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THE LEVI PROBLEM IN STEIN SPACES

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1. Introduction.

Let X denote a Stein space and let Ω be an open set in X. Assume that for every $p \in \partial\Omega$ there exists an open neighborhood $U(p)$ such that $\Omega \cap U(p)$ is Stein.

THE LEVI PROBLEM. *Is Ω necessarily Stein?*

In case X is a complex manifold this was solved affirmatively by Docquier and Grauert [4], and in case X has at most isolated singularities it was solved affirmatively by Andreotti and Narasimhan [1].

THE UNION PROBLEM. *If $\Omega^{\mathrm{open}} \subset X^{\mathrm{Stein}}$ and $\Omega_1 \subset \Omega_2 \subset \ldots \subset \bigcup \Omega_n^{\mathrm{open}} = \Omega$ with each Ω_n Stein, is Ω Stein?*

This was proved to be true if $X = \mathbf{C}^k$ by Behnke and Stein [2]. The case when X is a Stein manifold follows from the work of Docquier and Grauert [4] via the embedding of X as a closed complex submanifold of some $\mathbf{C}^l$, Remmert [13], Bishop [3] and Narasimhan [9].

If one drops the assumption that X is Stein, the result is not true, Fornæss [6].

Suppose next that $\{\Omega_t\}_{t\in\mathbf{R}}$ is a family of Stein open subsets of X and that $\bigcup_{\tau<t}\Omega_\tau$ is a union of connected components of Ω_t and that Ω_t is a union of connected components of $\mathrm{int}\bigcap_{\tau>t}\Omega_\tau$ for each $t \in \mathbf{R}$.

THE RUNGE PROBLEM. *Is Ω_r Runge in Ω_s whenever $r<s$.*

When X is complex manifold this was answered affirmatively by Docquier and Grauert [4].

In this short note we will solve affirmatively the above problems in the Stein space

$$X = Z \times \mathbf{C}, \quad Z = \{(z_1,z_2,z_3,z_4) \in \mathbf{C}^4 \; ; \; z_1z_2 = z_3z_4\} .$$

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The reason that we find the space Z interesting is the following observation by Grauert and Remmert [7]. The map $\Phi\colon \mathbf{C} \times (\mathbf{C}^2 - (0)) \to Z$ by

$$\Phi(t, w, \eta) = (w, t\eta, \eta, tw)$$

is biholomorphic onto the open set $\Omega = Z - \{z_1 = z_3 = 0\}$. For every $p \in \partial\Omega$, $p \neq 0$, there exists an open neighborhood $U(p)$ in Z such that $U(p)$ is Stein. However Ω is obviously not Stein. This can be compared to the following theorem by Grauert and Remmert [7].

THEOREM. *If $\Omega^{\text{open}} \subset \mathbf{C}^n$ and for every $p \in \partial\Omega$, $p \neq 0$, there exists an open neighborhood $U(p)$ such that $U(p) \cap \Omega$ is Stein, then Ω is Stein, unless $\Omega \cup (0)$ is open (in which case $\Omega \cup (0)$ is Stein).*

A function $f\colon X \to \mathbf{R} \cup \{-\infty\}$ where X is a (reduced) complex space will be said to be plurisubharmonic if for every $x \in X$ there is an open neighborhood $U(x)$ which can be realized as a closed complex subvariety $Y \subset V^{\text{open}} \subset \mathbf{C}^n$, $\Phi\colon U(x) \xrightarrow{\cong} Y$ such that $f \circ \Phi^{-1}$ is the restriction to Y of a plurisubharmonic function on V. The function f is continuous (smooth) and plurisubharmonic if in addition $f \circ \Phi^{-1}$ can be chosen to be continuous (smooth). Also f is said to be (continuous/smooth) strongly plurisubharmonic if $f \circ \Phi^{-1} + \varepsilon\tau$ is (continuous/smooth) plurisubharmonic for all $\varepsilon \geqq 0$ sufficiently small whenever $\tau \in \mathscr{C}_0^\infty(V)$, $\tau\colon V \to \mathbf{R}$.

It is a theorem by Richberg [14] that strongly plurisubharmonic functions which are continuous are continuous strongly plurisubharmonic.

The results and proofs in this paper are equally valid in Stein spaces $X' = Z' \times M$ where M is any Stein manifold and

$$Z' = \{(z_1, \ldots, z_n, w_1, \ldots, w_n) \in \mathbf{C}^{2n}\ ;\ z_i w_j = z_j w_i \text{ for all } i, j\}\ .$$

2. Preliminary remarks.

We would like here to briefly recall a few results which we will need.

THEOREM 1. (Narasimhan [10, 11]). *Let X be a complex space. Then X is Stein if and only if there exists a continuous strongly plurisubharmonic function $\varphi\colon X \to \mathbf{R}$ such that $X_\alpha = \{x \in X\ ;\ \varphi(x) < \alpha\}$ is relatively compact in X for all $\alpha \in \mathbf{R}$.*

THEOREM 2. (Narasimhan [10, 11]). *Let X be a Stein space and let $\varphi\colon X \to \mathbf{R}$ be a continuous plurisubharmonic function. Then $X_\alpha = \{x \in X\ ;\ \varphi(x) < \alpha\}$ is Stein and Runge in X for all $\alpha \in \mathbf{R}$.*

A particularly useful consequence of the two above theorems is the following well known result:

COROLLARY 3. *If X is a Stein space and $K=\hat{K}$ is a compact set in X, then K has a neighborhood basis of Stein open sets which are Runge in X.*

We also need the following theorem due to Richberg [14].

THEOREM 4. *If ϱ is a continuous strongly plurisubharmonic function on a countably compact complex manifold M and $\tau\colon M \to \mathbf{R}^+$ is a strictly positive continuous function, then there exists a smooth strongly plurisubharmonic function ϱ^* on M such that $\varrho<\varrho^*<\varrho+\tau$. If σ is a continuous nonnegative plurisubharmonic function on a countably compact complex space X, $\sigma\equiv 0$ in a neighborhood of the singular locus of X and there exists a bounded continuous strongly plurisubharmonic function on X, then for every $\varepsilon>0$ there exists a smooth plurisubharmonic function σ^* on X with $\sigma<\sigma^*<\sigma+\varepsilon$.*

Let us consider the Stein space $Z\times\mathbf{C}$ where $Z=\{z\in\mathbf{C}^4\ ;\ z_1z_2=z_3z_4\}$. If $\Omega^{\text{open}}\subset Z\times\mathbf{C}$, we can define a distance function $\delta\colon \Omega\to\mathbf{R}\cup\{\infty\}$ as follows. For any $q=(p,c)\in\Omega$, we let

$$\delta(q) \;=\; \sup\{r\ ;\ (p,c+z)\in\Omega \text{ for all } z\in\mathbf{C},\ |z|<r\}\ .$$

PROPOSITION 5. *The function $-\log\delta\colon \Omega\to\mathbf{R}\cup\{-\infty\}$ is plurisubharmonic if Ω is Stein, except on those connected components of Ω where $-\log\delta\equiv-\infty$.*

PROOF. By the theorem of Siu [15] there exists a domain of holomorphy, $\tilde{\Omega}$, in $\mathbf{C}^4$ such that $\tilde{\Omega}\cap(Z\times\mathbf{C})=\Omega$. If we define $\tilde{\delta}\colon\tilde{\Omega}\to\mathbf{R}\cup\{\infty\}$ in the same way as δ, we obtain a plurisubharmonic function $-\log\tilde{\delta}\colon\tilde{\Omega}\to\mathbf{R}\cup\{-\infty\}$ such that $-\log\tilde{\delta}\,|\,\Omega=-\log\delta$.

3. Z as a branched Riemann domain.

In the paper of Andreotti and Narasimhan [1] they make fundamental use of the fact that a pure n-dimensional Stein space X may be realized as a branched Riemann domain over $\mathbf{C}^n$ in many different ways. Although the singular points of X necessarily are branch points, one can always make the branch locus avoid any given regular point.

Let $Z=\{(z_1,z_2,z_3,z_4)\in\mathbf{C}^4\ ;\ z_1z_2=z_3z_4\}$. We consider two holomorphic maps $\Phi_1,\Phi_2\colon\mathbf{C}^3\to Z$, by

$$\Phi_1(t,w,\eta) \;=\; (w,t\eta,\eta,tw) \quad\text{and}\quad \Phi_2(t,w,\eta) \;=\; (tw,\eta,t\eta,w)\ .$$

The following lemma is easily verified.

LEMMA 6. $\Phi_1(t,0,0)\equiv\Phi_2(t,0,0)\equiv 0$ *and if*

$$U = \{(t,w,\eta)\in \mathbf{C}^3\ ;\ (w,\eta)\neq(0,0)\}$$

then $\Phi_i\,|\,U$ *is biholomorphic onto the open set* $\Phi_i(U)$, $i=1,2$. *Moreover* $Z-(0) =\Phi_1(U)\cup\Phi_2(U)$.

We will now define four holomorphic maps π_i: $Z\to\mathbf{C}^3$, $i=1,2,3,4$. More precisely $\pi_1(z_1,z_2,z_3,z_4)=(z_1,z_2,z_3+2z_4)$, $\pi_2(z)=(z_1,z_2,2z_3+z_4)$, $\pi_3(z)=(z_1+2z_2,z_3,z_4)$ and $\pi_4(x)=(2z_1+z_3,z_3,z_4)$. Also define the holomorphic functions f_j: $Z\to\mathbf{C}$ by $f_1(z)=z_3-2z_4$, $f_2(z)=2z_3-z_4$, $f_3(z)=z_1-2z_2$ and $f_4(z) =2z_1-z_2$.

LEMMA 7. *The holomorphic map* π_j: $Z\to\mathbf{C}^3$ *makes* Z *into a doubly sheeted branched covering of* $\mathbf{C}^3$. *The set of branch points is precisely* $S_j=\{f_j=0\}$. *If* $p\in S_j-\{0\}$ *one can find local holomorphic coordinates* $w=(w_1,w_2,w_3)$, $p=0$ *on* Z *and local holomorphic coordinates* $\eta=(\eta_1,\eta_2,\eta_3)$, $\pi_j(p)=0$ *on* $\mathbf{C}^3$ *such that*

$$\pi_j(w_1,w_2,w_3) = (w_1,w_2,w_3^2) = (\eta_1,\eta_2,\eta_3)\,.$$

The proof of this is a straightforward computation and will be omitted. We should also point out that in the coordinatesystem of the lemma, f_j/w_3 is a nonzero holomorphic function.

Following the argument of Andreotti and Narasimhan [1] we obtain plurisubharmonic functions on open subsets Ω of $Z\times\mathbf{C}$ which are locally Stein away from $(0)\times\mathbf{C}$. First we define $\tilde\pi_j$: $Z\times\mathbf{C}\to\mathbf{C}^4=\mathbf{C}^3\times\mathbf{C}$ by $\tilde\pi_j(p,c) =(\pi_j(p),c)$. Clearly this defines $Z\times\mathbf{C}$ as a branched Riemann domain over $\mathbf{C}^4$ with branch locus $\tilde S_j=\{\tilde f_j=0\}$ where $\tilde f_j(p,c)=f_j(p)$.

For any $j\in\{1,2,3,4\}$, $\tilde\pi_j$: $\Omega-\tilde S_j\to\mathbf{C}^4$ realizes this as an unbranched Riemann domain. From the classical theory on the Levi problem one now has that $-\log d_j$: $\Omega-\tilde S_j\to\mathbf{R}$ is continuous and plurisubharmonic. Here $d_j(q)$ is the radius of the largest ball centered at $\tilde\pi_j(q)$ onto which $\tilde\pi_j$ maps a neighborhood of q biholomorphically. Let us define φ_j: $\Omega\to\mathbf{R}\cup\{-\infty\}$ by

$$\tilde\varphi_j = -\log d_j+3\log|\tilde f_j| \text{ on } \Omega-\tilde S_j,\qquad \tilde\varphi_j\equiv-\infty \text{ on } \Omega\cap\tilde S_j\,.$$

PROPOSITION 8. *The function* $\varphi_j=\max\{\tilde\varphi_j,0\}$ *is continuous and plurisubharmonic on* Ω. *Moreover, if* $q\in\partial\Omega-\tilde S_j$, $\varphi_j(p)\to\infty$ *when* $p\in\Omega$ *and* $p\to q$.

PROOF. It is clear that if $q\in\partial\Omega-\tilde S_j$, then $\varphi_j(p)\to\infty$ when $p\in\Omega$ and $p\to q$. In fact $\varphi_j(p)$ grows like $-\log\operatorname{dist}(p,\partial\Omega)$ measured in any smooth Hermitian metric defined on $Z\times\mathbf{C}$ near q.

Near a point $q\in\Omega\cap\tilde S_j$, $q\neq 0$, $-\log d_j$ grows like $2\log|\tilde f_j|$. Hence φ_j is plurisubharmonic across $\tilde{S}_j$ away from $(0) \times \mathbf{C}$. To complete the proof it suffices to show that if $q = (0, c) \in \Omega$, then $\varphi_j \equiv 0$ in a neighborhood of q, because then φ_j is locally on Ω the restriction to Ω of a plurisubharmonic function defined on an open set in $\mathbf{C}^4$.

Let us consider $j = 1$. Then one computes that the image of the branch locus is

$$\{\tau = (\tau_1, \tau_2, \tau_3, \tau_4) \; ; \; \tau_1 \tau_2 = \tau_3^2/8\} \; = \; S'_1 \; .$$

Moreover $|\tilde{f}_1|^2(\tilde{q}) = |\tau_3^2 - 8\tau_1\tau_2|$ if $\tilde{\pi}_1(\tilde{q}) = \tau$. Hence already $-\log d_1 + 2\log|\tilde{f}_1|$ approaches $-\infty$ when $\tilde{q} \to q$. The same argument applies to $j = 2, 3, 4$.

We remark that we could have defined $\varphi_j = \max\{-\log d_j + 2\log|\tilde{f}_j|, 0\}$ without altering the conclusion of Proposition 8.

4. Another distance function.

We have in the preceding sections described two sorts of plurisubharmonic functions on a Stein open set Ω of $Z \times \mathbf{C}$. One is the φ_j's which blow up at nonsingular boundary points and the other measures boundary distance in the $\mathbf{C}$-direction. In this section we want to construct plurisubharmonic functions which blow up the Z-direction when we approach a point $(0, c) \in \partial\Omega$.

Let us first define a holomorphic map $\Gamma \colon \mathbf{C}^4 \to Z$ by

$$\Gamma(w) \; = \; (w_1 w_2, w_3 w_4, w_1 w_3, w_2 w_4)$$

LEMMA 9. *The holomorphic map $\Gamma \colon \mathbf{C}^4 \to Z$ is onto. Furthermore*

$$\Gamma^{-1}(0) \; = \; \{w_1 = w_4 = 0\} \cup \{w_2 = w_3 = 0\}$$

while for every $w \in \mathbf{C}^4 - \Gamma^{-1}(0)$, we have

$$\Gamma^{-1}(\Gamma(w)) \; = \; \{(w_1\tau, w_2/\tau, w_3/\tau, w_4\tau) \; ; \; \tau \in \mathbf{C} - (0)\} \; .$$

The proof is straightforward and may be omitted. Let us now consider the map $\tilde{\Gamma} \colon \mathbf{C}^5 \to Z \times \mathbf{C}$ by $\tilde{\Gamma}(p, c) = (\Gamma(p), c)$. For any open set $\Omega \subset Z \times \mathbf{C}$ we can define the distance functions $\Delta_1, \Delta_2 \colon \tilde{\Omega} = \tilde{\Gamma}^{-1}(\Omega) \to \mathbf{R} \cup \{\infty\}$ as follows:

Let

$$\Delta_1(w, c) \; = \; \sup\{r \; ; \; (w_1 + \tau_1, w_2, w_3, w_4 + \tau_2, c) \in \tilde{\Omega} \quad \text{for all } (\tau_1, \tau_2) \in \mathbf{C}^2, \; |\tau_1|^2 + |\tau_2|^2 < r^2\}$$

and let

$$\Delta_2(w, c) \; = \; \sup\{r \; ; \; (w_1, w_2 + \tau_1, w_3 + \tau_2, w_4, c) \in \tilde{\Omega} \quad \text{for all } (\tau_1, \tau_2) \in \mathbf{C}^2, \; |\tau_1|^2 + |\tau_2|^2 < r^2\} \; .$$

LEMMA 10. *$\Delta_1 \cdot \Delta_2$ is constant on each fibre of $\tilde{\Gamma}$. Moreover, if $\tilde{\Gamma}(q) = (p,c)$ and if $(0,c) \notin \Omega$, then $\Delta_1 \cdot \Delta_2(q) \leqq 2\|p\|$.*

PROOF. We easily check that $\Delta_1(w_1\tau, w_2/\tau, w_3/\tau, w_4\tau, c) = |\tau|\Delta_1(w,c)$ and

$$\Delta_2(w_1\tau, w_2/\tau, w_3/\tau, w_4\tau, c) = \frac{1}{|\tau|}\Delta_2(w,c)$$

from which the first statement follows.

Next assume that $q = (w,c) \in \tilde{\Omega}$ and that $(0,c) \notin \Omega$. Let $(z,c) = \tilde{\Gamma}(w,c)$, and assume $|z_1| = \max_{j=1,\dots,4}\{|z_j|\}$. The argument is similar for the other possibilities. By the first statement, we may suppose that we have chosen $w_1 = \sqrt{z_1}$. Thus $w_2 = \sqrt{z_1}$, $w_3 = z_3/\sqrt{z_1}$ and $w_4 = z_4/\sqrt{z_1}$ as one deduces from the definition of Γ. In particular, this implies that $|w_3|, |w_4| \leqq |w_1| = |w_2| = \sqrt{|z_1|}$. Since $\tilde{\Gamma}^{-1}(0,c) \subset \mathbf{C}^5 - \tilde{\Omega}$, one obtains that

$$\Delta_1(w,c) \leqq (|w_1|^2 + |w_4|^2)^{\frac{1}{2}} \leqq \sqrt{2}\sqrt{|z_1|}$$

and similarly $\Delta_2(w,c) \leqq \sqrt{2}\sqrt{|z_1|}$. Hence $\Delta_1 \cdot \Delta_2(w,c) \leqq 2|z_1| \leqq 2\|z\|$ as desired.

DEFINITION 11. Assume $\Omega \subset Z \times \mathbf{C}$ is an open subset. Then $\Delta(q)\colon \Omega \to \mathbf{R} \cup \{\infty\}$ is defined as $\Delta(q) = \Delta_1(\tilde{q}) \cdot \Delta_2(\tilde{q})$ for any $\tilde{q} \in \tilde{\Gamma}^{-1}(q)$.

LEMMA 12. *Assume $\Omega \subset Z \times \mathbf{C}$ is Stein or an increasing union of Stein open sets or is locally Stein. Then $\Delta^* = \max\{-\log \Delta, 0\}\colon \Omega \to \mathbf{R}$ is plurisubharmonic. Moreover $\Delta^* \equiv 0$ in an open neighborhood of* $(0,c)$ if $(0,c) \in \Omega$. *Also if* $(0,c) \notin \Omega$,

$$\Delta^*(p,c) \geqq -\log \Delta \geqq -\log \|p\| - \log 2$$

whenever $(p,c) \in \Omega$.

PROOF. The set $\tilde{\Omega} = \tilde{\Gamma}^{-1}(\Omega)$ is a domain of holomorphy. This implies that $-\log \Delta_1$ and $-\log \Delta_2$ are plurisubharmonic on $\tilde{\Omega}$. Therefore $-\log \Delta_1\Delta_2$ is also plurisubharmonic. Clearly $-\log \Delta_1\Delta_2 \equiv -\infty$ on $\tilde{\Gamma}^{-1}(0,c)$ if $(0,c) \in \Omega$. The Lemma now follows from Lemma 10 and the observation that for every $(z^0,c) \in (Z - (0)) \times \mathbf{C}$ there exists a holomorphic map T defined in an open neighborhood of (z^0,c) in $Z \times \mathbf{C}$ to $\mathbf{C}^5$ such that $\tilde{\Gamma} \circ T = \mathrm{Id}$. For example, if $z_1^0 \neq 0$, we can define

$$T(z,c) = (1, z_1, z_3, z_4/z_1, c)\,.$$

LEMMA 13. *If $\Omega \subset Z \times \mathbf{C}$ is locally Stein or an increasing union of Stein open sets and if $\partial\Omega$ is smooth away from $0 \times \mathbf{C}$, then Δ^* is continuous.*

This is proved using the result by Kerzman [8] that smoothly bounded domains of holomorphy are taut. Let us just point out that if $(w^0, c) \in \mathbf{C}^5$, $\tilde{\Gamma}(w^0, c) = (z^0, c) \neq (0, c)$, and if say

$$\Delta_1(w^0, c) = \sqrt{w_1^0 \bar{w}_1^0 + w_4^0 \bar{w}_4^0} = \delta ,$$

then we can show that Δ_1 is upper semicontinuous at (w^0, c) by contradiction as follows. Assume for some $\delta' > 0$ that there exists a sequence $\{(w^n, c_n)\}_{n=1}^{\infty} \in \tilde{\Gamma}^{-1}(\Omega) = \tilde{\Omega}$ such that $(w^n, c_n) \to (w^0, c)$ and

$$\{(w_1, w_2^n, w_3^n, w_4, c_n) \ ; \ \sqrt{|w_1 - w_1^n|^2 + |w_4 - w_4^n|^2} < \delta + \delta'\}$$

is contained in $\tilde{\Omega}$ for all n. By tautness away from $\tilde{\Gamma}^{-1}(0 \times \mathbf{C})$ it follows that $\tilde{\Omega}$ contains

$$\{(w_1, w_2^0, w_3^0, w_4, c) \ ; \ \sqrt{|w_1 - w_1^0|^2 + |w_4 - w_4^0|^2} < \delta + \delta' \quad \text{and} \quad (w_1, w_4) \neq (0,0)\} .$$

Since $\tilde{\Omega}$ is a domain of holomorphy, $(0, w_2^0, w_3^0, 0, c) \in \tilde{\Omega}$ as well and hence $\Delta_1(w^0, c) \geqq \delta + \delta'$ which gives a contradiction.

5. The Levi problem.

Assume Ω is an open subset of $X = Z \times \mathbf{C}$, $Z = \{z \in \mathbf{C}^4 \ ; \ z_1 z_2 = z_3 z_4\}$.

THEOREM 14. *If Ω is locally Stein,* i.e. *for every point $p \in \partial\Omega$ there is an open neighborhood $U(p)$ such that $U(p) \cap \Omega$ is Stein, then Ω is Stein.*

PROOF. The function $\|z\|^2 + c\bar{c}$, $z \in Z$, $c \in \mathbf{C}$ is a continuous plurisubharmonic function on X. Hence it follows from Theorem 2 that we may assume that $\Omega \subset\subset Z \times \mathbf{C}$. The maps $\tilde{\pi}_j$: $\Omega - \tilde{S}_j \to \mathbf{C}^4$ realize $\Omega - \tilde{S}_j$ as a locally Stein unbranched Riemann domain over $\mathbf{C}^4$, $j = 1, 2, 3, 4$. By Oka's [12] solution of the Levi Problem it follows that $\Omega - \tilde{S}_j$ is Stein. Therefore the functions φ_j constructed in Proposition 8 are continuous plurisubharmonic functions on Ω which are identically zero in a neighborhood of $\Omega \cap \{0 \times \mathbf{C}\}$.

Hence by Theorem 4 there is a smooth plurisubharmonic function φ: $\Omega \to \mathbf{R}$ such that $|\varphi - \sup \varphi_j| < 1$ on Ω. In particular $\varphi(p) \to \infty$ if $p \in \Omega$ approaches any point $q \in \partial\Omega - (0 \times \mathbf{C})$.

This implies, by Sard's theorem, that there exists arbitrarily large $\alpha \in \mathbf{R}$ such that $\Omega_\alpha = \{\varphi < \alpha\}$ has smooth boundary away from $(0) \times \mathbf{C}$.

By Theorem 2 it suffices to prove that any such Ω_α is Stein. So we fix an Ω_α with the above boundary property in the rest of the proof.

From Lemma 13, applied to Ω_α, it follows that Δ^* is a continuous nonnegative plurisubharmonic function on Ω_α which is identically zero in an open set containing $\Omega_\alpha \cap (0 \times \mathbf{C})$. Hence using Theorem 4 again, we find a smooth plurisubharmonic function $\hat{\Delta}: \Omega_\alpha \to \mathbf{R}$ such that $|\hat{\Delta} - \Delta^*| < 1$ on Ω_α.

Let $\Omega^\beta = \{q \in \Omega_\alpha ; \hat{\Delta}(q) < \beta\}$ for $\beta \in \mathbf{R}$. From Sard's theorem it follows that $\partial\Omega^\beta$ is smooth away from $\partial\Omega_\alpha$ and $\Omega_\alpha \cap (0 \times \mathbf{C})$ for arbitrarily large β. We fix such an Ω^β in the rest of the proof and observe that by Theorem 2 it suffices to show that Ω^β is Stein.

We will construct a continuous strongly plurisubharmonic exhaustion function on Ω^β. Since, if $\varphi_j: \Omega^\beta \to \mathbf{R}$ is as in Proposition 8, $\max_{j=1,2,3,4}\{\varphi_j\} + \|z\|^2 + c\bar{c}$ is a nonnegative strongly plurisubharmonic function which blows up at every boundary point of Ω^β, except along $(0) \times \mathbf{C}$, it suffices to find a continuous nonnegative plurisubharmonic function on Ω^β which blows up at every boundary point of Ω^β on $(0) \times \mathbf{C}$. In fact, we will prove that $\max\{-\log\delta, \gamma\} = \delta^*$ is such a function if δ is as in Proposition 5, and if γ is sufficiently large.

The local Stein-ness of Ω^β follows from Theorem 2 and implies via Proposition 5 that δ^* is plurisubharmonic if γ is sufficiently large. It remains to prove that δ is continuous and that $\delta \to 0$ when we approach $\partial\Omega^\beta \cap (0 \times \mathbf{C})$.

Let $U = \Omega^\beta \cap (0 \times \mathbf{C})$ and consider a point $(0, c) \notin U$. First of all, we observe that $(0, c) \notin \Omega_\alpha$ since we may assume $\beta \gg 1$. If $(p, c) \in \Omega^\beta$, then $\hat{\Delta}\ (p, c) < \beta$ and hence $\Delta^*(p, c) < \beta + 1$. From Lemma 12 it now follows that $-\log\|p\| - \log 2 < \beta + 1$, and so $\|p\| > e^{-\beta-2}$. Therefore, if $(p, c) \in \Omega^\beta$ and $\|p\| < e^{-\beta-2}$, we must necessarily have $(0, c) \in U$. This implies that $\delta \to 0$ when we approach $\partial\Omega^\beta \cap (0 \times \mathbf{C})$. Also, this implies that δ is continuous at every point in U.

Fix a point $(p, c) \in \Omega^\beta$, $p \neq 0$. We will show that δ is continuous at this point. Since Ω^β is open, δ is lower semicontinuous. Assume δ is not upper semicontinuous. Let (p, c') be a point on $\partial\Omega^\beta$ with $|c' - c| = \delta(p, c)$. There exists an $\varepsilon > 0$ and a sequence $\{p^n\}_{n=1}^\infty$ such that $p^n \to p$ and

$$\Delta^n = \{(p^n, c'') ; |c'' - c| < \delta(p, c) + \varepsilon\} \subset \Omega^\beta$$

for all n. Let $\Delta = \{(p, c'') ; |c'' - c| < \delta(p, c) + \varepsilon\}$, and observe that since $\Delta^n \subset \Omega_\alpha$ and Ω_α is taut at smooth boundary points, Kerzman [8], it follows that $\Delta \subset \Omega_\alpha$. This implies that $(p, c') \in \partial\Omega^\beta \cap \Omega_\alpha$, which contradicts the same result of Kerzman since $\partial\Omega^\beta$ is smooth away from $\partial\Omega_\alpha$ and $(0) \times \mathbf{C}$. Hence δ is upper semicontinuous at (p, c) as well.

6. The union problem.

Let $\{\Omega_n\}$ be a sequence of Stein open subsets of

$$X = Z \times \mathbf{C}, \quad Z = \{z \in \mathbf{C}^4 ; z_1 z_2 = z_3 z_4\}.$$

THEOREM 15. *If* $\Omega_1 \subset \Omega_2 \subset \ldots$ *and* $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$, *then* Ω *is Stein.*

We will first prove a standard Lemma which reduces the proof to an estimate of the hulls of compact subsets of Ω.

LEMMA 16. *If for every compact set* $K \subset \Omega$ *there exists a compact set* $F \subset \Omega$ *such that* $\hat{K}_{\Omega_n} \subset F$ *for all* $\Omega_n \supset K$, *then* Ω *is Stein.*

PROOF OF THE LEMMA. Choose compact sets $\{K_n\}_{n=1}^{\infty}$ such that $K_n \subset \operatorname{int} K_{n+1}$ for all n and $\Omega = \bigcup K_n$. Let $\{F_n\}_{n=1}^{\infty}$ be the corresponding compact sets given by the hypothesis of the Lemma. We may assume that $F_n \subset F_{n+1}$ for all n. To show that Ω is Stein, it suffices to prove that for every sequence $\{p_n\} \subset \Omega$ without cluster point in Ω there exists a holomorphic function $f\colon \Omega \to \mathbf{C}$ such that $\sup_n |f(p_n| = \infty$.

Taking suitable subsequences, we may assume that $p_n \in K_{n+1} \subset F_{n+1} \subset \Omega_n$ and that $p_n \notin F_n$. Choose inductively a sequence $\{f_n\}$ of holomorphic functions, $f_n\colon \Omega_n \to \mathbf{C}$ with the property that

(i) $|f_{n+1} - f_n| < 1/2^n$ on K_{n+1}
(ii) $f_n(p_k) = k, \quad k = 1, \ldots, n.$

This clearly is possible. If $f = \lim_{n\to\infty} f_n$, then f has the desired properties.

PROOF OF THE THEOREM. Let us fix a compact set $K \subset \Omega$ and show that there exists a compact set $F \subset \Omega$ such that $\hat{K}_n := \hat{K}_{\Omega_n} \subset F$ for all n. By Lemma 16 this will complete the proof.

Let $\varphi = \max_{j=1,2,3,4} \varphi_j$ be the function constructed in Proposition 8. Since the Union Problem has been solved on unbranched Riemann domains, the function φ is plurisubharmonic on Ω.

By Theorem 2 we may assume that Ω is bounded. Using Theorem 4, we find a smooth plurisubharmonic function φ^* on Ω such that $|\varphi - \varphi^*| < 1$.

If $m = \max_K \varphi^*$, it follows that $\hat{K}_n \subset \{q \in \Omega\ ;\ \varphi^*(q) < m+1\} = \Omega^{m+1}$ for all n such that $\Omega_n \supset K$. This is clear because $\{\varphi^* < \alpha\} \cap \Omega_n$ is Runge in Ω_n for all α by Theorem 2.

We fix an $\alpha > m+1$ such that $\partial\Omega^\alpha$ is smooth away from $(0) \times \mathbf{C}$. Next we consider the function $\Delta^*\colon \Omega^\alpha \to \mathbf{R}$ constructed in Lemma 12. By Lemma 13, Δ^* is continuous and plurisubharmonic on Ω^α.

Let m' be the maximum value of Δ^* on K. We fix a $\beta > m'$. Let us denote by Ω_n^α the set $\Omega^\alpha \cap \Omega_n$ and by $\Omega_{n,\beta}$ the set $\{q \in \Omega_n^\alpha\ ;\ \Delta^* < \beta\}$. Then $\Omega_{n,\beta} \supset K$ and is Runge in Ω_n. Therefore $\hat{K}_{\Omega_n} = \hat{K}_{\Omega_{n,\beta}} \subset \Omega_{n,\beta}$. In particular

$$\hat{K}_n \subset \{q \in \Omega^\alpha\ ;\ \Delta^* < \beta\} = \Omega_\beta^\alpha \quad \text{for all large } n\,.$$

The sets $U=\Omega\cap(0\times\mathbf{C})$ and $\Omega^\alpha\cap(0\times\mathbf{C})$ and $\Omega^\alpha_\beta\cap(0\times\mathbf{C})$ are all equal since φ and Δ^* are 0 in an open set containing U.

We obtain from Lemma 12 that if $(p,c)\in\Omega^\alpha_\beta$ and $(0,c)\notin U$, then $\|p\| >e^{-\beta-1}$. Now the sets $\Omega^\alpha_{\beta,n}=\Omega^\alpha_\beta\cap\Omega_n$ are Stein and

$$\Omega^\alpha_{\beta,1}\subset\Omega^\alpha_{\beta,2}\subset\ldots\subset\bigcup_{n=1}^{\infty}\Omega^\alpha_{\beta,n}=\Omega^\alpha_\beta.$$

Let n_0 be some index such that $K\subset\Omega^\alpha_{\beta,n_0}$. If $n\geqq n_0$ and f is a holomorphic function on $\Omega^\alpha_{\beta,n}$, then $\partial^r f/\partial c^r$ is also holomorphic on $\Omega^\alpha_{\beta,n}$. Moreover, choose a positive number $\varepsilon>0$ such that $(p,c)\in K$ and $c'\in\mathbf{C}$, $|c'|<\varepsilon$ implies that $(p,c+c')\in\Omega^\alpha_{\beta,n_0}$. It follows that if $(p,c)\in\hat{K}_{\Omega^\alpha_{\beta,n}}$, $n\geqq n_0$ then $(p,c+c')\in\Omega^\alpha_{\beta,n}$ for all $c'\in\mathbf{C}$, $|c'|<\varepsilon$. In particular, if $\|p\|\leqq e^{-\beta-1}$, then $(0,c)\in U$ and $(0,c+c')\in U$ for all $c'\in\mathbf{C}$, $|c'|<\varepsilon$.

In conclusion, we have shown that if K is a compact subset of Ω, then there exists a compact subset F of Ω such that $\hat{K}_{\Omega_n}\subset F$ whenever $K\subset\Omega_n$. This completes the proof of the Theorem.

7. The Runge problem.

As always, let $X=Z\times\mathbf{C}$ with $Z=\{z\in\mathbf{C}^4\,;\,z_1z_2=z_3z_4\}$. Let $\{\Omega_t\}_{t\in\mathbf{R}}$ be a family of Stein open subsets of X such that Ω_t is a union of connected components of the interior of $\bigcap_{\tau>t}\Omega_\tau$ and such that $\bigcup_{\tau<t}\Omega_\tau$ is a union of connected components of Ω_t for each $t\in\mathbf{R}$.

THEOREM 17. *If $t_1<t_2$ are real numbers, then Ω_{t_1} is Runge in Ω_{t_2}.*

PROOF. We fix a $t\in\mathbf{R}$ and a compact set $K\in\Omega_t$. To arrive at a contradiction, let us assume that for some $\tau>t$ the set $\hat{K}_\tau=\hat{K}_{\Omega_\tau}$ is not contained in Ω_t. From Corollary 3 it follows that $\hat{K}_\tau$ is contained in the union $\bigcup_{\lambda<\tau}\Omega_\lambda$. Hence there exists a number t', $t\leqq t'<\tau$ such that $\hat{K}_\tau\subset\Omega_\lambda$ when $\lambda>t'$ and $\hat{K}_\tau\not\subset\Omega_\lambda$ when $\lambda<t'$. Let us assume that $\hat{K}_\tau\cap\partial\Omega_{t'}=\emptyset$. It would then follow from Corollary 3 that $\hat{K}_\tau\subset\Omega_{t'}$. This implies that $t'>t$ since $\hat{K}_\tau\not\subset\Omega_t$. Therefore $\hat{K}_\tau\subset\bigcup_{\lambda<t'}\Omega_\lambda$, again by Corollary 3. Hence $\hat{K}_\tau\subset\Omega_\lambda$ for some $\lambda<t'$ contradicting the choice of t'.

We may assume therefore that $\hat{K}_\tau\cap\partial\Omega_{t'}\neq\emptyset$, and hence we may also assume that $t=t'$.

Summarizing, we assume that $t<\tau$ and that K is a compact subset of Ω_t such that $\hat{K}_\tau\cap\partial\Omega_t\neq\emptyset$ while $\hat{K}_\tau\in\Omega_\lambda$ for all $\lambda>t$. We denote $\hat{K}_\tau\cap\partial\Omega_t$ by F. Let us first prove that $F\subset(0)\times\mathbf{C}$.

Pick a point $(p,c)\in F$ with $p\neq0$. There exists a $j\in\{1,2,3,4\}$ such that $(p,c)\notin\tilde{S}_j$. Hence the map $\tilde{\pi}_j\colon Z\times\mathbf{C}\to\mathbf{C}^4$ is regular at (p,c). Let d^λ_j be the distance function on $\Omega_\lambda - \tilde{S}_j$ obtained from viewing this set as an unbranched Riemann domain over $\mathbf{C}^4$.

If $\lambda > t$, the functions $\varphi_j^\lambda = \max\{-\log d_j^\lambda + 3\log|f_j|, 0\}$ are continuous and plurisubharmonic on Ω_λ as was established in Proposition 8. Moreover they are uniformly bounded on K. By Corollary 3, $\hat{K}_\lambda = \hat{K}_\tau$ for all $\lambda \in (t,\tau)$. Therefore, by Theorem 2, the functions φ_j^λ are uniformly bounded at (p,c), $\lambda \in (t,\tau)$. Hence (p,c) is an interior point of $\bigcap_{\lambda>t}\Omega_\lambda$. This contradicts that $(p,c) \in \partial\Omega_t$ and that Ω_t consists of connected components of the interior of $\bigcap_{\lambda>t}\Omega_\lambda$. This shows that we must have $F \subset (0)\times\mathbf{C}$.

Let now $U = \Omega_t \cap ((0)\times\mathbf{C})$ and let $\varepsilon > 0$ be a number such that $(p, c+c') \in \Omega_t$ whenever $(p,c) \in K$ and $|c'| < 4\varepsilon$.

We will show that there exists a positive number $\delta > 0$ such that if $(p,c) \in \Omega_t$, $p \neq 0$ and $\|p\| < \delta$ and moreover $(0,c) \notin U$ or has distance from ∂U, U thought of as an open set in $\mathbf{C}$, less than ε, then $(p,c) \notin \hat{K}_\tau$.

Let $\varphi_j^\lambda\colon \Omega_\lambda \to \mathbf{R}$ be the continuous plurisubharmonic function constructed in Proposition 8, $\lambda \in (t,\tau)$. From the inclusions we have the obvious estimate that $\varphi_j^\lambda \leqq \varphi_j^t$ on Ω_t.

Using Theorem 4 we find smooth plurisubharmonic functions $\varphi^\lambda\colon \Omega_\lambda \to \mathbf{R}$ such that

$$\left|\varphi^\lambda - \max_{j=1,\ldots,4}\{\varphi_j^\lambda\}\right| < 1 \qquad \text{on } \Omega_\lambda\,.$$

Here we use again the observation that by Theorem 2 we may assume that the sets $\Omega_\lambda \subset\subset X$ for all λ.

We choose a number m such that $\Omega_t' = \{q \in \Omega_t\,;\ \varphi^t(q) < m\}$ has smooth boundary away from $0\times\mathbf{C}$ and such that if $(p,c) \in K$ and $|c'| < 3\varepsilon$, then $(p,c+c') \in \Omega_t'$. Next we define Ω_λ' for $\lambda \in (t,\tau)$ as $\{q \in \Omega_\lambda\,;\ \varphi^\lambda(q) < m_\lambda\}$ where $m_\lambda \in (m+2, m+3)$ is chosen such that Ω_λ' has smooth boundary away from $\{0\}\times\mathbf{C}$. Then each Ω_λ' is Stein and we have the estimates

(i) if $(p,c) \in K$ and $|c'| < 3\varepsilon$, then $(p,c+c') \in \Omega_\lambda'$
(ii) $\hat{K}_\lambda \subset \Omega_\lambda'$ and
(iii) For any positive number $\eta > 0$ there exists a $\lambda(\eta) > t$ such that if $\lambda \in (t,\lambda(\eta))$ and $(p,c) \in \partial\Omega_t$, $\|p\| > \eta$, then $(p,c) \notin \Omega_\lambda'$.

In fact (i) follows since $\Omega_\lambda' \supset \Omega_t'$, (ii) follows from Corollary 3 and (iii) follows since $(p,c) \in \partial\Omega_t$ cannot be interior points of $\bigcap_{\lambda>t}\Omega_\lambda$.

Now let $\Delta_\lambda^*\colon \Omega_\lambda' \to \mathbf{R}$ be the functions constructed in Lemma 12. From Lemma 13 it follows that Δ_λ^* is continuous and plurisubharmonic. We have the obvious estimate $\Delta_t^* \geqq \Delta_\lambda^*$ on Ω_t' for all $\lambda \in (t,\tau)$.

We choose a $k \in \mathbf{R}$ such that if $(p,c) \in K$ and $|c'|<2\varepsilon$, then $(p,c+c') \in \Omega'_t$ and $\Delta_t^*(p,c)<k$. If we let

$$\Omega''_\lambda = \{q \in \Omega'_\lambda \ ; \ \Delta_\lambda^* < k\}, \qquad \lambda \in [t,\tau)\ ,$$

then if $(p,c) \in K$ and $|c'|<2\varepsilon$, then $(p,c+c') \in \Omega''_\lambda$. Furthermore, by Corollary 3, $\hat{K}_\tau \subset \Omega''_\lambda$ for all $\lambda \in (t,\tau)$.

We let $\delta = \frac{1}{4}e^{-k}$ and choose a point $(p,c) \in \Omega_t$ with $p \neq 0$ and $\|p\| < \delta$. To arrive at a contradiction, let us assume that $(p,c) \in \hat{K}_\tau$ and that $(0,c) \notin U$ or has distance from ∂U less than ε. Since each Ω''_λ is Stein it follows that if $|c'|<2\varepsilon$, then $(p,c+c') \in \Omega''_\lambda$, $\lambda \in (t,\tau)$. Hence we may find a possibly different point $(p,c) \in \Omega_t$ with $p \neq 0$ and $\|p\| < \delta$ such that $(0,c) \notin U$ and $(p,c) \in \Omega''_\lambda$ for all $\lambda \in (t,\tau)$. We consider a point $(w^0,c) \in \mathbf{C}^5$ such that $\tilde{\Gamma}(w^0,c)=(p,c)$ in the notation of section 4. Let Σ_1 be the two dimensional complex plane

$$\Sigma_1 = \{(w,c)\ ;\ w_1 = w_1^0,\ w_4 = w_4^0\}$$

and similarly let

$$\Sigma_2 = \{(w,c)\ ;\ w_2 = w_2^0,\ w_3 = w_3^0\}\ .$$

It is possible to choose w^0 so that $\max|w_i^0| < \sqrt{\delta}$, as is seen from the definition of $\tilde{\Gamma}$. Let B_1 be the open ball in Σ_1 centered at (w^0,c) with radius $\sqrt{2}\sqrt{\delta}$. Then the point $q=(w_1^0,0,0,w_4^0,c) \in B_1$. We let $\tilde{\Omega}_t$ be the pull back to $\mathbf{C}^5$ of Ω_t. Since $(0,c) \notin \Omega_t$, it follows that $q \notin \tilde{\Omega}_t$. Since $\tilde{\Omega}_t$ is a domain of holomorphy, it follows that $\tilde{\Omega} \cap \Sigma_1$ is also a domain of holomorphy in $\Sigma_1 \cong \mathbf{C}^2$. This implies that there must exist a point $q' \neq q$ in B_1 such that $q' \in \partial\tilde{\Omega}_t$ and hence $p' = \tilde{\Gamma}(q') \subset \partial\Omega_t$. Therefore, (iii) gives that $p' \notin \Omega'_\lambda$ for all sufficiently small $\lambda > t$. In particular we get that $q' \notin \tilde{\Gamma}^{-1}(\Omega'_\lambda)$ for all such λ. Let $\Delta_1^\lambda, \Delta_2^\lambda$ be the distance functions on $\tilde{\Gamma}^{-1}(\Omega'_\lambda)$ used in Lemma 10. We have now the estimate $\Delta_2^\lambda(w^0,c) < \sqrt{2}\sqrt{\delta}$, and by the same argument applied to Σ_2, $\Delta_1^\lambda(w^0,c) < \sqrt{2}\sqrt{\delta}$ also. Since

$$\Delta_\lambda^*(p,c) = \max\{-\log \Delta_1^\lambda \Delta_2^\lambda(w^0,c), 0\}$$

we get

$$\Delta_\lambda^*(p,c) > -\log 2 - \log\delta = \log 2 + k > k\ .$$

This contradicts that $(p,c) \in \Omega''_\lambda$ because $\Omega''_\lambda = \{q \in \Omega'_\lambda\ ;\ \Delta_\lambda^* < k\}$.

Let us fix a number $\delta > 0$ such that if $(p,c) \in \Omega_t$, $p \neq 0$ and $\|p\| < \delta$ and moreover $(0,c) \notin U$ or has distance from ∂U less than ε, then $(p,c) \notin \hat{K}_\tau$.

Denote by F_0 the set $\hat{K}_\tau \cap \partial U$. We know now that if F_0 is empty, then $\hat{K}_\tau \subset U$ by Corollary 3. So we assume that there exists at least one point $(0,c') \in \hat{K}_\tau \cap \partial U$. Let H be those points in $U \cap \hat{K}_\tau$ with distance from ∂U in $[\varepsilon/3, \varepsilon/2]$. Then H is a compact set, and we consider this as a subset of $\mathbf{C} \cong \{0\} \times \mathbf{C}$. Hence c' is in a connected component V of $\mathbf{C} - H$.

Let us first show that $V \not\subset \Omega_\lambda$ if $\lambda > t$ is small enough. The set $\partial V \subset H \subset \Omega_t$, and so there exists a $v > 0$ such that if $(q, c) \in Z$, $\|q\| \leqq v$ and $c \in \partial V$, then $(q, c) \in \Omega_t$. If $V \in \Omega_\lambda$, it therefore follows that

$$\{(q, c)\ ;\ \|q\| \leqq v \text{ and } c \in \bar{V}\} \subset \Omega_\lambda .$$

If this holds for all $\lambda > t$, $(0, c')$ must be an interior point of $\bigcap_{\lambda > t} \Omega_\lambda$ which contradicts that $(p, 0) \in \partial \Omega_t$. Hence there exists a point $(0, c'') \notin \Omega_\lambda$ whenever $\lambda > t$ is small enough, $c'' \in V$.

By the well known Runge theorem in one complex variable there exists a rational function $P(c)\colon \mathbf{C} \to \mathbf{C}$ with poles at c'' only such that $|P(c')| > 1 > \max_H |P|$. Since $\Omega_\lambda \cap (0 \times \mathbf{C})$ is a closed subvariety of a Stein space, we may find a holomorphic extension $\tilde{P}\colon \Omega_\lambda \to \mathbf{C}$. Since F_0 is compact, we may find a $\lambda_0 > t$ and a finite collection of holomorphic function $\tilde{P}_1, \ldots, \tilde{P}_l\colon \Omega_{\lambda_0} \to \mathbf{C}$ such that

$$\max_{j=1,\ldots,l} |\tilde{P}_j|(0, c) > 1 \qquad \text{for all } c \in F_0$$

and each $|\tilde{P}_j| < 1$ on H.

Let W be a Stein open set in Ω_{λ_0} containing $\hat{K}_\tau$ which is Runge in Ω_{λ_0}. We can by Corollary 3 assume that W is contained in any given neighborhood of $\hat{K}_\tau$. In what follows we will assume W is sufficiently small.

Let $\chi\colon \mathbf{R} \to \mathbf{R}$ be a continuous convex function such that $\chi(x) > 0$ when $x > 0$, $\chi(x) \equiv 0$ when $x \leqq 0$. We define a continuous plurisubharmonic function, ϱ, on W by

(i) $\varrho \equiv 0$ for those points in W which lie near points in $\hat{K}_\tau \cap \Omega_t$ except near those in $(0) \times U$ whose distance to ∂U is less than $\varepsilon/2$.

(ii) $\varrho = \chi \circ \max_j |\tilde{P}_j|$ otherwise.

Then by Theorem 2 the set F_0 must be empty since W is Runge in Ω_{λ_0} and $\varrho \,|\, F_0 > 0$ while $\varrho \,|\, K \equiv 0$.

8. Some remarks.

We will list a few other problems than the ones mentioned in the introduction, but which are suggested by the preceding proofs.

PROBLEM 1. *Assume $\varphi\colon X \to \mathbf{R}$ is a continuous function on a complex space X such that $\varphi \circ \psi\colon \Delta \to \mathbf{R}$ is subharmonic whenever $\psi\colon \Delta \to X$ is a holomorphic map of the open unit disc into X. Is φ necessarily plurisubharmonic?*

This problem was posed in Narasimhan [10]. Clearly there are other similar problems with other regularity conditions on the functions.

PROBLEM 2. *If* $\{\varphi_n\}_{n=1}^{\infty}: X \to \mathbf{R}$ *is a sequence of continuous plurisubharmonic functions on* X *converging uniformly to* $\varphi: X \to \mathbf{R}$ *on compact subsets of the complex space* X. *Is* φ *plurisubharmonic?*

This problem was posed in Richberg [14]. Again similar problems arise with other regularity conditions on the functions. Theorem 4 of Richberg suggests the following type of problem:

PROBLEM 3. *If* $\varphi: X \to \mathbf{R}$ *is a plurisubharmonic function on a complex space* X, *does there exist a sequence of smooth plurisubharmonic functions* $\{\varphi_n\}: X \to \mathbf{R}$ *such that* $\varphi_n \searrow \varphi$ *when* $n \to \infty$.

This is of course true if X is a Stein manifold.

PROBLEM 4. *Assume* $\varphi: X \to \mathbf{R}$ *is a strongly plurisubharmonic function on a complex space* X *such that* $\{\varphi < \alpha\} \subset\subset X$ *for all* $\alpha \in \mathbf{R}$. *Is* X *Stein?*

If we assume in addition that φ is continuous, this is Theorem 2 by Narasimhan [10]. Problem 4 is still open if X is a complex manifold. If $\varphi: X \to \mathbf{R}$ is a plurisubharmonic exhaustion function and there exists a continuous strongly plurisubharmonic function $\psi: X \to \mathbf{R}$, then X is Stein if it is a complex manifold, Richberg [14], Suzuki [16] and Elenzwajg [5].

PROBLEM 5. *Assume* $\varphi: X \to \mathbf{R}$ *is a plurisubharmonic function on a Stein space* X. *Is* $X_\alpha = \{\varphi < \alpha\}$ *Runge in* X *and/or Stein for any* $\alpha \in \mathbf{R}$?

This is true if X is a Stein manifold. Also if φ is continuous it reduces via Richberg's theorem to Theorem 2 by Narasimhan [10].

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