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A REMARK ON SINGULAR SUPPORTS OF CONVOLUTIONS

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For distributions $f, g \in \mathscr{E}'(\mathbf{R}^n)$ it was proved in [2] that

$$\text{ch sing supp}\,(\varphi f) * (\psi g) \subset \text{ch sing supp}\, f * g \tag{1}$$

when φ and ψ are polynomials. Here ch denotes convex hull. The question of the validity of (1) for all $\varphi, \psi \in C^\infty$ was also raised in [2], and in [1] an extension to entire analytic functions φ, ψ of exponential type was stated. (Dr. Dostal has informed the author that the published proof is not correct.) The purpose of this note is to show that the methods of [3] give fairly complete information concerning the validity of (1):

THEOREM 1. *Let $f, g \in \mathscr{E}'(\mathbf{R}^n)$ and assume that $\varphi, \psi \in C^\infty(\mathbf{R}^n)$ are real analytic near* sing supp f *and* sing supp g *respectively. Then it follows that* (1) *is valid.*

THEOREM 2. *In any Denjoy–Carleman class of C^∞ functions which is strictly larger than the analytic class it is possible to find a function φ such that for some $f, g \in \mathscr{E}'$, with* $\text{sing supp}\, f = \text{sing supp}\, g = \{0\}$

$$\text{sing supp}\, f * g = \emptyset, \quad \text{ch sing supp}\,(\varphi f) * g = \{0\}\,.$$

In the proof we shall use the notations of [3]. In particular, we write

$$v_f(z, \xi) = (\log |\hat{f}(\xi + z \log |\xi|)|)/\log |\xi|\,.$$

Recall that every sequence $\xi_j \to \infty$ in $\mathbf{R}^n$ has a subsequence for which $v_f(z, \xi_{j_k})$ converges to a plurisubharmonic function (possibly $-\infty$) having a supporting function H in the sense of [3, section 3]. The set of such supporting functions is denoted by $\mathscr{H}(f)$. We write $\mathscr{H}(f, g)$ for the set of pairs of supporting functions corresponding to simultaneous limits of v_f and v_g.

THEOREM 3. *Let $f \in \mathscr{E}'$ and assume that $\varphi \in C^\infty$ is real analytic near* sing supp f. *Then $(h_1, h) \in \mathscr{H}(\varphi f, f)$ implies $h_1 \leqq h$.*

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PROOF THAT THEOREM 3 IMPLIES THEOREM 1. By Theorem 5.1 and Lemma 5.2 in [3] the supporting function of the left hand side of (1) is the supremum of all sums h_1+h_2 with $(h_1,h_2) \in \mathscr{H}(\varphi f, \psi g)$. By Lemma 5.1 in [3] one can find h_3, h_4 so that $(h_1,h_2,h_3,h_4) \in \mathscr{H}(\varphi f, \psi g, f, g)$, thus $(h_1,h_3) \in \mathscr{H}(\varphi f, f)$ and $(h_2,h_4) \in \mathscr{H}(\psi g, g)$. Hence Theorem 3 gives $h_1 \leqq h_3$ and $h_2 \leqq h_4$ so $h_1+h_2 \leqq h_3+h_4 \in \mathscr{H}(f*g)$, by Theorem 5.1 in [3], so Lemma 5.2 in [3] proves Theorem 1.

PROOF OF THEOREM 3. Let ξ_j be a sequence such that $v_f(z,\xi_j)$ and $v_{\varphi f}(z,\xi_j)$ converge to plurisubharmonic functions with supporting functions h and h_1 respectively. Choose C so that the limit of $v_f(z,\xi_j)$ is $\leqq C-1+h(\operatorname{Im} z)$. For every $R>0$ it follows than that for $j>j(R)$

$$v_f(z,\xi_j) \leqq C+h(\operatorname{Im} z), \qquad |z|<R ,$$

that is,

$$|\hat{f}(z+\xi_j)| \leqq |\xi_j|^C e^{h(\operatorname{Im} z)}, \qquad |z| < R\log|\xi_j| . \tag{2}$$

In addition

$$|\hat{f}(z)| \leqq C_1(1+|z|)^{C_2} e^{C_3|\operatorname{Im} z|}, \qquad z \in \mathbf{C}^n , \tag{3}$$

for some *positive* constants C_1, C_2, C_3. It is no restriction to assume that φ is analytic in a neighborhood of $\operatorname{supp} f$, for f may be replaced by χf where $\chi \in C_0^\infty$ is equal to 1 near $\operatorname{sing\,supp} f$ and φ is analytic near $\operatorname{supp}\chi$. We can then use [4, Proposition 2.4] to choose a sequence of functions $\varphi_k \in C_0^\infty$ equal to φ near $\operatorname{supp} f$ so that for every k

$$|\hat{\varphi}_k(\xi)| \leqq C, \; |\hat{\varphi}_k(\xi)| \leqq C^{k+1}(k/(k+|\xi|))^k, \qquad \xi \in \mathbf{R}^n . \tag{4}$$

If $F=\varphi f$ then

$$\hat{F}(\xi_j+z) = (2\pi)^{-n}\int \hat{f}(\xi_j+z-\theta)\hat{\varphi}_k(\theta)\,d\theta ,$$

and we shall estimate this when $|z|<\gamma R\log|\xi_j|$ where $\gamma \in (0,1/2)$ will be chosen later on. By (2) the integral over the set where $|\theta|<(R/2)\log|\xi_j|$ can be estimated by

$$|\xi_j|^{C+1} e^{h(\operatorname{Im} z)} \tag{5}$$

and the remaining part of the integral can be estimated by

$$C_1(1+|\xi_j|+|z|)^{C_2} e^{C_3|\operatorname{Im} z|}\int_{|\theta|>R/2\log|\xi_j|}(1+|\theta|)^{C_2}|\hat{\varphi}_k(\theta)|\,d\theta .$$

If $a=Ce$ where C is the constant in (4), we have for large k

$$\int_{|\theta|>ka}(1+|\theta|)^{C_2}|\hat{\varphi}_k(\theta)|\,d\theta \leqq C'C^k(ak)^{C_2+n}a^{-k} \leqq C''e^{-k}k^{C_2+n} .$$

We choose k equal to the integral part of $(R/2a)\log|\xi_j|$ and obtain

$$e^{-k} < e|\xi_j|^{-R/2a} .$$

Without restriction we may assume that $h \geqq 0$. If $\gamma < 1/2aC_3$ and $C_4 > \max(C+1, C_2+1)$ we then obtain for large j

$$(5)' \qquad |\hat{F}(\xi_j+z)| \leqq |\xi_j|^{C_4} e^{h(\operatorname{Im} z)}, \qquad |z| < \gamma R \log|\xi_j| .$$

This implies that $h_1 \leqq h$ which completes the proof of Theorem 3.

Before passing to the proof of Theorem 2 we recall the basic definitions involved. By a Denjoy–Carleman class C^M where $M=(M_0, M_1, \ldots)$ is an increasing logarithmically convex sequence with $M_0=M_1=1$ we mean the space of C^∞ functions φ such that for every compact set K there is a constant C_K such that for all multiindices α

$$|D^\alpha \varphi(x)| \leqq C_K^{|\alpha|+1} M_{|\alpha|}, \qquad x \in K .$$

We assume $M_k \geqq k!$ so that C^M contains the real analytic class. Set

$$M(t) = \sum_0^\infty t^k/M_k$$

which is then convergent. It is obvious that for all α

$$|D^\alpha \varphi| \leqq (2\pi)^{-n} M_{|\alpha|} \int |\hat{\varphi}(\xi)| M(|\xi|)\, d\xi$$

so C^M contains the Banach space B of all $\varphi \in \mathscr{S}'$ with $\hat{\varphi} \in L^1$ and the norm

$$\|\varphi\|_B = (2\pi)^{-n} \int |\hat{\varphi}(\xi)| M(|\xi|)\, d\xi$$

finite. It is well known that C^M is the analytic class if and only if

$$e^{ct} \leqq CM(t)$$

for some $c>0$ and C. If this is not the case we therefore have

$$\varliminf_{t\to\infty} M(t) e^{-t/j} = 0$$

for every positive integer j. Hence we can choose a sequence $a_j \to \infty$, increasing as rapidly as we please, so that

$$(6) \qquad jM(a_j) < \exp(a_j/j) .$$

Choose a sequence $\xi_j \in \mathbf{R}^n$ with

$$\log|\xi_j| = a_j/j .$$

If a_j increases sufficiently rapidly then the balls

$$\{\xi \in \mathbf{R}^n \; ; \; |\xi - \xi_j| \leqq a_j\}$$

are disjoint and $a_j/|\xi_j| \to 0$. Set

$$E = \{\xi \; ; \; |\xi - \xi_j| \geqq a_j/2 \text{ for all } j\} .$$

Then we have

$$|\xi - \xi_j| \geqq a_j/2 = (j/2)\log|\xi_j|, \qquad \xi \in E ,$$

so it follows from [3, Theorem 5.2] that we can find $f \in \mathscr{E}'$ with $\operatorname{sing\,supp} f = \{0\}$ so that

$$v_f(z; \xi) \to -\infty \qquad \text{when } E \ni \xi \to \infty$$

but $v_f(z; \xi_j)$ does not converge to $-\infty$. Choose $\eta_j \in E$ with $|\eta_j - \xi_j| = a_j$.

PROPOSITION 4. *If M is not the analytic class and f is chosen as just described then $v_{\varphi f}(z, \eta_j)$ does not converge to $-\infty$ for all $\varphi \in B$.*

PROOF. If $v_{\varphi f}(z, \eta_j)$ converges to $-\infty$ then

$$\sup_j |\eta_j|^N |(\varphi f)\hat{}\,(\eta_j)|$$

is finite for every N, and if this is true for every $\varphi \in B$, then

$$\sup_j |\eta_j|^N |(\varphi f)\hat{}\,(\eta_j)| \leqq C_N \|\varphi\|_B$$

by the closed graph theorem. Thus

$$|\eta_j|^N \left| \int \hat{\varphi}(\xi) \hat{f}(\eta_j - \xi)\, d\xi \right| \leqq C_N \int |\hat{\varphi}(\xi)| M(|\xi|)\, d\xi$$

which means that

$$(7) \qquad |\eta_j|^N \sup_\xi |\hat{f}(\eta_j - \xi)|/M(|\xi|) \leqq C_N .$$

Now there is a subsequence ξ_{j_k} such that

$$v_f(z, \xi_{j_k})$$

converges to a plurisubharmonic function which is not $-\infty$ identically and therefore constant since the supporting function is 0. (See [3, Lemma 3.6].) Hence

$$v_f(z, \xi_{j_k}) \to C$$

in $L^1_{\text{loc}}(\mathbf{C}^n)$ and also in $L^1_{\text{loc}}(\mathbf{R}^n)$. It follows that we can find $\theta_{j_k} \in \mathbf{R}^n$ so that for large k

$$|\theta_{j_k} - \eta_{j_k}| \leqq a_{j_k}, |\hat{f}(\theta_{j_k})| > |\xi_{j_k}|^{C-1}.$$

With $\xi = \eta_{j_k} - \theta_{j_k}$ and $j = j_k$ in (7) we obtain

$$|\xi_{j_k}|^{N+C-1} \leqq C'_N M(a_{j_k}).$$

Choose N so that $N+C>2$. Then we obtain

$$\exp(a_{j_k}/j_k) \leqq C'_N M(a_{j_k})$$

which contradicts (6), so the proof is complete.

Proof of Theorem 2. Assuming that M is not the analytic class we use Proposition 4 to choose $\varphi \in B_M$ so that $v_{\varphi f}(z, \eta_j)$ does not tend to $-\infty$. Let η_{j_k} be a subsequence for which there is a finite limit. With

$$E_1 = \{\xi \ ; \ |\xi - \eta_{j_k}| \geqq a_{j_k}/2 \text{ for all } k\}$$

we choose according to [3, Theorem 5.2] a distribution $g \in \mathscr{E}'$ with sing supp $g = \{0\}$ so that $v_g(z, \eta_{j_k})$ does not tend to $-\infty$ but $v_g(z, \xi) \to -\infty$ when $\xi \to \infty$ in E_1. Then $f*g \in C^\infty$ by [3, Lemma 5.2 and Theorem 5.1], for every sequence $\to \infty$ in $\mathbf{R}^n$ contains a subsequence in E or one in E_1, so for $(h_f, h_g) \in \mathscr{H}(f, g)$ we always have $h_f = -\infty$ or $h_g = -\infty$. On the other hand, for a subsequence of η_{j_k} we know that both $v_{\varphi f}(z, \eta_{j_k})$ and $v_g(z, \eta_{j_k})$ have finite limits, so $(0, 0) \in \mathscr{H}(\varphi f, g)$. Hence $(\varphi f)*g$ is not in C^∞ so the singular support is $\{0\}$.

It is clear that by a slight modification of the preceding construction one can modify the statement of Theorem 2 so that sing supp $f*g$ and sing supp $(\varphi f)*g$ are two arbitrary convex compact sets.

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