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ON EXTENSIONS OF LOCALLY COMPACT GROUPS AND UNITARY GROUPS

ESBEN T. KEHLET

1.

We construct a topological group G containing a given topological group K as a closed normal subgroup with $K\backslash G$ a given topological group Q, corresponding to a given cocycle (c,α), under the conditions that Q is locally compact, K is isomorphic and homeomorphic to a subgroup of the unitary group on a Hilbert space, and c and α satisfy suitable conditions of measurability in the sense of Lusin.

For discrete groups the construction is due to O. Schreier [14]. It generalizes work of G. W. Mackey [10] (Q and K locally compact with countable basis for the topology, K abelian), and M. A. Rieffel [13] (Q and K locally compact, K abelian and σ-compact), and is related to work of L. G. Brown [4] (Q and K polish).

We topologize G by means of the representation of G induced from the given representation of K, or in other terms by the strong operator topology on the twisted crossed product of (the von Neumann algebra spanned by) the image of K with Q [15]. This is related to Varadarajan's definition of Weil topology [16, p. 48]. Although the construction is thus well known, I believe the observation that it solves the extension problem is new.

In Section 2 we give some measurability results; in Section 3 we state and prove our theorem; in Section 4 we introduce a notion of measurability of a map between locally compact spaces, useful in the treatment in Section 5 of the case where K is locally compact.

We use freely [1], [2], [3], [5].

All measures occurring will be positive Radon measures on locally compact spaces. By measurable we mean measurable in the sense of Lusin (or Bourbaki [1]). If A is a locally compact space, μ a measure on A, and h a Hilbert space, a map $T\colon A \to \mathscr{L}(h)$ is called scalarly measurable if $a \mapsto (T(a)\xi \mid \eta)$ is measurable for all $\xi,\eta \in h$, and a measurable field if $a \mapsto T(a)\xi$ is measurable

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for each $\xi \in h$. We denote by $\mathcal{O}(h,\mu)$ the set of bounded measurable fields $T: A \to \mathscr{L}(h)$, for which $T^*: a \mapsto T(a)^*\xi$ is also a measurable field, and by $\mathcal{M}(h,\mu)$ the subset of fields also satisfying: for each compact subset L of A and $\xi \in h$, $T(L)\xi$ and $T(L)^*\xi$ are separable. Then $\mathcal{O}(h,\mu)$ and $\mathcal{M}(h,\mu)$ are weakly sequentially closed C* algebras in their natural representations on $l^2(A,h)$. If T is a bounded and scalarly measurable map, we denote by $T(\mu)$ the operator on $L^2(\mu,h)$ defined by

$$(T(\mu)f \mid g) = \int_A (T(a)(f(a)) \mid g(a))\, d\mu(a), \qquad f,g \in \mathscr{L}^2(\mu,h)\,.$$

If $T \in \mathcal{O}(h,\mu)$, then $(T(\mu)f)(a) = T(a)(f(a))$, $f \in \mathscr{L}^2(\mu,h)$, $a \in A$. See [17], [9].

I am indebted to Niels Vigand Pedersen for suggesting that the methods used here might apply for non abelian and non locally compact K, cf. [4], [12].

2.

LEMMA 1. *Assume given a compact space A, a measure μ on A, a Hilbert space h, and a map $T \in \mathcal{O}(h,\mu)$. There exists a family $(h_i, M_i)_{i\in I}$, where $(h_i)_{i\in I}$ is a family of pairwise orthogonal separable closed subspaces of h with Hilbert sum h, and M_i is a subset of A with $\mu(M_i)=0$ for each $i \in I$, such that h_i is invariant under $T(a)$ and $T(a)^*$ for each $a \in A \setminus M_i$.*

PROOF. Choose a map $S \in \mathcal{M}(h,\mu)$ and a family $(h_i)_{i\in I}$ of pairwise orthogonal separable closed subspaces of h with Hilbert sum h, such that $S(a)h_i \subseteq h_i$, $i \in I$, $a \in A$, and $S(\mu) = T(\mu)$, cf. [9]. Choose $M_i \subset A$ with $\mu(M_i)=0$ such that $S(a)\xi = T(a)\xi$ and $S(a)^*\xi = T(a)^*\xi$ for each ξ in a dense countable subset of h_i and $a \in A \setminus M_i$.

PROPOSITION 1. *Assume given locally compact spaces A and B with measures μ and ν resp., a Hilbert space h and a map $D \in \mathcal{O}(h, \mu\times\nu)$. Assume that $D(\cdot,b)$ is scalarly measurable for each $b \in B$ and define $D(\mu,b)$ as above. Then*

$$D(\mu,\cdot) \in \mathcal{O}(h \otimes L^2(\mu), \nu)\,.$$

PROOF. For the case $h = \mathbf{C}$, see [8, Lemma 3]. Since

$$(D(\mu,b)\xi\otimes\varphi \mid \eta\otimes\psi) = \int_A (D(a,b)\xi \mid \eta)\varphi(a)\overline{\psi(a)}\, d\mu(a)$$

is a measurable function of b by the theorem of Fubini when $\xi,\eta \in h$ and $\varphi,\psi \in \mathscr{K}(A)$, we see that $D(\mu,\cdot)$ is scalarly measurable.

In the rest of the proof we may and shall assume that A and B are compact.

First assume that h is separable. Then $D(\mu\times\nu)$ belongs to the von Neumann tensor product $\mathscr{L}(h)\otimes L^\infty(\mu)\otimes L^\infty(\nu)$ acting on $h\otimes L^2(\mu)\otimes L^2(\nu)$. This tensor product is of countable type, since $\mathscr{L}(h)$, $L^\infty(\mu)$ and $L^\infty(\nu)$ are, so its unit ball is metrizable in the strong operator topology. By the density theorem of Kaplansky $D(\mu\times\nu)$ can be approximated strongly by a bounded sequence $(D_n)_{n\in\mathbf{N}}$ of operators in the algebraic tensor product, and we may even assume that $D_n(a,b)\to D(a,b)$ strongly $\mu\times\nu$ almost everywhere, by [5, Ch. II, § 2, Prop. 4].

For each $n\in\mathbf{N}$ we can define $D_n(\mu,b)$, and $D_n(\mu,\cdot)$ is trivially a measurable field. There exists a subset N of B with $\nu(N)=0$, such that $D_n(a,b)\to D(a,b)$ strongly μ a.e. and hence $D_n(\mu,b)\to D(\mu,b)$ strongly for $b\notin N$. It follows that $D(\mu,\cdot)$ is a measurable field.

$D(\mu,\cdot)^*$ is treated the same way.

We now drop the assumption that h is separable. Choose by Lemma 1 a family $(h_i,M_i)_{i\in I}$, where $(h_i)_{i\in I}$ is a family of pairwise orthogonal closed separable subspaces of h with Hilbert sum h, and for each $i\in I$ M_i is a subset of $A\times B$ with $\mu\times\nu(M_i)=0$ and h_i is invariant under $D(a,b)$ and $D(a,b)^*$ for $(a,b)\notin M_i$. Define $D_i(a,b)=D(a,b)$, $(a,b)\notin M_i$, and $D_i(a,b)=1$, $(a,b)\in M_i$, and $D_{i0}(a,b)=D_i(a,b)\,|\,h_i$, $(a,b)\in A\times B$. Then $D_{i0}\in\mathcal{O}(h_i,\mu\times\nu)$, and $D_{i0}(\mu,\cdot)\in\mathcal{O}(h_i\otimes L^{42}(\mu),\nu)$, and

$$D_{i0}(\mu,b) \;=\; D_i(\mu,b)\,|\,h_i\otimes L^2(\mu)\,.$$

Also $D_i(\mu,b)=D(\mu,b)$ for b outside a subset N_i of B with $\nu(N_i)=0$ chosen such that $D_i(a,b)=D(a,b)$ for μ almost all $a\in A$ when $b\notin N_i$. It follows that $D(\mu,\cdot)\in\mathcal{O}(h\otimes L^2(\mu),\nu)$.

The next proof is just a variation, written up for completeness, of the proof that the regular representation of a locally compact group Q on the space of Haar locally square integrable functions on Q is continuous.

LEMMA 2. *Assume given a locally compact group Q with right Haar measure μ and modular function Δ, a Hilbert space h and a bounded measurable map $f\colon Q\to h$. For each compact subset M of Q, the map*

$$q\mapsto\int_M \|f(qr)-f(r)\|^2\,d\mu(r)$$

is continuous.

PROOF. It is enough to prove continuity at e. Assume a compact subset M of Q and $\varepsilon>0$ given. Choose a compact neighbourhood L of e in Q, and choose a compact subset N of LM such that $\mu(LM\setminus N)<\varepsilon$ and $f\,|\,N$ is continuous. Choose a neighbourhood $P\subseteq L$ of e in Q such that $\|f(qr)-f(r)\|^2<\varepsilon$ for $q\in P$ and $r\in M\cap N\cap q^{-1}N$. Then

$$\int_M \|f(qr)-f(r)\|^2 \, d\mu(r) \leqq \varepsilon\mu(M)+4 \sup_{s\in LM} \|f(s)\|^2(1+\sup_{r\in L} \Delta(r))\varepsilon .$$

3.

Let Q be a locally compact group with right Haar measure μ. Let h be a Hilbert space, and $U(h)$ the topological group of unitary operators on h with strong operator topology. Let K be a topological group, and v an isomorphism and homeomorphism of K with a subgroup of $U(h)$.

Let Aut (K) denote the group of topological automorphisms of K. For $f \in K$ let In (f) denote the inner automorphism $k \mapsto fkf^{-1}$, $k \in K$.

By a cocycle on Q we mean a pair (c,α), where c is a map $Q \times Q \to K$ and α is a map $Q \to \text{Aut}(K)$, satisfying:

$$\forall q,r \in Q\colon\ \alpha(q)\alpha(r) = \text{In}(c(q,r))\alpha(qr), \text{ and}$$

$$\forall q,r,s \in Q\colon\ \alpha(q)(c(r,s))c(q,rs) = c(q,r)c(qr,s), \text{ and}$$

$$\alpha(e) = e, \text{ and}$$

$$c(e,e) = e .$$

Then $c(q,e)=c(e,q)=e$ for all $q \in Q$.

We shall say that α is continuous at $e \in K$ almost uniformly over compact subsets of Q, if to each compact subset L of Q and $\varepsilon>0$ and neighbourhood N of e in K there exist a compact subset M of L and a neighbourhood P of e in K such that $\mu(L\setminus M)<\varepsilon$ and $\alpha(M)(P)\subseteq N$. To obtain this it is enough to assume that to each compact subset L of Q and $\varepsilon>0$ and $\xi \in h$ there exist a compact subset M of L and a neighbourhood P of e in K, such that $\mu(L\setminus M)<\varepsilon$ and

$$\|v(\alpha(q)(k))\xi-\xi\| < 1 \qquad \text{for } q \in M,\ k \in P .$$

Similar remarks apply to $\alpha^{-1}\colon (k,q) \mapsto \alpha(q)^{-1}(k)$.

By an extension we understand a triple (G,i,p), where G is a topological group, i is an isomorphism and homeomorphism of K with a subgroup of G, and p is a continuous open homomorphism of G onto Q with kernel $i(K)$. If ϱ is a cross section for p with $\varrho(e)=e$, then (c,α) defined by

$$c(q,r) = i^{-1}[\varrho(q)\varrho(r)\varrho(qr)^{-1}] \quad \text{and}$$

$$\alpha(q)(k) = i^{-1}[\varrho(q)i(k)\varrho(q)^{-1}], \qquad q,r \in Q,\ k \in K ,$$

is a cocycle, which we shall say corresponds to ϱ. We say that a cocycle (c,α) is associated to (G,i,p), if it corresponds to some cross section ϱ for p with $\varrho(e)=e$.

Remark 1. Let (G,i,p) be an extension. Assume there exist a faithful continuous unitary representation V of G on a Hilbert space H and a cross section ϱ for p with $\varrho(e)=e$, such that $V\circ\varrho \in \mathcal{O}(H,\mu)$. Let (c,α) be the corresponding cocycle. Then $V\circ i\circ c \in \mathcal{O}(H,\mu\times\mu)$, $V\circ i\circ c(\cdot,r) \in \mathcal{O}(H,\mu)$ for each $r \in Q$, and $V\circ i(\alpha(\cdot)(k)) \in \mathcal{O}(H,\mu)$ for each $k \in K$. If also $V\circ i$ is a homeomorphism, then α and α^{-1} are continuous at $e \in K$ almost uniformly over compact subsets of Q; indeed to L compact in Q, $\xi \in H$ and $\varepsilon>0$ we can choose a compact subset M of L, such that $\mu(L\setminus M)<\varepsilon$ and $V\circ\varrho(\cdot)^{-1}\xi$ is continuous on M and $V\circ\varrho(\cdot)V\circ\varrho(q_0)^{-1}\xi$ is continuous on M for each $q_0 \in M$; for $q_0 \in M$ we have

$$\begin{aligned}
&\|V\circ\varrho(q)V\circ i(k)V\circ\varrho(q)^{-1}\xi-\xi\|\\
\leqq\ &\|V\circ\varrho(q)^{-1}\xi - V\circ\varrho(q_0)^{-1}\xi\| + \|[V\circ i(k)-1]V\circ\varrho(q_0)^{-1}\xi\| +\\
&+\|[V\circ\varrho(q)-V\circ\varrho(q_0)]V\circ\varrho(q_0)^{-1}\xi\| \ < \ 1
\end{aligned}$$

when $q \in M$ is close to q_0 and k is close to e; the result for α follows by compactness of M, and α^{-1} is treated similarly.

THEOREM 1. *Assume given a locally compact group Q, a topological group K, a Hilbert space h and an isomorphism and homeomorphism v of K with a subgroup of $U(h)$, and a cocycle (c,α) on Q. Assume that*

$v\circ c \in \mathcal{O}(h,\mu\times\mu)$, *and*

$v(c(\cdot,r)) \in \mathcal{O}(h,\mu)$ *for each* $r \in Q$, *and*

$v(\alpha(\cdot)(k)) \in \mathcal{O}(h,\mu)$ *for each* $k \in K$, *and*

α *and* α^{-1} *are continuous at* $e \in K$ *almost uniformly over compact subsets of* Q.

Then there exists an extension (G,i,p) such that (c,α) is an associated cocycle.

There exists an isomorphic and homeomorphic unitary representation V of G on a Hilbert space H and a cross section ϱ for p with $\varrho(e)=e$, such that (c,α) is the corresponding cocycle and $V\circ\varrho \in \mathcal{O}(H,\mu)$.

PROOF. Let G denote $K\times Q$ organized as a group under the product

$$(k_1,q_1)(k_2,q_2) \;=\; (k_1\alpha(q_1)(k_2)c(q_1,q_2),q_1q_2)\,.$$

Define $i(k)=(k,e)$, $p(k,q)=q$, and $\varrho(q)=(e,q)$; then (G,i,p) is an extension of the underlying (discrete) groups of K and Q, ϱ is a cross section for p, and (c,α) is the corresponding cocycle [14], [4].

Let H denote the Hilbert space $L^2(Q,h)$, identified with $h\otimes L^2(Q)$ whenever convenient. We define a homomorphism $V\colon G \to U(H)$ by

$$(V(k_1, q_1)f)(q) = v(\alpha(q)(k_1)c(q, q_1))(f(qq_1)) .$$

It is easy to see that $V \circ i$ is continuous; in fact it is enough to show that given $\xi \in h$ and $\psi \in \mathscr{K}(Q)$ there exists a neighbourhood L of e in K such that $k \in L$ implies

$$\int_Q \|v(\alpha(q)(k))\xi - \xi\|^2 |\psi(q)|^2 \, d\mu(q) < 1 ,$$

and for this it is enough to choose a compact subset P of $\operatorname{supp}\psi$ and L such that

$$4\|\xi\|^2 \int_{\operatorname{supp}\psi \setminus P} |\psi(q)|^2 \, d\mu(q) \leqq \tfrac{1}{2} \quad \text{and}$$

$$\|v(\alpha(q)(k))\xi - \xi\| \, \|\psi\|_2 < \tfrac{1}{2}$$

for $q \in P$ and $k \in L$.

We next show that $V \circ i$ is open and injective. Assume given a neighbourhood L of e in K; we show the existence of a neighbourhood W of e in $U(H)$ such that $k \notin L$ implies $V(i(k)) \notin W$.

Choose a compact set M in Q of positive measure, and a neighbourhood P of e in K such that $\alpha(M)^{-1}(P) \subseteqq L$, then choose $\xi_1, \xi_2, \ldots, \xi_n \in h$ such that

$$\{k \in K \mid \|v(k)\xi_\nu - \xi_\nu\| < 1, \ \nu = 1, 2, \ldots, n\} \subseteqq P .$$

Define

$$W = \left\{ w \in U(H) \;\middle|\; \sum_{\nu=1}^{n} \|w\xi_\nu \otimes 1_M - \xi_\nu \otimes 1_M\|^2 < \mu(M) \right\} .$$

Then $k \notin L$ implies that $\alpha(q)(k) \notin P$ and so

$$\sum_{\nu=1}^{n} \|v(\alpha(q)(k))\xi_\nu - \xi_\nu\|^2 \geqq 1 \qquad \text{for each } q \in M;$$

thus

$$\sum_{\nu=1}^{n} \|V \circ i(k)\xi_\nu \otimes 1_M - \xi_\nu \otimes 1_M\|^2$$

$$= \sum_{\nu=1}^{n} \int_Q \|v(\alpha(q)(k))\xi_\nu - \xi_\nu\|^2 1_M(q)^2 \, d\mu(q) \geqq \mu(M)$$

and $V \circ i(k) \notin W$.

We now show that the kernel of V is contained in $i(K)$ and that $p \circ V^{-1}$ is continuous. Let a neighbourhood N of e in Q be given. Choose a compact neighbourhood P of e in Q with $P^{-1}P \subseteqq N$ and a unit vector $\xi \in h$; let W

denote $\{w \in U(H) \mid \|w\xi\otimes 1_P - \xi\otimes 1_P\|^2 < \mu(P)\}$. When $r \in Q\setminus N$ and $k \in K$, then

$$\begin{aligned}&\|V(k,r)\xi\otimes 1_P - \xi\otimes 1_P\|^2\\ &= \int_{Pr^{-1}} \|v(\alpha(q)(k)c(q,r))\xi\|^2\, d\mu(q) + \int_P \|\xi\|^2\, d\mu(q)\\ &= 2\mu(P)\,,\end{aligned}$$

so $V(k,r) \notin W$. Since $V\circ i$ is injective, V is injective.

We topologize G by defining the family of open sets in G as the family of counter images under V of open sets in $U(H)$. Then V is an isomorphism and homeomorphism of G with a subgroup of $U(H)$, G is a Hausdorff topological group, i is an isomorphism and homeomorphism of K with a subgroup of G, and p is a continuous homomorphism of G on Q with kernel $i(K)$.

Define $R \in \mathcal{O}(H,\mu)$ by $(R(r)f)(q) = f(qr)$; then $V\circ\varrho(q) = c(\mu,q)R(q)$. As $v\circ c(\mu,\cdot) \in \mathcal{O}(H,\mu)$ by Proposition 1, $V\circ\varrho \in \mathcal{O}(H,\mu)$.

We complete the proof of Theorem 1 by showing that p is open.

So assume given a neighbourhood S of e in G; choose $\xi_1, \xi_2, \ldots, \xi_n \in h$ and $\psi_1, \psi_2, \ldots, \psi_n \in \mathcal{K}(Q)$ such that

$$\{(k,q) \in G \mid \|V(k,q)\xi_\nu\otimes\psi_\nu - \xi_\nu\otimes\psi_\nu\| < 1,\ \nu = 1,2,\ldots,n\} \subseteq S\,.$$

Choose a compact subset M of Q with positive measure. Choose by Lemma 2 a neighbourhood L of e in Q, such that

$$\begin{aligned}&\forall q \in L\ \forall \nu \in \{1,2,\ldots,n\}:\\ &\int_M \|V(e,r)^*\xi_\nu\otimes\psi_\nu - V(e,qr)^*\xi_\nu\otimes\psi_\nu\|^2\, d\mu(r) < n^{-1}\mu(M);\end{aligned}$$

then with

$$A_\nu(q) = \{r \in M \mid \|V(e,r)^*\xi_\nu\otimes\psi_\nu - V(e,qr)^*\xi_\nu\otimes\psi_\nu\|^2 \geqq 1\}$$

we have $\mu(A_\nu(q)) < n^{-1}\mu(M)$ and $\mu(\bigcup_{\nu=1}^n A_\nu(q)) < \mu(M)$ for $q \in L$; so

$$\begin{aligned}&\forall q \in L\ \exists r \in M\ \forall \nu \in \{1,2,\ldots,n\}:\\ &\qquad \|V(e,r)^*\xi_\nu\otimes\psi_\nu - V(e,qr)^*\xi_\nu\otimes\psi_\nu\| < 1\,.\end{aligned}$$

From $(c(q,r)^{-1},q)(e,r) = (e,qr)$ we get

$$V(e,qr)V(e,r)^{-1} = V(c(q,r)^{-1},q)\,,$$

so with $k = c(q,r)^{-1}$ we have

$$\forall q \in L\ \exists k \in K\ \forall \nu \in \{1,2,\ldots,n\}:\ \|V(k,q)\xi_\nu\otimes\psi_\nu - \xi_\nu\otimes\psi_\nu\| < 1\,.$$

This shows that $L \subseteq p(S)$.

REMARK 2. Assume given Q, K, h and v as in Theorem 1. If (c,α) is a cocycle on Q, and $v\circ c \in \mathscr{M}(h,\mu\times\mu)$ [9], and $v(\alpha(\cdot)(k)) \in \mathscr{O}(h,\mu)$ for each $k \in K$, then $v\circ c(\cdot,r) \in \mathscr{M}(h,\mu)$ for each $r \in Q$.

PROOF (cf. [8, p. 129]). Without lack of generality we assume Q is σ-compact.

Assume first that h is separable. Then $v\circ c$ is a measurable map into $U(h)$. Define

$$Q_0 = \{r \in Q \mid v(c(\cdot,r)) \in \mathscr{M}(h,\mu)\} .$$

Then Q_0 is a subgroup of Q because

$$c(q,rs) = \alpha(q)(c(r,s)^{-1})c(q,r)c(qr,s) \text{ and}$$
$$c(q,r^{-1}) = \alpha(q)(c(r^{-1},r))c(qr^{-1},r)^{-1} ,$$

and $\mu(Q\setminus Q_0)=0$ by the theorem of Fubini, so $Q_0=Q$.

In the general case there exists a family $(h_i)_{i\in I}$ of pairwise orthogonal closed separable subspaces of h invariant under $v(c(q,r))$, $q,r \in Q$, with $h=\bigoplus_{i\in I} h_i$. Then also

$$v(\alpha(q)(c(r,s)))h_i = h_i, \quad i \in I,\ q,r,s \in Q .$$

By the first part of the proof $v\circ c(\cdot,r)\mid h_i \in \mathscr{M}(h_i,\mu)$ for each $i \in I$ so $v\circ c(\cdot,r) \in \mathscr{M}(h,\mu)$, $r \in Q$.

4.

In this section we assume given a locally compact space T and a measure ν on T. Let a be a map of T into a locally compact space S. We say that a is D measurable, if a satisfies the conditions

A. For each compact subset L of T, $a(L)$ is contained in a σ-compact set, and
1. For each Baire set [7, § 51] B in $S, a^{-1}(B)$ is measurable.

and we say that a is F meâsurable if a satisfies 1. and

B. To each compact subset L of T there exists a subset N of L with $\nu(N)=0$ such that $a(L\setminus N)$ is contained in a σ-compact set.

Condition 1 is equivalent to

2. For each $f \in \mathscr{K}(S)$, $f\circ a$ is measurable.

LEMMA 3. *Let S be a locally compact space and a an F measurable map $T\to S$. Then a satisfies*

C. *To each compact subset L of T there exists a sequence $N, L_1, L_2,\ldots$ of* *measurable subsets of L with* $L=\bigcup_{i=1}^{\infty} L_i \cup N$, *such that* $\nu(N)=0$ *and* $a(L_i)$ *is relatively compact for* $i=1,2,\dots$, *and*

3. *For each continuous map f of S into a metrizable space,* $f\circ a$ *is measurable, and*

4. *For each Hilbert space h and strongly continuous map* $f\colon S \to \mathscr{L}(h)$, $f\circ a$ *is a measurable field.*

PROOF. Condition C is satisfied because each compact set in S is contained in a compact Baire set.

Now let f be a continuous complex function on S, and L a compact subset of T. Choose $N, L_1, L_2, \dots$ according to Condition C. For each n choose $\varphi_n \in \mathscr{K}(S)$ such that $\varphi_n | a(L_n) = f | a(L_n)$. Then for each open subset U of $\mathbf{C}$

$$(f\circ a)^{-1}(U) \cap L = \bigcup_{n=1}^{\infty} ((\varphi_n \circ a)^{-1}(U) \cap L_n) \cup ((f\circ a)^{-1}(U) \cap N)$$

is measurable, so $f\circ a$ is measurable.

Conditions 3 and 4 now follow from [1, Chap. IV, § 5, Théorème 4].

PROPOSITION 2. *Assume given a locally compact group G, a faithful strongly continuous unitary representation v of G on a Hilbert space h, and a map* $a\colon T \to G$. *Assume* $v\circ a$ *is scalarly measurable. If v tends to zero at infinity in weak operator topology and* $v\circ a$ *is a measurable field, or if v is a homeomorphism with* $v(G)$ *and a satisfies Condition* B, *or if a satisfies Condition* C, *then a is F measurable. If v tends to zero at infinity and* $v\circ a \in \mathscr{M}(h, \nu)$, *then a is D measurable.*

PRROF. The set R of bounded continuous complex functions f on G with $f\circ a$ measurable is a C* algebra with unit, separating points in G since v is faithful. By the Stone–Weierstrass theorem, for each compact subset M of G,

$$\{f | M \mid f \in R\} = C(M),$$

and if v tends to zero at infinity $R \supseteqq C_0(G)$.

If a satisfies Condition C, we can proceed as in the proof of Lemma 3 to show that a is F measurable, utilizing R in stead of $\mathscr{K}(G)$.

If v is open, the topology of G has a basis consisting of relatively compact open sets W with $a^{-1}(W)$ measurable, and then Condition B implies Condition C.

Now assume that v tends to zero at infinity (and so is open). For each unit vector $\xi \in h$ there exists a σ-compact open subgroup G_0 of G such that $(v(g)\xi | \xi)=0$ for all $g \notin G_0$; then $(v(g_1)\xi | v(g_2)\xi)=0$ whenever g_1 and g_2 belong to different cosets modulo G_0. Thus $v(a(L))\xi$ is non-separable when $a(L)$ is not contained in any σ-compact set, and if $v \circ a$ is a measurable field, a is F measurable, and if $v \circ a \in \mathscr{M}(h, v)$, a is D measurable.

COROLLARY. *Let V denote the right regular representation of G. Then a is F measurable if and only if $V \circ a \in \mathcal{O}(L^2(G), v)$, and a is D measurable if and only if $V \circ a \in \mathscr{M}(L^2(G), v)$.*

LEMMA 4. *Assume given a locally compact space S, a measure μ on S, and a map $a\colon T \to S$. Assume that a is F measurable, that μ is the image of v, i.e. the essential measure $v^{\bullet}(a^{-1}(B)) = \mu(B)$ for each Baire set B of S, and that μ is completion regular* [7, § 52]. *Then $a^{-1}(M)$ is v measurable for each μ measurable set $M \subseteqq S$ with $v^{\bullet}(a^{-1}(M)) = \mu^{\bullet}(M)$ (a is v adéquate). If also a is bijective and a^{-1} is F measurable, then for each Baire set D in T, $a(D)$ is μ measurable with $\mu^{\bullet}(a(D)) = v(D)$.*

PROOF. Let M be a measurable subset of S, and let L be a member of the v dense family of compact subsets of T [1, Chap. IV, § 5, no. 8] with relatively compact image under a. Choose a compact Baire set R with $a(L) \subseteqq R$, and choose Baire sets B and C with $B \subseteqq M \cap R \subseteqq C \subseteqq R$ and $\mu(C \setminus B) = 0$. Then

$$a^{-1}(B) \cap L \subseteqq a^{-1}(M) \cap L \subseteqq a^{-1}(C) \cap L,$$

and

$$v([a^{-1}(C) \cap L] \setminus [a^{-1}(B) \cap L]) \leqq v^{\bullet}(a^{-1}(C \setminus B)) = 0, \quad \text{and}$$

$$v(a^{-1}(M) \cap L) \leqq \mu(C) = \mu(M \cap R) \leqq \mu^{\bullet}(M).$$

Thus $a^{-1}(M)$ is measurable and $v^{\bullet}(a^{-1}(M)) \leqq \mu^{\bullet}(M)$. For each Baire set B contained in M,

$$\mu(B) = v^{\bullet}(a^{-1}(B)) \leqq v^{\bullet}(a^{-1}(M)),$$

so $\mu^{\bullet}(M) = v^{\bullet}(a^{-1}(M))$.

When also a is bijective and a^{-1} is F measurable, then for each Baire set D in T, $a(D)$ is measurable with $\mu^{\bullet}(a(D)) = v^{\bullet}(a^{-1}(a(D))) = v(D)$.

COROLLARY. *Assume given locally compact groups T and G with Haar measures v and μ respectively, and a bijective map a of T on G. Assume that a and a^{-1} are F measurable, and that μ is the image of v. Then a subset M of G is measurable if and only if $a^{-1}(M)$ is measurable, and when M is measurable $v^{\bullet}(a^{-1}(M)) = \mu^{\bullet}(M)$. Thus $f \mapsto f \circ a$ defines an isometric isomorphism of $L^{\infty}(G)$ on $L^{\infty}(T)$.*

PROOF. Combine Lemma 4 and [7, § 64].

LEMMA 5. *Assume given two locally compact groups Q and G and an F measurable map $\varrho: Q \to G$. Define $c(q,r)=\varrho(q)\varrho(r)\varrho(qr)^{-1}$, $q,r \in Q$. Then c is D measurable if and only if ϱ is D measurable.*

PROOF. Assume c is D measurable. We assume that Q is σ-compact. Since ϱ is F measurable there exist a subset N of Q with Haar measure zero and an open σ-compact subgroup G_0 of G with $\varrho(Q \setminus N) \subseteqq G_0$. Let π denote the natural map of G on G/G_0. Then $\pi(c(Q \times Q))$ is countable. To $n \in N$ we choose $q \notin N \cup n^{-1}N$; then $\pi(\varrho(n)) = \pi(c(n,q))$. Thus $\pi(\varrho(N))$ is countable, so $\varrho(N)$ and $\varrho(Q)$ are contained in σ-compact sets, and ϱ is D measurable.

The converse is trivial.

5.

In this section Q and K denote locally compact groups with right Haar measures μ and $\varkappa$ respectively.

PROPOSITION 3. *Let G be a locally compact group, H a closed subgroup, K a locally compact group with countable basis for neighbourhoods of e, and ψ a continuous homomorphism of H into K.*

There exists a measurable map P of G into K extending ψ and satisfying:

$$P(hg) = \psi(h)P(g), \qquad h \in H,\ g \in G\,.$$

PROOF. Choose an open σ-compact subgroup K_0 of K, and let H_0 denote the relatively open subgroup $\psi^{-1}(K_0)$ of H. Then K_0 has countable basis for the topology, and there exists a measurable extension R of $\psi \mid H_0$ to G satisfying

$$R(hg) = \psi(h)R(g), \qquad h \in H_0,\ g \in G$$

[8, Corollary 2 of Theorem 1]. Choose a neighbourhood $W = H_0 W$ of e in G with $WW^{-1} \cap H = H_0$. Define P on HW by $P(hw) = \psi(h)R(w)$; then P is well defined and measurable. To define P everywhere on G, utilize that G is disjoint union of an H saturated local null set and a locally countable family of right translates of measurable subsets of HW.

From this follow generalizations of Corollaries 3 and 4 of Theorem 1 in [8]. We note especially

COROLLARY (cf. [6]). *Let G be a locally compact group, K a closed subgroup with countable basis for neighbourhoods of e. There exists a measurable cross section for the natural map of G on the quotient space.*

REMARK 3. Let (G,i,p) be an extension. Then G is locally compact by [11, p. 52]. If there exists an F measurable cross section ϱ_0 for p, then there also exists an F measurable cross section ϱ with the property that $\varrho(L)$ is relatively compact for each compact subset L of Q; in fact we can choose an open set W in G such that $p(W)=Q$ and $W \cap p^{-1}(L)$ is relatively compact for each compact L, and then use that the set of compact subsets L of Q for which there exists $k \in K$ such that $\varrho(L)i(k) \subseteq W$ is μ dense. Now assume ϱ is a D measurable cross section and let (c,α) be the corresponding cocycle. Then c is D measurable, $\alpha(\cdot)(k)$ is D measurable for each $k \in K$, α and α^{-1} are continuous at $e \in K$ almost uniformly over compact subsets of Q, and the family of compact subsets M of Q, for which $\alpha(M)^{-1}(L)$ is relatively compact for each compact set L in K, is μ dense. Also $q \mapsto c(q^{-1},q)$, $q \mapsto c(q,r)$, and $q \mapsto c(r,q)$, $r \in Q$, are D measurable.

Define a map P of G into K by $P(g)=i^{-1}[g\varrho(p(g))^{-1}]$, and define a map φ of the product topological group $K \times Q$ onto G by $\varphi(k,q)=i(k)\varrho(q)$. Then $\varphi^{-1}(g) = (P(g),p(g))$, and P, φ and φ^{-1} are F measurable. Right Haar measure λ on G can be chosen such that

$$\int_G f(g)\,d\lambda(g) = \int_Q \int_K f(i(k)g)\,d\varkappa(k)\,d\mu(p(g))$$

$$= \int_Q \int_K f(i(k)\varrho(q))\,d\varkappa(k)\,d\mu(q), \qquad f \in \mathscr{K}(G);$$

by the Corollary of Lemma 4 a subset M of G is measurable if and only if $\varphi^{-1}(M)$ is measurable, and if M is measurable $\lambda^{\bullet}(M) = \varkappa \times \mu^{\bullet}(\varphi^{-1}(M))$.

THEOREM 2. *Assume given two locally compact groups Q and K, and a cocycle (c,α) on Q. Assume that c is D measurable, that $\alpha(\cdot)(k)$ is F measurable for each $k \in K$, and that α and α^{-1} are continuous at $e \in K$ almost uniformly over compact subsets of Q. Assume also that to each compact set $L \subseteq K$ and compact set $M \subseteq Q$ and $\varepsilon > 0$, there exists a compact subset N of M such that $\mu(M \setminus N) < \varepsilon$ and $\alpha(N)^{-1}(L)$ is relatively compact.*

Then there exist an extension (G,i,p) and a D measurable cross section ϱ for p with $\varrho(e)=e$ such that (c,α) is the corresponding cocycle.

PROOF. Let v denote the right regular representation of K. Since c is D measurable, $v \circ c \in \mathscr{M}(L^2(K), \mu \times \mu)$. By Remark 2, $v(c(\cdot,r)) \in \mathscr{M}(L^2(K),\mu)$ for each $r \in Q$. Define (G,i,p), ϱ, H and V as in the proof of Theorem 1. Then $V \circ i$ tends to zero at infinity in weak operator topology; it is enough to check that

$$(V \circ i(k)\xi \otimes \varphi \mid \eta \otimes \psi) \to 0$$

at infinity when $\xi,\eta \in L^2(K)$, $\varphi,\psi \in K(Q)$; to $\varepsilon>0$ we can choose L compact in K such that

$$2|(v(k)\xi \mid \eta)| \, \|\varphi\|_2 \, \|\psi\|_2 < \varepsilon$$

when $k \notin L$, and a compact subset N of $M=\operatorname{supp}(\varphi\psi)$, such that

$$2\int_{M\setminus N} |\varphi\bar{\psi}| \, d\mu \, \|\xi\| \, \|\eta\| < \varepsilon \quad \text{and}$$

$\alpha(N)^{-1}(L)$ is relatively compact; when $k \notin \alpha(N)^{-1}(L)$ we then have

$$\left|\int_Q (v(\alpha(q)(k))\xi \mid \eta)\varphi(q)\overline{\psi(q)} \, d\mu(q)\right| < \varepsilon .$$

Let λ denote the right Haar measure on G corresponding to given right Haar measures $\varkappa$ on K and μ on Q. Define a map P of G into K by

$$P(g) = i^{-1}[g\varrho(p(g))^{-1}] .$$

Now $V \circ i \circ P \in \mathcal{O}(H,\lambda)$ since $V \circ \varrho \in \mathcal{O}(H,\mu)$, and by Proposition 2 P is F measurable.

We show that ϱ satisfies Condition B in Section 4. Let a compact subset L of Q be given; choose a continuous function $h\colon G \to [0,\infty[$ such that $\int_K h(i(k)g)\,d\varkappa(k) \equiv 1$ and $M=p^{-1}(L)\cap \operatorname{supp}(h)$ is compact; choose a σ-compact subset T of M such that $\lambda(M\setminus T)=0$ and $P(T)$ is contained in a σ-compact set. Then $p(T)$ is a measurable subset of L, and $\varrho(p(T)) \subseteq i(P(T))^{-1}M$ is contained in a σ-compact set, and

$$\mu(L\setminus p(T)) = \int_G h(g) 1_{p^{-1}(L)\setminus KT}(g) \, d\lambda(g)$$

$$\leqq \int_{M\setminus T} h(g) \, d\lambda(g) = 0 .$$

As V is open, ϱ is F measurable by Proposition 2, and D measurable by Lemma 5.

COROLLARY. *Assume given locally compact groups* K *and* Q *and a cocycle* (c,α) *on* Q.

Assume that c *is* D *measurable, that* $\alpha(\cdot)(k)$ *is* F *measurable for each* $k \in K$, *and that* α *and* α^{-1} *are continuous at* $e \in K$ *almost uniformly over compact subsets of* Q. *The following conditions are equivalent*

1. $q \mapsto c(q^{-1},q)$ *is* F *measurable*
2. *For each* $k \in K$, $\alpha(\cdot)^{-1}(k)$ *is* F *measurable*
3. *For each compact set* $L \subseteq K$, *the family of compact sets* $M \subseteq Q$ *with* $\alpha(M)^{-1}(L)$ *relatively compact is* μ *dense.*

PROOF. The implication 1. ⇒ 2. follows from

$$\alpha(q)^{-1}(k) = c(q^{-1},q)^{-1}\alpha(q^{-1})(k)c(q^{-1},q), \qquad q \in Q,\ k \in K .$$

By a short compactness argument 2. implies 3. That 3. implies 1. follows from Theorem 2.

Let (G,i,p) be an extension, λ a right Haar measure on G, v the right regular representation of K, and $\operatorname{ind} v$ the induced representation of G. Then $\operatorname{ind} v$ is a homeomorphism since by the theorem on induction in stages it is unitarily equivalent with the right regular representation of G.

It is known (see [8]) that there exists a map $P \in \mathcal{O}(L^2(K),\lambda)$ (even $\in \mathcal{M}(L^2(K),\lambda)$), such that

$$P(i(k)g) = v(k)P(g) \quad \text{and} \quad P(e) = 1 .$$

Define a unitary map $\tilde{P}$ of $L^2(Q,L^2(K))$ on h $(\operatorname{ind} v)$ by $(\tilde{P}f)(g)=P(g)(f(p(g)))$, and define

$$R(g) = \tilde{P}^{-1}\circ(\operatorname{ind} v)(g)\circ\tilde{P};$$

that is

$$(R(g)f)(p(w)) = P(w)^{-1}P(wg)[f(p(w)p(g))];$$

then R is a homeomorphism.

If e.g. ϱ is an F measurable cross section for p with $\varrho(e)=e$, and (c,α) the corresponding cocycle on Q, and $P(g)=v\circ i^{-1}(g\varrho(p(g))^{-1})$, then

$$(R(i(k_1)\varrho(q_1))f)(q) = v(\alpha(q)(k_1)c(q,q_1))(f(qq_1)) .$$

By an extension with a cross section we understand a quadruple (G,i,p,ϱ) where (G,i,p) is an extension and ϱ is a cross section for p with $\varrho(e)=e$. By an algebraic equivalence between two extensions with cross sections (G_1,i_1,p_1,ϱ_1) and (G_2,i_2,p_2,ϱ_2) we understand an isomorphism Φ of G_1 on G_2 satisfying $\Phi\circ i_1 = i_2$, $p_2\circ\Phi=p_1$ and $\Phi\circ\varrho_1=\varrho_2$. If Φ is also a homeomorphism, we call it a topological equivalence.

Cocycles corresponding to algebraically equivalent extensions with cross sections are equal.

PROPOSITION 4. *Assume given extensions with cross sections* (G_n,i_n,p_n,ϱ_n), $n=1,2$. *If* ϱ_1 *and* ϱ_2 *are* F *measurable, then any algebraic equivalence* Φ *is topological.*

PROOF. Define $P_n(g_n) = v \circ i_n^{-1}(g_n \varrho_n(p_n(g_n))^{-1})$, and define R_n as above, $n = 1, 2$; then $P_2 \circ \Phi = P_1$ and $R_2 \circ \Phi = R_1$.

So we have established a bijection between topological equivalence classes of extensions with D measurable cross sections and cocycles (c, α) satisfying that c is D measurable, that $\alpha(\cdot)(k)$ and $\alpha(\cdot)^{-1}(k)$ are D measurable for each $k \in K$, and that α and α^{-1} are continuous at $e \in K$ almost uniformly over compact subsets of Q.

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MATEMATISK INSTITUT
KØBENHAVNS UNIVERSITET
UNIVERSITETSPARKEN 5
DK-2100 KØBENHAVN Ø
DENMARK