively (w)) holds for $(X\times\mathbf{R}, Y\times\mathbf{R})$ along $0\times\mathbf{R}$ in $\mathbf{R}^n\times\mathbf{R}$. However (r) does not have this property.

PROPOSITION 1. *Let* (X, Y) *be a pair of strata in* $\mathbf{R}^n$ *not* (w)*-regular at* 0 (*but possibly satisfying* (r)) *and let* Y *be linear. Then* $(X\times\mathbf{R}, Y\times\mathbf{R})$ *fails to satisfy* (r) *at any point of* $0\times\mathbf{R}$ *in* $\mathbf{R}^n\times\mathbf{R}$.

PROOF. Let X, Y have dimensions m, p respectively and identify the set of one dimensional subspaces of T_0Y with the Grassmannian G_1^p.

Define three subsets of $\mathbf{R}^n\times\mathbf{R}^n\times G_m^n\times G_1^p\times\mathbf{R}$:

$$V_1 = \{(x, \pi_Y(x), T_xX) : x\in X\}\times G_1^p\times\mathbf{R}$$

$$V_2 = \{(x, y, T, \langle v\rangle, \varepsilon) : |x-y| < \varepsilon d(\langle v\rangle, T)\}$$

$$V_3 = \mathbf{R}^n\times\mathbf{R}^n\times\{(T, \langle v\rangle) : d(\langle v\rangle, T) = d(T_0Y, T)\}\times\mathbf{R}$$

V_1 is subanalytic using Verdier [8, Lemma 1.6] (by restricting to a compact neighbourhood of 0 in $\mathbf{R}^n$ if necessary), V_2 is semialgebraic, and V_3 is algebraic, Hence $V = V_1\cap V_2\cap V_3$ is a subanalytic set.

We have that (w) fails for the pair (X, Y) at 0, which is equivalent to the existence of $\tau \in G_m^n$ and $v \in T_0 Y$ with $\|v\| = 1$ such that

$$(0, 0, \tau, \langle v \rangle, 0) \in \bar{V} \subset \mathbf{R}^n \times \mathbf{R}^n \times G_m^n \times G_1^p \times \mathbf{R} .$$

By curve selection [2] we can find an analytic arc

$$\alpha\colon [0, 1] \to \mathbf{R}^n \times \mathbf{R}^n \times G_m^n \times G_1^p \times \mathbf{R} ,$$

such that $\alpha(0) = (0, 0, \tau, \langle v \rangle, 0)$ and such that $\alpha(t) \in V$ if $t \neq 0$. Write

$$\alpha(t) = (x_t, \pi_y(x_t), T_{x_t}X, \langle v_t \rangle, \varepsilon_t)$$

where $v_t \in T_0 Y$, $\|v_t\| = 1$ and $v_t \to v$ as $t \to 0$. Then

$$\frac{d(\langle v_t \rangle, T_{x_t}X)}{|x_t - \pi_y(x_t)|}$$

is unbounded as t tends to 0. We assert that

$$d(\langle v \rangle, T_{x_t}X) \geqq \tfrac{1}{2} d(\langle v_t \rangle, T_{x_t}X)$$

for t sufficiently small. This is a consequence of the definition of V_3, as follows:

Let $v = v_t \cos \varphi_t + u_t \sin \varphi_t$ where $\|u_t\| = 1$, $v_t \perp u_t$ and φ_t is the positive angle between v and v_t, we can assume $0 \leqq \varphi_t < \pi/2$. Let π_t denote the orthogonal projection onto $T_{x_t}X$. Then

$$\begin{aligned} d(\langle v \rangle, T_{x_t}X) &= |v - \pi_t(v)| = |(v_t - \pi_t(v_t)) \cos \varphi_t + (u_t - \pi_t(u_t)) \sin \varphi_t| \\ &\geqq |v_t - \pi_t(v_t)| \cos \varphi_t - |u_t - \pi_t(u_t)| \sin \varphi_t \\ &\qquad \text{(using the triangle inequality)} \\ &\geqq |v_t - \pi_t(v_t)| (\cos \varphi_t - \sin \varphi_t) \\ &\qquad \text{(By definition of } V_3, |v_t - \pi_t(v_t)| \geqq |u_t - \pi_t(u_t)|) \\ &= d(\langle v_t \rangle, T_{x_t}X)(\cos \varphi_t - \sin \varphi_t) \end{aligned}$$

Since φ_t tends to 0 as t tends to 0, it follows that, for t sufficiently small,

$$d(\langle v \rangle, T_{x_t}X) \geqq \tfrac{1}{2} d(\langle v_t \rangle, T_{x_t}X) .$$

We deduce that $d(\langle v \rangle, T_{x_t}X)/|x_t - \pi_y(x_t)|$ is also unbounded as t tends to 0. After reparametrisation we can suppose that

$$\frac{d(\langle v \rangle, T_{x_t}X)}{|x_t - \pi_y(x_t)|} \sim t^{-k} \quad \text{for some } k \geqq 1$$

In $\mathbf{R}^n \times \mathbf{R}$ consider the curve $q(t) = (x_t, t_0 + t)$. Using the canonical inclusion $T_0 Y \subset T_{(0, t_0)}(Y \times \mathbf{R})$, we can consider v as a unit vector of $T_{(0, t_0)}(Y \times \mathbf{R})$. Then

$$\frac{d(\langle v\rangle, T_{q(t)}(X \times \mathbf{R}))\cdot|q(t)-(0,t_0)|}{|q(t)-\pi_{Y\times\mathbf{R}}(q(t))|}$$

$$= \frac{d(\langle v\rangle, T_{x_t}X)\cdot|(x_t,t)|}{|x_t-\pi_y(x_t)|}$$

$$\geqq \frac{d(v, T_{x_t}X)\cdot t}{|x_t-\pi_Y(x_t)|} \sim t^{-(k-1)},$$

which does not tend to zero as t approaches zero since $k \geqq 1$. Hence the ratio test (r) fails for the pair $(X \times \mathbf{R}, Y \times \mathbf{R})$ at every point $(0, t_0)$ of $0 \times \mathbf{R}$ in $\mathbf{R}^n \times \mathbf{R}$, completing the proof of Proposition 1.

EXAMPLE 1. Let $V = \{y^3 = z^2x^3 + x^5\} \subset \mathbf{R}^3$, and let Y be the z-axis and $X = V - Y$.

$(z^2x^3+x^5)^{1/3}$ is a C^1 function of x and z, and so V, as the graph of a C^1 map, is a C^1 submanifold of $\mathbf{R}^3$. Hence X is ($\breve{b}$)-regular over Y. By Theorem 2 of [3] we deduce that (X, Y) satisfies (r) at 0, since dim $Y = 1$.

Consider the curve $p(t) = (t^3, \sqrt[3]{2}\cdot t^5, t^3)$ from the origin into X. The normal direction to X at (x, y, z) is $(3x^2z^2+5x^4 : -3(z^2x^3+x^5)^{2/3} : 2zx^3)$. At $p(t)$ this becomes

$$(8t^2 : -3\cdot 2^{2/3} : 2t^2).$$

So

$$d(T_0Y, T_{p(t)}X) = \frac{2t^2}{(68t^4+18\sqrt[3]{2})^{\frac{1}{2}}}$$

and

$$\frac{d(T_0Y, T_{p(t)}X)}{|p(t)-\pi_Y(p(t))|} \sim \frac{t^2}{t^3} \sim \frac{1}{t},$$

which is unbounded as t approaches zero, so that (w) fails for (X, Y) at 0.

Now let

$$V' = V \times \mathbf{R} = \{y^3 = z^2x^3 + x^5\} \subset \mathbf{R}^4 = \{(x, y, z, u)\}.$$

Let

$$Y' = Y \times \mathbf{R} = \{y = x = 0\} \subset \mathbf{R}^4 \quad \text{and} \quad X' = V' - Y'.$$

By Proposition 1, (X', Y') fails to satisfy (r) at any point of $0 \times \mathbf{R}$ (for example consider the curve $q(t) = (p(t), t)$ from 0 into X'). But since V' is a C^1 submanifold, (X', Y') is (b)-regular.

Example 1 describes the first example of a pair (X, Y) satisfying (b) but not (r) where X is the regular part of an algebraic variety and Y the singular locus. Contrast this with the complex hypersurface case where (b)-regularity, the ratio test, and (w)-regularity are equivalent. This is a consequence of the equivalence of (b)-regularity with Teissier's (c)-cosecance [5] (references for the implications giving this equivalence may be found in [1]); (c)-cosecance trivially implies (w)-regularity, and hence also the ratio test. It remains to be seen whether (b), (r) and (w) are distinct when V is a complex analytic variety of codimension greater than 1.

EXAMPLE 2 (from [7]). $V \equiv \{y^4 = z^4 x + x^3\} \subset \mathbf{R}^3$, $Y = \{z\text{-axis}\}$, $X = V \setminus Y$. Here y is not a C^1 function of x and z, but V is still a C^1 submanifold of $\mathbf{R}^3$, so that (b) holds for (X, Y). (w) fails along the curve $p(t) = (t^4, \sqrt[4]{2} \cdot t^3, t^2)$. As with Example 1 we can apply Proposition 1 to show that $(X \times \mathbf{R}, Y \times \mathbf{R})$ fails to satisfy (r) on $0 \times \mathbf{R}$ in $\mathbf{R}^4$, but (b) clearly holds.

EXAMPLE 3 (due to Kuo [4]). $V \equiv \{y^4 = z^2 x^5 + x^7\} \subset \mathbf{R}^3$, Y the z-axis, $X = V - Y$. V is no longer a C^1 submanifold – for each z, $y^4 = z^2 x^5 + x^7$ defines a plane curve of "cusp type" near 0. However (b) does hold and (w) fails. We can apply Proposition 1 as before.

Examples 1 and 2, and indeed the second discordant horn of [6], show that (r) and (w) are not invariant under C^1 diffeomorphisms. So (b) is more natural in differential topology; it is a C^1 invariant.

Looking closely at the proofs in [3] we see why it is not surprising that (r) is strictly stronger than (b) when $\dim Y \geqq 2$. It is proved in [3] that (b) is equivalent to the conjunction of (a) and (r′) defined as follows.

(r′) If $p(t)$, $t \in [0,1]$ is an analytic arc in $\mathbf{R}^n$ with $p(0) = 0$ and $p(t) \in X$ for $t \neq 0$, then

$$\lim_{t \to 0} \frac{|\pi_t(v)||p(t)|}{|p(t) - \pi_Y(p(t))|} = 0,$$

where v is the tangent at 0 to the arc $\pi_Y \circ p([0,1])$ on Y, and π_t is projection onto the normal space to X at $p(t)$.

It is obvious that (r) implies (a) + (r′) and that (a) + (r′) implies (r) when Y has dimension one. Being able to choose a vector v in $T_0 Y$ and a curve whose tangent at 0 is orthogonal to v suggested the counterexample in [6], and gives rise to the examples here too.

Rugose vector fields.

Given a (b)-regular stratification, one might hope to be able to find rugose vector fields tangent to the strata. Verdier shows that these exist on (w)-regular stratifications [8] and derives rugose trivialisations. However it can be impossible to extend a constant vector field on a base stratum Y to a rugose vector field on an attaching stratum X when (X, Y) is (b)-regular. This is a consequence of our next proposition and the existence of (b)-regular examples which do not satisfy (w).

We refer to [8] for the definition of rugose vector field. (Note the misprint in the definition of rugose function on page 307 of [8], as described below).

PROPOSITION 2. *Let X be a C^2 submanifold of $\mathbf{R}^n$ and let $Y = \mathbf{R}^m \times 0 \subset \mathbf{R}^n$. Suppose that each of the constant vector fields $\{\partial/\partial y_i\}$, $i = 1, \ldots, m$, on Y extends to a rugose vector field on $X \cup Y$. Then X is* (w)*-regular over Y.*

PROOF. Let $\tilde{v}_i$ denote the extension of $\partial/\partial y_i$. For each i there exists a constant C and a neighbourhood U of 0 such that

$$\left|\tilde{v}_i(x) - \frac{\partial}{\partial y_i}\right| \leqq C|x-y|$$

for all $x \in U \cap X$, $y \in U \cap Y$. We can assume that C and U are the same for all i. Let $x \in U$. Then

$$d\left(\frac{\partial}{\partial y_i}, T_x X\right) \leqq \left|\frac{\partial}{\partial y_i} - \tilde{v}_i(x)\right|,$$

hence

$$(*) \qquad d\left(\frac{\partial}{\partial y_i}, T_x X\right) \leqq C|x-y| \quad \text{for all } x \in X \cap U,\ y \in Y \cap U.$$

Take $v \in T_y Y$ with $|v| = 1$.

$$v = \sum_{i=1}^{m} a_i \frac{\partial}{\partial y_i}, \quad \text{with } \sum_{i=1}^{m} a_i^2 = 1.$$

Let $N_x X$ denote the orthogonal complement of $T_x X$ in $\mathbf{R}^n$ and $\pi_x \colon \mathbf{R}^n \to N_x X$ the orthogonal projection.

$$\begin{aligned} d(v, T_x X) = |\pi_x(v)| &= \left|\sum_{i=1}^{m} a_i \pi_x\left(\frac{\partial}{\partial y_i}\right)\right|. \\ &\leqq \sum_{i=1}^{m} \left|\pi_x\left(\frac{\partial}{\partial y_i}\right)\right| \\ &= \sum_{i=1}^{m} d\left(\frac{\partial}{\partial y_i}, T_x X\right). \\ &\leqq mC|x-y| \quad \text{by } (*). \end{aligned}$$

Hence

$$d(T_yY, T_xX) = \sup_{\substack{|v|=1 \\ v\in T_yY}} d(v, T_xX) \leqq mC|x-y| \quad \text{for all } x \in X\cap U,\ y \in Y\cap U\ ,$$

i.e. X is (w)-regular over Y at 0. Repeating the above argument for each $y \in Y$, we obtain that X is (w)-regular over Y, completing the proof of Proposition 2.

COROLLARY. *Let $A = X \cup B$ be a closed subset of $\mathbf{R}^n$, $B \cap X = \emptyset$, X a C^2 submanifold, B a closed subset, and let (B,Σ) be a* (w)*-regular stratification, with each stratum a C^2 submanifold. Then the stratification Σ' of A given by adding X to Σ is* (w)*-regular if and only if every rugose vector field on B tangent to Σ can be extended to a rugose vector field on A tangent to Σ'.*

PROOF. "Only if" is proved by Verdier [8]. "If" follows from Proposition 2 above by making the stratum containing a given point y, affine near y, by a C^2 change of local coordinates.

WARNING. The definition of rugosity in [8] should read "for all $x \in S_\alpha$, there is a constant C and a neighbourhood V of x such that for all $x' \in V \cap S_\alpha$ and all $y \in V \cap A$,

$$(**) \qquad |f(x')-f(y)| \leqq C|x'-y|\text{"}$$

and *not*

$$(***) \qquad \text{"}|f(x')-f(y)| \leqq C|x-y|\text{"}\ .$$

To see that these are effectively distinct notions in the case of vector fields we can use Example 2. (w) fails, so by Proposition 2 no lift of $\partial/\partial z$ satisfies (**). However the canonical lift of $\partial/\partial z$ (namely the vector field $v(x,y,z)$ on V defined by projecting $\partial/\partial z$ onto the tangent space to X at each point of X) satisfies (***) as follows.

Let $f(x,y,z) = -y^4 + z^4x + x^3$. Then

$$v(x,y,z) = (0,0,1) - \frac{(f_x, f_y, f_z)}{|\text{grad } f|}\cdot\frac{f_z}{|\text{grad } f|}\ .$$

Hence

$$|v(x,y,z)-(0,0,1)| = \frac{|f_z|}{|\text{grad } f|}\ .$$

We must check that $|v(x,y,z)-(0,0,1)|/|(x,y,z)|$ is bounded as (x,y,z) tends to 0 on X.

$$\frac{|v(x,y,z)-(0,0,1)|}{|(x,y,z)|} = \frac{|f_z|}{|\text{grad } f|\cdot|(x,y,z)|}$$

$$= \frac{|4z^3x|}{|(z^4+3x^2,\ -4(z^4x+x^3)^{3/4}, 4z^3x)|\cdot|(x,\ (z^4x+x^3)^{1/4}, z)|}$$

CASE 1. $|x/z^2| \leqq 1$. Dividing through by z^5, gives

$$\frac{|4x/z^2|}{|(1+(3x^2/z^4), .., .)|\cdot|(x/z, .., 1)|}$$

which is at most 4.

CASE 2. $|z^2/x| \leqq 1$. Dividing through by x^2z, gives

$$\frac{|4z^2/x|}{|(z^4/x^2+3, .., 4z^3/x)|\cdot|(x/z, .., 1)|}$$

which is at most 4/3.

We have shown that (***) is satisfied.

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