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A CARLEMAN THEOREM FOR CURVES IN $\mathbf{C}^n$

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1.

Let Γ be the image in $\mathbf{C}^n$ of the real axis under a proper continuous imbedding (a curve without self-intersections, "going to infinity in both directions"). Can every continuous function on Γ be asymptotically approximated by entire functions? That is, given a continuous complex valued function f on Γ and a positive continuous function ε on Γ, does there exist an entire function F such that $|f(x)-F(x)|<\varepsilon(x)$ for each x in Γ?

For $n=1$, Carleman [3] posed and answered this question affirmatively when Γ is the real axis and, more generally, his proof applied if Γ is merely assumed to be locally rectifiable. Subsequently, Keldych and Lavrentieff [4] characterized the plane continua E on which every continuous function can be asymptotically approximated. They showed, in particular, that approximation is always possible on the plane curves Γ defined above.

For $n>1$, a new phenomenon intervenes. Clearly, if asymptotic approximation is to hold on Γ, then for each compact subarc γ of Γ, every continuous function on γ must admit uniform approximation by polynomials. This necessary condition will not always hold, for Wermer [8] has given an example of an arc γ in C^3 (and Rudin [6] in $\mathbf{C}^2$) on which polynomial approximation fails. We conjecture that this necessary condition is also sufficient.

The Wermer–Rudin arcs are highly non-smooth. In fact, work of Wermer [9], Bishop [2] and Stolzenberg [7] shows that polynomial approximation is always possible on a smooth arc. This has led E. L. Stout to restrict the question to the setting of smooth Γ. He and, independently, B. Aupetit have obtained special cases of our principal result.

THEOREM. *For a smooth properly imbedded image Γ, in $\mathbf{C}^n$, of the real axis, asymptotic approximation is always possible; i.e., given a continuous function f on Γ and a positive continuous function ε on Γ there exists an entire function F on $\mathbf{C}^n$ such that $|f(x)-F(x)|<\varepsilon(x)$ for each x in Γ.*

The proof of the Theorem depends heavily on the Wermer–Bishop–

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Stolzenberg theory of analytic structure in certain polynomially convex hulls. The formulation of Stolzenberg [7], where the local aspect of the theory is exposed, is most suitable for our purposes. For the reader's convenience, we shall recall the statement of Stolzenberg's theorem below. In order to apply the results of [7] we shall take "smooth" in the theorem to mean "piecewise $\mathscr{C}^1$". However we have not attempted to state the most general result. For example, because the methods of [1] allow one to replace "$\mathscr{C}^1$" in Stolzenberg's theorem by "rectifiable", we could replace "smooth" in the present Theorem by "locally rectifiable". Similarly it is clear from the proof that Γ could be allowed to have certain self-intersections or that Γ could be taken to be a countably infinite locally finite family of disjoint smooth unbounded Jordan arcs.

Of course, one can also try to approximate functions on real submanifolds of $\mathbf{C}^n$ of higher dimension. Nunemacher [5] has shown that continuous functions on a $\mathscr{C}^1$ totally real submanifold M of $\mathbf{C}^n$ can be asymptotically approximated by functions holomorphic in a neighborhood of M. It appears to be difficult to get approximation by global (i.e., entire) functions in this setting. However Stout has obtained some results in this direction.

2.

To fix some notation, let X be a compact subset of $\mathbf{C}^n$. Then $C(X)$ will denote the Banach algebra of all continuous complex-valued functions on X with the supremum norm; its subalgebras $P(X)$ and $R(X)$ are the closures of the polynomials and the rational functions holomorphic on X, respectively. The polynomially convex hull $\hat{X}$ of X is

$$\{z \in \mathbf{C}^n : |h(z)| \leq |h|_X \text{ for every polynomial } h\} .$$

The complement of a set B in a set A will be denoted by $A \setminus B$. The ball $B(r)$ is $\{z \in \mathbf{C}^n : \|z\| < r\}$ where we are using the standard Euclidean norm. Without further qualification, an "arc" will be a homeomorphic image of the closed unit interval. We can now state Stolzenberg's theorem [7].

THEOREM. *Let X be polynomially convex and let K be a finite union of compact smooth ($\mathscr{C}^1$) curves in $\mathbf{C}^n$.*

A. *$(K \cup X)\hat{} \setminus (K \cup X)$ is a (possibly-empty) one-dimensional analytic subset of $\mathbf{C}^n \setminus K \cup X$.*

B. *If $g \in C(K \cup X)$ and the restriction $g|X \in P(X)$, then $g \in R(K \cup X)$.*

C. *If the map $\check{H}^1(K \cup X, \mathbf{Z}) \to \check{H}^1(X, \mathbf{Z})$ induced by $X \subseteq K \cup X$ is injective, then $K \cup X$ is polynomially convex.*

The hypothesis of part C involving Čech cohomology groups is equivalent to the following: If a continuous function on $K \cup X$ is nowhere zero and has a continuous logarithm on X, then it has a continuous logarithm on $K \cup X$. We shall summarize the conclusion of part A by saying, for a point $p \in (K \cup X)\hat{} \setminus (K \cup X)$, that "$(K \cup X)\hat{}$ is analytic at p".

Without loss of generality, we may assume that Γ contains the origin. Define $\gamma(r)$ to be the subarc of $\Gamma \cap B(r)$ which contains 0; then $\partial\gamma(r)$ consists of two points on $\partial B(r)$. Let $\sigma(r)$ be the set

$$\Gamma \setminus [\text{the two unbounded components of } \Gamma \setminus B(r)];$$

$\gamma(r)$ and $\sigma(r)$ are bounded open arcs in $\mathbf{C}^n$.

Define a sequence $\{r_k\}_0^\infty$ inductively as follows: Put $r_0 = 1$. Given r_{k-1} for $k > 0$, choose $r_k > r_{k-1} + 1$ such that

$$\sigma(r_{k-1}) \subseteqq B(r_k), \quad \text{and} \tag{1}$$

$$(\bar{B}(r_{k-1}) \cup \bar{\sigma}(r_{k-1}))\hat{} \cap (\bar{\sigma}(r_k) \setminus \gamma(r_k)) = \emptyset . \tag{2}$$

Note that (1) is possible because $\sigma(r)$ is bounded and (2) because $\bar{\sigma}(r) \setminus \gamma(r) \to \infty$ since $\gamma(r) \to \Gamma$ as $r \to \infty$. Now write γ_k for $\gamma(r_k)$, σ_k for $\sigma(r_k)$ and B_k for $B(r_k)$.

Next define two sets for each $k \geqq 2$:

$$X_k = (\bar{B}_{k-2} \cup \bar{\gamma}_{k-1})\hat{} \quad \text{and} \quad Y_k = X_k \cup \bar{\gamma}_k .$$

The crux of our proof lies in the description of X_k which is given in Lemma 1a.

LEMMA 1. (a) *$X_k = (\bar{B}_{k-2} \cup \bar{\sigma}_{k-2})\hat{} \cup \tau_k$ where τ_k is $\bar{\gamma}_{k-1} \setminus \sigma_{k-2}$.*

(b) *$Y_k = X_k \cup \alpha_k \cup \beta_k$ where α_k and β_k are smooth disjoint arcs each intersecting X_k in a single point.*

PROOF. (a) Let T_k be the set on the right hand side of the equality asserted in (a). Clearly, we have $T_k \subseteqq X_k \subseteqq \hat{T}_k$ (since $\hat{X}_k = X_k$; the second inclusion is in fact equality). Thus it suffices to show that T_k is polynomially convex. Arguing by contradiction, we suppose otherwise. By Stolzenberg's theorem A, $\hat{T}_k \setminus T_k$ is a 1-dimensional analytic subvariety of $\mathbf{C}^n \setminus T_k$. Let V be a non-empty irreducible analytic component of $\hat{T}_k \setminus T_k$.

We claim that $\bar{V} \setminus (\bar{B}_{k-2} \cup \bar{\gamma}_{k-1})\hat{}$ is an analytic subvariety of $\mathbf{C}^n \setminus (\bar{B}_{k-2} \cup \bar{\gamma}_{k-1})$. From the definition of T_k, it suffices to verify this locally at a point $x \in \bar{V} \cap Q$ where

$$Q = (\bar{B}_{k-2} \cup \bar{\sigma}_{k-2})\hat{} \setminus (\bar{B}_{k-2} \cup \bar{\sigma}_{k-2}) .$$

By Stolzenberg's theorem A, both X_k and Q are analytic near x, where "near x" refers to germs of sets at x, here and below. Furthermore, near x, $\bar{V} \subseteqq X_k$,

$V \subseteq X_k \setminus Q$ and $Q \subseteq X_k$. It follows that near x, $\bar{V}$ is just a union of some of the local analytic components of X_k at x; in fact, near x, $\bar{V} = V \cup \{x\}$.

Put

$$W = \bar{V} \setminus (\bar{B}_{k-2} \cup \bar{\gamma}_{k-1}) .$$

Then W is an irreducible subvariety of $\mathbf{C}^n \setminus (\bar{B}_{k-2} \cup \bar{\gamma}_{k-1})$ and moreover,

$$\bar{W} \setminus W \subseteq \bar{B}_{k-2} \cup \bar{\sigma}_{k-2} \cup \tau_k .$$

Thus $\bar{W} \subseteq (\bar{B}_{k-2} \cup \bar{\sigma}_{k-2} \cup \tau_k)\hat{}$ by the maximum principle. Fix a point $p \in V \subseteq W$. Since $p \notin T_k$, we have $p \notin (\bar{B}_{k-2} \cup \bar{\sigma}_{k-2})\hat{}$ and therefore there exists a polynomial h such that $h(p) = 0$ and $\operatorname{Re} h < 0$ on $(\bar{B}_{k-2} \cup \bar{\sigma}_{k-2})\hat{}$. By the open mapping theorem, either $h(W)$ is an open neighborhood of 0 or $h \equiv 0$ on W. In the latter case, $h \equiv 0$ on $\bar{W}$ and so $\bar{W} \setminus W$ is disjoint from $\bar{B}_{k-2} \cup \bar{\sigma}_{k-2}$. This implies that $W \subseteq \hat{\tau}_k$, a contradiction, because, being the disjoint union of two smooth arcs, τ_k is polynomially convex. Hence, the former case holds. As $h(\tau_k)$ is nowhere dense in the plane, there is a small complex number $\alpha \in h(W)$ such that $h \neq \alpha$ on τ_k. Now put $g = h - \alpha$. If α is sufficiently small, we conclude that (i) $\operatorname{Re} g < 0$ on $(\bar{B}_{k-2} \cup \bar{\sigma}_{k-2})\hat{}$, (ii) $g(q) = 0$ for some $q \in W$ and (iii) $g \neq 0$ on τ_k.

Now (i) implies that the polynomial g has a continuous logarithm on $(\bar{B}_{k-2} \cup \bar{\sigma}_{k-2})\hat{}$ and so, by restriction, on $\bar{B}_{k-2} \cup \bar{\sigma}_{k-2}$. Continuing (by iii) this logarithm from $\bar{B}_{k-2} \cup \bar{\sigma}_{k-2}$ along the two arcs which comprise τ_k, we conclude that g has a continuous logarithm on $\bar{B}_{k-2} \cup \bar{\sigma}_{k-2} \cup \bar{\tau}_k$. But this last set contains $\bar{W} \setminus W$. Applying the argument principle to g on W gives a contradiction to (ii). (Cf. the proof of part C of [7])

(b) We can write $\bar{\gamma}_k \setminus \gamma_{k-1}$ as a disjoint union of two arcs α_k and β_k, each joining ∂B_{k-1} to ∂B_k. We claim that $X_k \cap \alpha_k$ is the one point set $\partial\alpha_k \cap \partial B_{k-1}$ and likewise for β_k. In fact,

$$\alpha_k \cap \mathbf{B}_{k-1} \subseteq (\bar{\gamma}_k \setminus \gamma_{k-1}) \cap B_{k-1} \subseteq \bar{\sigma}_{k-1} \setminus \gamma_{k-1}$$

and by part (a), we have

$$X_k = (\bar{B}_{k-2} \cup \bar{\sigma}_{k-2})\hat{} \cup \bar{\gamma}_{k-1} .$$

Intersecting and using (2) we conclude that $X_k \cap \alpha_k \cap B_{k-1}$ is empty. Hence $X_k \cap \alpha_k \subseteq \partial B_{k-1}$ and so

$$X_k \cap \alpha_k = \partial\gamma_{k-1} \cap \alpha_k = \partial\alpha_k \cap \partial B_{k-1} ,$$

as asserted.

LEMMA 2. *Let X be a compact polynomially convex subset of $\mathbf{C}^n$ and let α and β be disjoint smooth arcs in $\mathbf{C}^n$ such that $\alpha \cap X$ and $\beta \cap X$ each contain a single point.*

(a) *Then* $X \cup \alpha \cup \beta$ *is polynomially convex.*

(b) *If* $g \in C(X \cup \alpha \cup \beta)$ *and* $g \equiv 0$ *on* X, *then* $g \in P(X \cup \alpha \cup \beta)$. *That is, given* $\varepsilon > 0$, *there exists a polynomial* p *such that* $|g-p| < \varepsilon$ *on* $X \cup \alpha \cup \beta$; *moreover we can choose* p *such that* $g = p$ *at the two point set* $(\partial\alpha \cup \partial\beta) \setminus X$.

PROOF. (a) By the geometric hypotheses on X, α and β, every continuous function on $X \cup \alpha \cup \beta$ which is nowhere zero and which has a continuous logarithm on X also has a continuous logarithm on $X \cup \alpha \cup \beta$. This is exactly the condition of Theorem C of Stolzenberg which asserts that $X \cup \alpha \cup \beta$ is polynomially convex.

(b) By Theorem B of Stolzenberg, $g \in R(X \cup \alpha \cup \beta)$. By part (a) and the Oka–Weil theorem, $P(X \cup \alpha \cup \beta) = R(X \cup \alpha \cup \beta)$. Thus $g \in P(X \cup \alpha \cup \beta)$. Now an approximating polynomial for g can easily be adjusted so as to equal g at the two required points.

We shall apply Lemma 2b to do polynomial approximation on the sets $Y_k = X_k \cup \alpha_k \cup \beta_k$ which, by Lemma 1b, satisfy the hypotheses of Lemma 2. The rest of the proof of the Theorem is standard and, in fact, essentially, follows the lines of Carleman's original argument.

Choose a sequence of positive real numbers $\{\varepsilon_k\}_1^\infty$ such that

$$\sum_{k=n}^{\infty} \varepsilon_k \leqq \min\{\varepsilon(z) : z \in \Gamma \cap \bar{B}_n\} \equiv \eta_n \tag{3}$$

for each $n \geqq 1$; $\varepsilon_n = \min\{\eta_k 2^{-(n+1)} : 1 \leqq k \leqq n\}$ will do. Since $C(\bar{\gamma}_1) = P(\bar{\gamma}_1)$, there is a polynomial p_1 such that $|f - p_1|_{\gamma_1} < \varepsilon_1$ and

$$f - p_1 = 0 \quad \text{on } \partial\gamma_1 . \tag{4}$$

Now define g_2 on Y_2 by

$$g_2 = \begin{cases} 0 & X_2 \\ f - p_1 & \text{on } \bar{\gamma}_2 \setminus \gamma_1 = \alpha_2 \cup \beta_2 . \end{cases}$$

By (4), g_2 is continuous on Y_2. Applying Lemma 2 to g_2 on Y_2, we get a polynomial p_2 such that $|g_2 - p_2|_{Y_2} < \varepsilon_2$ and $g_2 - p_2 = 0$ on $(\partial\alpha_2 \cup \partial\beta_2) \cap \partial B_2$. Proceeding inductively, given polynomials $p_1, p_2, \ldots, p_{n-1}$ and functions $g_2, g_3, \ldots, g_{n-1}$ with $g_k \in C(Y_k)$ satisfying

$$(5)_k \quad \begin{aligned} &(a)\ g_k = 0 && \text{on } X_k \\ &(b)\ g_k = f - \sum_{j=1}^{k-1} p_j && \text{on } \alpha_k \cup \beta_k \\ &(c)\ |g_k - p_k| < \varepsilon_k && \text{on } Y_k \\ &(d)\ g_k - p_k = 0 && \text{on } (\partial\alpha_k \cup \partial\beta_k) \cap \partial B_k \end{aligned}$$

for $2 \leqq k \leqq n-1$, we define p_n and g_n satisfying $(5)_n$ as follows. On Y_n define

$$g_n = \begin{cases} 0 & \text{on } X_n \\ f - \sum_{j=1}^{n-1} p_j & \text{on } \alpha_n \cup \beta_n . \end{cases}$$

Then $(5a)_n$ and $(5b)_n$ hold and g_n is continuous because of $(5b)_{n-1}$ and $(5d)_{n-1}$. Now Lemma 2 yields a polynomial p_n satisfying $(5c)_n$ and $(5d)_n$.

We claim that $\sum_1^\infty p_n$ converges uniformly on compact subsets of $\mathbf{C}^n$ to an entire function F satisfying $|f(x)-F(x)| < \varepsilon(x)$ for each x in Γ.

To see the convergence we note that $(5)_n$ implies $|p_n| < \varepsilon_n$ on X_n which contains B_{n-2}. Thus, if K is compact in $\mathbf{C}^n$, since $K \subseteqq B_{n-2}$ for n sufficiently large, we have $|p_n| < \varepsilon_n$ on K for such n. As $\sum_1^\infty \varepsilon_n < \infty$, the uniform convergence on K follows.

Finally, let $x \in \Gamma$. Choose n to be minimal such that $x \in \bar{\gamma}_n$. Then $x \in \bar{\gamma}_n \setminus \gamma_{n-1}$ implies

$$g_n(x) = f(x) - \sum_1^{n-1} p_k(x) .$$

We have

$$|f(x)-F(x)| \leqq \left| \left(f(x) - \sum_1^{n-1} p_k(x) \right) - p_n(x) \right| + \sum_{n+1}^{\infty} |p_k(x)|$$

$$= |g_n(x) - p_n(x)| + \sum_{n+1}^{\infty} |p_k(x)| .$$

As $x \in \bar{\gamma}_n$, it follows that $x \in X_k$ for $k \geqq n+1$ and so $|p_k(x)| < \varepsilon_k$ for $k \geqq n+1$. Thus

$$|f(x)-F(x)| < \varepsilon_n + \sum_{n+1}^{\infty} \varepsilon_k \leqq \eta_n \leqq \varepsilon(x) .$$

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