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IMBEDDING THEOREMS OF SOBOLEV TYPE IN POTENTIAL THEORY

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0. Introduction.

In this paper we are going to study imbeddings of spaces of potentials of functions and measures into Banach spaces of functions which are partially ordered in the sense that $f, g \in \boldsymbol{B}$ and $|f| \leqq |g|$ implies that $\|f\| \leqq \|g\|$ for a norm $\|\cdot\|$ of $\boldsymbol{B}$. We are considering two cases. In the first case we study potentials of the form $u(x)=\int G(x,y)\,d\mu(y)$, where G is a positive definite kernel on some space Ω and μ is a signed measure such that the energy integral $\iint G\,d\mu \times d\mu$ is finite. Secondly we treat potentials of the form

$$u(x) = \int K(x,y) f(y)\,d\omega(y)\,,$$

where K is a positive kernel on some measure space (Ω, ω) and $f \in L^p(\omega)$ for $1<p<\infty$. In both cases we are able to identify the smallest partially ordered Banach space which contains the potentials of the types above provided the potentials of the kernels G and K satisfy a weak form of a maximum principle. The spaces in the imbedding theorems are characterized in terms of the capacities defined by the kernels, and our main results are that the spaces consist of those quasicontinuous functions f which satisfy the following boundedness conditions:

0.1 $$\int_0^\infty C(\{x : |f(x)|>t\})t\,dt < \infty$$

respectively

0.2 $$\int_0^\infty C_p(\{x : |f(x)|>t\})t^{p-1}\,dt < \infty\,.$$

Here C denotes the classical capacity defined by the kernel G and C_p is the L^p-capacity [13] defined by K.

As the main tool in proving these imbedding results we use strong type capacitary estimates of the type proved in the case if Riesz and Bessel

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potentials by Adams [1], [2], Mazýa [12] and Dahlberg [6]. Our proofs of these estimates are however different and can be applied in more general situations.

In the last section we show a relation between our imbedding results and the well known imbedding theorems of Sobolev and others. Other applications are given in [8].

This paper is an outgrowth of the first part of the authors thesis [8], and he is indebted to professor K. O. Widman for his support and valuable criticism.

1. Hilbert space theory.

Let Ω denote a locally compact metric space such that every open set in Ω is a countable union of compact sets, and let G be a symmetric and positive definite kernel which satisfies a boundedness condition. By this we mean that G is a positive and lower semicontinuous function on $\Omega \times \Omega$ taking values in $[0, \infty]$ and satisfying the following four conditions.

1.1 $G(x,y) = G(y,x)$ for all x,y in Ω.

1.2 $I(\mu) = \iint G\,d\mu \times d\mu > 0$ for every signed measure $\mu \neq 0$ on Ω.

1.3 G is continuous in $\Omega \times \Omega \setminus \{(x,y) : x = y\}$, and in case of a non compact Ω it is also required that the function $G(.,y)$ tend to zero uniformly on compact sets as $y \to \infty$.

1.4 There is a constant A such that the following boundedness principle holds for the potentials $U\mu(x) = \int G(x,y)\,d\mu(y)$ of positive measures μ with compact support $S\mu$.

$$\sup_{\Omega} U\mu \leqq A \sup_{S\mu} U\mu .$$

Let μ be a signed measure on Ω and let $U\mu$ be the potential of μ with respect to a symmetric and positive definite kernel G. We denote by $\boldsymbol{H}$ the space of potentials $U\mu$ such that $I(|\mu|) < \infty$ where $|\mu|$ is the total variation of μ. In view of 1.2 there is a natural inner product on $\boldsymbol{H}$ defined as the mutual energy of the potentials

$$(U\mu, U\nu) = \iint G\,d\mu \times d\nu .$$

In the following we are going to study Banach spaces $\boldsymbol{B}$ of realvalued functions (or function classes) on Ω into which $\boldsymbol{H}$ can be continuously imbedded. By this we mean that $\boldsymbol{H} \subset \boldsymbol{B}$ holds in the sense of set inclusion, and if $\|\cdot\|$ is a norm for $\boldsymbol{B}$ we should have

$$\sup_{f \in \boldsymbol{H}} \|f\|^2/(f,f) < \infty .$$

We will restrict ourselves to imbeddings of $\boldsymbol{H}$ into Banach spaces which are partially ordered, and we make the following definition.

DEFINITION 1.1. A normed space $\boldsymbol{B}$ of (equivalence classes of) realvalued functions on Ω is partially ordered if there is a norm $\|\cdot\|$ for $\boldsymbol{B}$ such that $f, g \in \boldsymbol{B}$ and $|f| \leqq |g|$ implies that $\|f\| \leqq \|g\|$.

The main purpose of this chapter is to characterize the smallest partially ordered Banach space into which H can be imbedded. This minimal space can be described in terms of the capacity which is defined by the kernel G. We first state a few notational conventions and, without proof, a well known theorem of potential theory.

The class of continuous functions f on Ω will be denoted by $\boldsymbol{C}$, and in the case of a non compact Ω we also require that $f(x) \to 0$ as $x \to \infty$. In the non compact case we also use the notation $\boldsymbol{C}_0$ for the functions in $\boldsymbol{C}$ with compact support.

In the proofs of the theorems that follow we suppose that Ω is non compact. In the compact case most proofs are simpler and the modifications are left to the reader. Typical examples of kernels G on spaces Ω that satisfies the conditions 1.1 to 1.4 above are the Riesz and Bessel kernels on $\mathbf{R}^d$, and the Green function of a second order differential operator on a regular domain Ω of $\mathbf{R}^d$.

THEOREM 1.2 (*Continuity principle*) *If G is a kernel with the properties* 1.1 *to* 1.4 *and if μ is a positive measure with compact support $S\mu$ such that the restriction of $U\mu$ to $S\mu$ is continuous then it follows that $U\mu \in C$.*

For a proof of theorem 1.2 see [4, p. 34].

For any subset E of Ω we define the capacity of E as follows.

$$C(E) = \sup_{\mu \in \Gamma(E)} \mu(E)$$

where $\Gamma(E)$ denotes the set of positive measures with compact support $S\mu$ such that $S\mu \subset E$ and $U\mu \leqq 1$ for all $x \in S\mu$.

We also define the exterior capacity of E as

$$C^*(E) = \inf C(V)$$

where the infimum is taken with respect to open sets V containing E.

THEOREM 1.3. 1. *$C^*(\bigcup E_n) \leqq \sum C^*(E_n)$ where E_n, $n=1,2,\ldots$ are arbitrary subsets of Ω.*

2. *If* $E_n \subset E_{n+1}$ *for* $n=1,2,\ldots$ *we have if* $E=\bigcup E_n$ *that* $\lim_{n\to\infty} C^*(E_n) = C^*(E)$.

3. *For any compact set E we have* $C(E)<\infty$ *and for any Borel set E we have* $C(E)=C^*(E)$.

4. *For any compact set E there is a unique positive measure* μ *with* $S\mu \subset E$ *such that* $\mu(E)=C(E)$, $U\mu(x)\leqq 1$ *on* $S\mu$ *and* $U\mu(x)\geqq 1$ *for all x in E outside a set of capacity zero.*

5. *For any open set V with* $C(V)<\infty$ *there is a positive measure* μ *concentrated on* $\bar{V}$ *such that* $\mu(\bar{V})=C(V)$, $U\mu\leqq 1$ *on* $S\mu$ *and* $U\mu(x)\geqq 1$ *for all* $x \in V$ *outside a set of capacity zero.*

For a proof of theorem 1.3 the reader is referred to [7, ch. I. 2]. From theorem 1.3.5 it follows that the exterior capacity of a set E can be written as $C^*(E)=\inf_\mu I(\mu)$ where μ is a measure such that the potential $U\mu$ is $\geqq 1$ C-a.e. on E. In analogy with this we define a seminorm on the real valued functions on Ω which generalizes to functions the notion of capacity of a set. Cf. [2, def. 3.1].

DEFINITION 1.4. $p(f)=\inf_\mu \sqrt{I(\mu)}=\inf_\mu \|U\mu\|$. where the infimum is taken over positive measures μ on Ω such that $|f|\leqq U\mu$ C-a.e. If $I(\mu)=\infty$ for every such measure we define $p(f)=\infty$.

Since $\|u\|=\sqrt{(u,u)}$ is a norm on $\boldsymbol{H}$ it is clear that $p(f)$ is a seminorm with the property that $|f|\leqq|g|$ implies that $p(f)\leqq p(g)$. In a few special cases $p(f)$ is easy to compute.

PROPOSITION 1.5. *For any set E the relation* $p(\chi_E)^2=C^*(E)$ *holds, and if* $\mu\geqq 0$ *and* $I(\mu)<\infty$ *it follows that* $p(U\mu)^2=\|U\mu\|^2=I(\mu)$.

PROOF. The first statement is clear, and the second will follow if we prove the estimate $p(U\mu)^2\geqq I(\mu)$. Suppose thst $U\mu\leqq U\sigma$ C-a.e. for some positive measure σ. It follows that

$$I(\mu) \leqq \int U\sigma\, d\mu \leqq \sqrt{I(\mu)}\sqrt{I(\sigma)} \Rightarrow I(\mu) \leqq I(\sigma)$$

and from the definition of the seminorm p it follows that $I(\mu)\leqq p(U\mu)^2$.

For an arbitrary function f we use the following strong capacitary estimate for $p(f)$, which in the Hilbert space situation generalizes a theorem of Adams [1, Thm. 1].

THEOREM 1.6. *For any real valued function f the following estimate for $p(f)$ holds*

$$p(f)^2 \leqq 2A \int_0^\infty C^*(\{x : |f(x)|>t\})t\,dt \leqq 4A^2 p(f)^2$$

where A is the constant in property 1.4 *of the kernel G.*

PROOF. We first prove the left inequality. Let the function f be given such that $\int_0^\infty C^*(|f|>t)t\,dt<\infty$, and let α be a real number >1. Choose for every integer n an open set V_n which contain the set $\{|f|>\alpha^n\}$ and is such that

$$C(V_n) < \varepsilon_n + C^*(|f|>\alpha^n)$$

where $\sum \alpha^{2n}\varepsilon_n \leqq 1$. By theorem 1.3.5 there are measures μ_n on $\bar{V}_n$ such that $\mu_n(\bar{V}_n)=C(V_n)$ and $U\mu_n \geqq 1$ C-a.e. on V_n. Define the measure μ_α as

1.5 $$\mu_\alpha = (\alpha-1)\sum_n \alpha^n \mu_n .$$

The convergence of 1.5 follows from the estimates $\mu_n(E) \leqq AC(E)$ for every n and every Borel set E, and

$$\alpha^{2n}C^*(|f|>\alpha^n) \leqq 2\int_0^\infty C^*(|f|>t)t\,dt .$$

Using this we find that

$$\begin{aligned}\mu_\alpha(E) &\leqq A(\alpha-1)\sum_{n\leqq 0}\alpha^n C(E) + (\alpha-1)\sum_{n>0}\alpha^n C(V_n)\\ &\leqq A(\alpha-1)\sum_{n\leqq 0}\alpha^n C(E) + (\alpha-1)\sum_{n>0}\alpha^n\varepsilon_n +\\ &\quad + (\alpha-1)\sum_{n>0}\alpha^{-n}2\int_0^\infty C^*(|f|>t)t\,dt < \infty\end{aligned}$$

if $C(E)<\infty$.

Let the integer $n(x)$ be defined by $\alpha^{n(x)}<|f(x)|\leqq\alpha^{n(x)+1}$. For C-a.a. x we have the inequality

1.6 $$\begin{aligned}U\mu_\alpha(x) &\geqq (\alpha-1)\sum_{n\leqq n(x)}\alpha^n U\mu_n(x) \geqq (a-1)\sum_{n\leqq n(x)}\alpha^n = \alpha^{n(x)+1}\\ &\geqq |f(x)| .\end{aligned}$$

The energy of the measure μ_α is estimated in the following way.

1.7 $$I(\mu_\alpha) = (\alpha-1)^2\sum_n\sum_m \alpha^{n+m}\int U\mu_m\,d\mu_n \leqq 2(\alpha-1)^2\sum_{m\leqq n}\sum \alpha^{n+m}\int U\mu_m\,d\mu_n$$

$$\leqq 2A(\alpha-1)^2 \sum_n \alpha^n C(V_n) \sum_{m \leqq n} \alpha^m = 2\alpha A(\alpha-1) \sum_n \alpha^{2n} C(V_n) .$$

We also have that

$$2\int_0^\infty C^*(|f|>t)t\,dt = 2\sum_n \int_{\alpha^{n-1}}^{\alpha^n} C^*(|f|>t)t\,dt \geqq (1-\alpha^{-2}) \sum_n C^*(|f|>\alpha^n)\alpha^{2n}$$

$$\geqq (1-\alpha^{-2}) \sum_n \alpha^{2n} C(V_n) - (1-\alpha^{-2}) \sum_n \alpha^{2n}\varepsilon_n$$

which together with the estimate 1.7 gives

1.8 $$I(\mu_\alpha) \leqq \frac{2\alpha^3}{\alpha+1} 2A \int_0^\infty C^*(|f|>t)t\,dt + 2\alpha A(\alpha-1) \sum_n \alpha^{2n}\varepsilon_n .$$

From 1.6 and 1.8 and the definition of $p(f)$ it follows that if we let $\alpha \downarrow 1$ we get

$$p(f)^2 \leqq 2A \int_0^\infty C^*(|f|>t)t\,dt .$$

To prove the right inequality we first suppose that $0 \leqq f \in \boldsymbol{C}_0$. Let $V_n = \{f > \alpha^n\}$ where as before α is a real number >1. Let also μ_n be the equilibrium measure for V_n, and define μ_α as in 1.5. Since $S\mu_n \subset \bar{V}_n \subset \{f \geqq \alpha^n\}$, we have the estimate $\alpha^{2n}C(V_n) = \alpha^{2n}\int d\mu_n \leqq \int f\alpha^n\,d\mu_n$ implies

$$(\alpha-1)\sum_n \alpha^{2n} C(V_n) \leqq \int f\,d\mu_\alpha \leqq \int U\sigma\,d\mu_\alpha \leqq \sqrt{I(\sigma)}\sqrt{I(\mu_\alpha)}$$

for every measure σ such that $f \leqq U\sigma$ C-a.e. Using 1.7 we get that

$$(\alpha-1)\sum_n \alpha^{2n} C(V_n) \leqq 2\alpha A I(\sigma) \Rightarrow (\alpha-1)\sum_n \alpha 2^n C(V_n) \leqq C(V_n) \leqq p(f)^2 \Rightarrow$$

$$2\int_0^\infty C(f>t)t\,dt \leqq (\alpha^2-1)\sum_n \alpha^{2n} C(V_n) \leqq \alpha(\alpha+1)2Ap(f)^2 .$$

From this the right inequality follows if we let $\alpha \downarrow 1$ and if $0 \leqq f \in \boldsymbol{C}_0$. The inequality is then true also for an arbitrary function f such that $p(f) < \infty$. For there is then a positive measure σ such that $|f| \leqq U\sigma$ C-a.e. and $p(U\sigma)^2 = I(\sigma) < \varepsilon + p(f)^2$. The potential $U\sigma$ is lower semicontinuous, and therefore we have $U\sigma = \lim_{n\to\infty} f_n$ for some non decreasing sequence f_n of non negative functions in $\boldsymbol{C}_0$. By theorem 1.3.2 we have that

$$C^*(|f|>t) \leqq C(U_\sigma > t) = \lim_{n\to\infty} C(f_n > t) ,$$

and by monotone convergence we find that

$$\int_0^\infty C^*(|f| > t)t\,dt \leqq \lim_{n\to\infty} \int_0^\infty C(f_n > t)t\,dt \leqq 2A \overline{\lim_{n\to\infty}}\, p(f_n)^2$$

$$\leqq 2Ap(U\sigma)^2 < 2A\varepsilon + 2Ap(f)^2$$

for every $\varepsilon > 0$, and the theorem is proved.

REMARK. That the constants $2A$ and $4A^2$ are optimal at least if $A=1$ is seen by the following examples. For the left inequality we need only take $f=\chi_E$, and for the right inequality we consider the classical Newton capacity on $\mathbf{R}^n$ where $G(x,y)=a|x-y|^{2-n}$ and the constant a is determined so that $I(\mu) = \int |\text{grad}\, U\mu|^2\,dx$.

The capacity of the ball $\{|x| \leqq R\}$ is $\omega_{n-1}(n-2)R^{n-2}$, and if we define the measure μ on $\mathbf{R}^n$ as

$$\mu(E) = (n-2)^2 \int_E |x|^{-2}\,dx$$

it follows that $\mu(E) \leqq C(E)$ for every Borel set E. This can be seen in the following way. Let E^* denote the ball with center in 0 which has the same measure as E. An easy calculation shows that $\mu(E) \leqq \mu(E^*) = C(E^*)$. The inequality $C(E^*) \leqq C(E)$ follows from the formula

$$C(E) = \inf\left\{\int |\text{grad}\, U|^2\,dx \;:\; U \geqq 1 \;C\text{-a.e. on } E\right\}$$

and the isoperimetric inequality for functions in W_1^2 proved in [10, Thm. 2]. Now suppose the estimate $\int_0^\infty C(|f|>t)t\,dt \leqq kp(f)^2$ holds for some constant k independent of f. If f is a function in W_1^2 we find that

$$\int |f|^2\,d\mu = 2\int_0^\infty \mu(|f|>t)t\,dt \leqq 2\int_0^\infty C(|f| > t)t\,dt \leqq 2kp(f)^2\,,$$

and since we also have

$$p(f)^2 = \inf\left\{\int |\text{grad}\, U|^2\,dx \;:\; U \geqq f\;C\text{-a.e.}\right\}$$

it follows that

$$\int |f|^2\,d\mu \leqq 2k \int |\text{grad}\, f|^2\,dx\,.$$

If we take $f=\min(1, |x|^{-\alpha})$ for $\alpha > (n-2)/2$ we get

$$\int |\text{grad}\, f|^2\,dx = \alpha^2 \int_{|x|\geqq 1} |f|^2 |x|^{-2}\,dx = \left(\frac{\alpha}{n-2}\right)^2 \int_{|x|\geqq 1} |f|^2\,d\mu$$

from which it follows that $2k(\alpha/(n-2))^2>1$, and letting $\alpha \downarrow (n-2)/2$ we find that $k \geqq 2$.

From theorem 1.6 we get a convergence result analogous to a well known theorem for Lebesgue classes.

THEOREM 1.7. *If $(f_n)_{n=1}^{\infty}$ is a sequence of real valued functions on Ω such that*

$$\sum_n p(f_n) < \infty$$

then the series $\sum_n f_n(x)$ converges absolutely for all x outside a set of exterior capacity zero. If $f(x)$ is defined as the sum $\sum_n f_n(x)$ it also follows that

$$\lim_{N\to\infty} p\left(f - \sum_{n=1}^{N} f_n\right) = 0$$

PROOF. Define $g=\sum_n |f_n|$. From theorem 1.3.2 it follows that

$$C^*(g > t) = \lim_{N\to\infty} C^*\left(\sum_{n=1}^{N} |f_n| > t\right)$$

and from theorem 1.6 and monotone convergence we get

$$\int_0^{\infty} C^*(g > t)t\,dt = \lim_{N\to\infty} \int_0^{\infty} C^*\left(\sum_{n=1}^{N} |f_n| > t\right)t\,dt$$

$$\leqq 2A \overline{\lim_{N\to\infty}}\, p\left(\sum_{n=1}^{N} |f_n|\right)^2 \leqq 2A\left[\sum_n p(f_n)\right]^2 < \infty .$$

This implies that $C^*(g=\infty)=0$ and by theorem 1.6 again and the estimate above we have that

$$p(f) \leqq p(g) \leqq 2A \sum_n p(f_n) < \infty .$$

The same argument also gives that

$$p\left(f - \sum_{n=1}^{N} f_n\right) = p\left(\sum_{n>N} f_n\right) \leqq 2A \sum_{n>N} p(f_n)$$

which proves the theorem.

From theorem 1.7 it follows that the quotient space defined by

$$\boldsymbol{S} = \{f:\ p(f)<\infty\}/\{f:\ p(f)=0\}$$

is complete. Since we intend to construct a Banach space into which $\boldsymbol{H}$ can be imbedded, and this space should be as small as possible, we must impose some regularity on the functions as well. From the next lemma we see that it is natural to consider the closure of $\boldsymbol{C}_0$ in the space $\boldsymbol{S}$ defined above.

LEMMA 1.8. *If μ is a positive measure on Ω such that $U\mu \in \boldsymbol{H}$ then there is a sequence $(f_n)_{n=1}^{\infty}$ of functions in $\boldsymbol{C}_0$ such that*

$$\lim_{n\to\infty} p(U\mu - f_n) = 0$$

PROOF. We first prove that $U\mu$ can be approximated by potentials in C of measures of compact support. Since $\Omega = \lim_n E_n$ where $(E_n)_{n=1}^{\infty}$ is a monotone sequence of compact sets we have, if μ_n is the restriction of μ to E_n, by proposition 1.5 that

$$p(U\mu - U\mu_n)^2 = I(\mu - \mu_n) = \iint_{(\Omega\setminus E_n)\times(\Omega\setminus E_n)} G\, d\mu \times d\mu \to 0$$

and we may therefore suppose that $S\mu$ is compact. If now $G_n \in \boldsymbol{C}_0(\Omega\times\Omega)$ and $G_n \uparrow G$ in every point of $\Omega\times\Omega$ it follows that $U_n \uparrow U\mu$ where $U_n = \int G_n(\cdot, y)\, d\mu(y)$, and by Egoroff's theorem there are open sets V_k such that $\mu(V_k) \to 0$ as $k \to \infty$ and the convergence $U_n \to U\mu$ is uniform on the set $\Omega\setminus V_k$. If $\mu_k = \mu|_{\Omega\setminus V_k}$ and $v_{n,k}(x) = \int G_n(x,y)\, d\mu_k(y)$ it follows that $v_{n,k} \to U\mu_k$ as $n \to \infty$ uniformly on $S\mu_k$ and from theorem 1.2 we conclude that $U\mu_k \in \boldsymbol{C}$, and from proposition 1.5 again it follows that

$$p(U\mu - U\mu_k)^2 = \iint_{V_k\times V_k} G\, d\mu \times d\mu \to 0 \quad \text{as } k \to \infty .$$

Now we finally suppose that $S\mu$ is compact and $U\mu \in C$. For $n = 1, 2, \ldots$ we have

$$\{x : U\mu(x) > 1/n\} \subset\subset \Omega$$

and if we choose functions $g_n \in \boldsymbol{C}_0$ such that $0 \leqq g_n \leqq 1$ and $g_n = 1$ if $U\mu > 1/n$ we get from theorem 1.6 that

$$p(U\mu - f_n)^2 \leqq 2A \int_0^\infty C((1-g_n)U\mu > t)t\, dt$$

$$\leqq 2A \int_0^{1/n} C(U\mu > t)t\, dt \to 0 \quad \text{as } n \to \infty$$

with $f_n = g_n U\mu \in \boldsymbol{C}_0$.

Motivated by remarks above we make the following definition.

DEFINITION 1.9. If G is a kernel with properties 1.1–1.4, the Banach space $\boldsymbol{B}(G)$ is the closure of $\boldsymbol{C}_0$ in the Banach space of function classes on Ω defined by $\{f: p(f)<\infty\}/\{f: p(f)=0\}$.

It is clear that $\boldsymbol{B}(G)$ is a separable Banach space and from lemma 1.8 it follows that every function in $\boldsymbol{H}$ generates a function class in $\boldsymbol{B}(G)$. To describe the regularity of the functions in $\boldsymbol{B}(G)$ we make the following standard definition analogous to the Luzin property of measurable functions.

DEFINITION 1.10. A real valued function f on Ω is quasi continuous with respect to a capacity C defined on the subsets of Ω if to every $\varepsilon>0$ there is an open set $V\subset\Omega$ such that $C(V)<\varepsilon$ and f restricted to $\Omega\setminus V$ is continuous.

We can now give the following characterization of the functions in $\boldsymbol{B}(G)$.

THEOREM 1.11. *If G is a kernel on Ω satisfying* 1.1–1.4 *then a real valued function f belongs to $\boldsymbol{B}(G)$ if and only if it is quasi-continuous with respect to the capacity C defined by the kernel G and satisfies the boundedness condition*

$$\int_0^\infty C^*(\{x: |f(x)|>t\})t\,dt < \infty .$$

PROOF. First suppose that $f\in\boldsymbol{B}(G)$. If $\varepsilon>0$ is given and if ε_n is a sequence of positive numbers such that $\sum\varepsilon_n<\varepsilon/2$ we can find functions $f_n\in\boldsymbol{C}_0$ such that

$$\varepsilon_n^2 C^*(|f-f_n|>\varepsilon_n) \leqq 2Ap(f-f_n)^2 \leqq \varepsilon_n^3$$

and open sets V_n containing $\{|f-f_n|>\varepsilon_n\}$ with

$$C(V_n) < \varepsilon_n+C^*(|f-f_n|>\varepsilon_n) .$$

With $V=\bigcup V_n$ we have $C(V)\leqq\sum C(V_n)<\varepsilon$ and on $\Omega\setminus V$ we have $|f-f_n|\leqq\varepsilon_n$ for all n, which proves the continuity of $f|_{\Omega\setminus V}$.

Now suppose that $\int_0^\infty C^*(|f|>t)t\,dt<\infty$ for a real valued quasicontinuous function f. If for any positive integer n we define $f_n(x)=f(x)$ if $|f(x)|\leqq n$ and $f_n(x)=n$ or $-n$ if $f(x)$ is $>n$ or $<-n$ respectively we get a sequence $(f_n)_{n=1}^\infty$ of bounded quasicontinuous functions and by the extension theorem of Tietze we can, for each n, find a continuous function g_n such that $|g_n|\leqq n$ and $C^*(f_n\neq g_n)<1/4n^4$. From theorem 1.6 we get the estimates

$$p(f_n-g_n)^2 \leqq 2A\int_0^{2n} C^*(|f_n-g_n|>t)t\,dt \leqq 2A\int_0^{2n} t\,dt/4n^4 = A/n^2$$

$$p(f-f_n)^2 \leqq 2A\int_0^\infty C^*(|f| > t+n)t\,dt \leqq 2A\int_n^\infty C^*(|f| > t)t\,dt$$

and it follows that $p(f-g_n) \to 0$ as $n \to \infty$.

We finally show that any bounded and continuous function g with $p(g)<\infty$ belongs to $\boldsymbol{B}(G)$.

Let $\Omega=\lim_n E_n$ where $(E_n)_{n=1}^\infty$ is an increasing sequence of compact sets and choose $f_n \in \boldsymbol{C}_0$ such that $0\leqq f_n\leqq 1$ and $f_n=1$ on E_n. We can also find a positive measure μ such that $|g|\leqq U\mu$ C-a.e. and $I(\mu)<\infty$.

We have that $f_n g \in \boldsymbol{C}_0$, and for any $h \in \boldsymbol{C}_0$ the inequality

$$|g-f_n g| \leqq (1-f_n)(U\mu-h)+(1-f_n)h$$

holds C-a.e. Since the last term is equal to zero if n is large enough this implies that

$$\overline{\lim_{n\to\infty}}\, p(g-f_n g) \leqq p(U\mu-h)$$

for all $h \in \boldsymbol{C}_0$ and since by lemma 1.8 $p(U\mu-h)$ can be made as small as we please the theorem follows.

To be able to use the Hahn–Banach theorem efficiently we also prove the following characterization of the dual space of $\boldsymbol{B}(G)$.

THEOREM 1.12. *Every continuous linear functional Λ on $\boldsymbol{B}(G)$ can be uniquely represented by a signed measure μ on Ω in the following way*

$$f \in L^1(|\mu|) \text{ and } \Lambda(f) = \int f\,d\mu \quad \text{for all } f \in \boldsymbol{B}(G)\,.$$

The norm of Λ is equal to $\sqrt{I(|\mu|)}$, where $|\mu|$ denotes the total variation of the measure that represents Λ.

PROOF. Suppose first that μ is a measure with $I(|\mu|)<\infty$. If $f \in \boldsymbol{C}_0$ and σ_n are positive measures such that $|f|\leqq U\sigma_n$ C-a.e. and $I(\sigma_n) \to p(f)^2$ as $n \to \infty$, then clearly we have

$$|\mu(f)| \leqq |\mu|(|f|) \leqq |\mu|(U\sigma_n) \leqq \sqrt{I(\sigma_n)}\sqrt{I(|\mu|)} \to p(f)\sqrt{I(|\mu|)}$$

and it follows that $f \to \mu(f)$ can be extended to a unique linear functional on $\boldsymbol{B}(G)$ with norm $\leqq\sqrt{I(|\mu|)}$.

Now suppose that $\Lambda \in \boldsymbol{B}^*(G)$. If E is a compact set and $g \in \boldsymbol{C}_0$ with $0\leqq g\leqq 1$ and $g=1$ on E we have for every f in $\boldsymbol{C}_0$ with support in E that $|f|\leqq g\|f\|_\infty$, and it follows that $p(f)\leqq p(g)\|f\|_\infty$. This proves that the restriction of Λ to $\boldsymbol{C}_0$

is continuous and by the Riesz representation theorem there is a measure μ on Ω such that $\mu(f)=\Lambda(f)$ for all $f \in \boldsymbol{C}_0$. Since $\boldsymbol{C}_0$ is dense in $\boldsymbol{B}(G)$ it follows that two different functionals must be represented by different measures, and it remains to prove that the energy of $|\mu|$ is finite and $\leqq \|\Lambda\|^2$. We have

$$|\mu|(f) = \sup\{|\mu(g)| : g \in \boldsymbol{C}_0 \text{ and } |g| \leqq f\} .$$

This implies that $|\mu|(f) \leqq \|\Lambda\| p(f)$ for non negative functions f in $\boldsymbol{C}_0$, and by approximation of $U\sigma$ from below with such functions we get the estimate

$$|\mu|(U\sigma) = \sigma(U_{|\mu|}) \leqq \|\Lambda\| \sqrt{I(\sigma)}$$

for all positive measures σ with $I(\sigma)<\infty$. It also follows that $|\mu|(E) \leqq \|\Lambda\| \sqrt{C(E)}$ for every Borel set E.

Define $\sigma=|\mu|\,|_{E \cap \{U_{|\mu|} \leqq t\}}$, where E is a compact set and $t \geqq 0$. We have that

$$I(\sigma) \leqq \sigma(U_{|\mu|}) \leqq t|\mu|(E) < \infty .$$

It then follows that

$$I(\sigma) \leqq \sigma(U_{|\mu|}) \leqq \|\Lambda\| \sqrt{I(\sigma)} \Rightarrow I(\sigma) \leqq \|\Lambda\|^2 ,$$

and it remains to prove that $I(\sigma) \uparrow I(|\mu|)$ as $t \uparrow \infty$ and $E \uparrow \Omega$. This will follow if we prove that $|\mu|(U_{|\mu|}>t) \rightarrow 0$ as $t \uparrow \infty$. Let E be a compact subset of $\{U_{|\mu|}>t\}$ with equilibrium measure ν. We have that

$$tC(E) = t \int d\nu \leqq \nu(U_{|\mu|}) \leqq \|\Lambda\| \sqrt{I(\nu)} = \|\Lambda\| \sqrt{C(E)}$$

$$\Rightarrow t^2 C(E) \leqq \|\Lambda\|^2 ,$$

and it follows that $t^2 C(U_{|\mu|}>t) \leqq \|\Lambda\|^2$. This implies that

$$|\mu|(U_{|\mu|} > t) \leqq \|\Lambda\|^2/t$$

and the theorem is proved.

In the next theorem we prove an imbedding result for the inner product space $\boldsymbol{H}$. We also prove that p is the largest monotone functional on $\boldsymbol{H}$ that can be majorized by the norm of $\boldsymbol{H}$.

THEOREM 1.13. *If $\bar{\boldsymbol{H}}$ is the completion of $\boldsymbol{H}$ then $\bar{\boldsymbol{H}}$ can be identified with a subset of $\boldsymbol{B}(G)$ and $p(f) \leqq \|f\| = \sqrt{(f,f)}$ for all $f \in \bar{\boldsymbol{H}}$.*

If q is a functional on $\boldsymbol{H}$ such that $q(f) \leqq \|f\|$ for all $f \in \boldsymbol{H}$ and $f, g \in \boldsymbol{H}$ and $|f| \leqq |g|$ C-a.e. implies that $q(f) \leqq q(g)$ then $q(f) \leqq p(f)$ for all f in $\boldsymbol{H}$.

For the proof we use the following lemma which generalizes the notion of equilibrium measure.

LEMMA 1.14. *To every non negative function f in $\boldsymbol{B}(G)$ there is a unique positive measure μ concentrated on the set $\{x: f(x)>0\}$ and such that $f \leqq U\mu$ C-a.e. and $\mu(f)=p(f)^2=I(\mu)$.*

PROOF. Define the convex set $\boldsymbol{K} \subset \boldsymbol{B}(G)^*$ as $\boldsymbol{K}=\{\mu: \mu(f)=p(f)^2\}$. Clearly we have

$$I(|\mu|) \geqq |\mu(f)/p(f)|^2 = p(f)^2$$

for every μ in $\boldsymbol{K}$, and by the Hahn–Banach theorem there is a measure μ in $\boldsymbol{K}$ such that $I(|\mu|)=p(f)^2=\mu(f)$. If $\nu=\mu|_{\{f>0\}}$ then $\nu \in \boldsymbol{K}$ and we have that

$$I(|\nu|) \geqq p(f)^2 = I(|\mu|) \geqq I(|\nu|)$$

which implies that $\nu=\mu$, and that μ is concentrated on $\{f>0\}$. We also have that $\mu^+(f) \geqq \mu(f)=p(f)^2$ and therefore $t\mu^+ \in \boldsymbol{K}$ for some t with $0<t \leqq 1$. As before we get

$$t^2 I(\mu^+) \geqq p(f)^2 = I(|\mu|) \geqq I(\mu^+) \Rightarrow t = 1$$
$$\Rightarrow \mu^+(f) = p(f)^2 = \mu(f) \Rightarrow \mu \geqq 0\,.$$

If ν is another positive measure such that $I(\nu)=p(f)^2=\nu(f)$ then $(\nu+\mu)/2 \in \boldsymbol{K}$ and

$$I(\nu+\mu) \geqq 4p(f)^2 = 2I(\nu)+2I(\mu) = I(\nu+\mu)+I(\nu-\mu)\,.$$

It follows that $I(\nu-\mu)=0 \Rightarrow \nu=\mu$ and the uniqueness is proved.

It remains to prove that $f \leqq U\mu$ C-a.e. Suppose $0 \leqq \sigma \in \boldsymbol{B}(G)^*$ and $\sigma(f)>0$. Define $\nu=[p(f)^2/\sigma(f)]\sigma$. We have $\nu \in \boldsymbol{K}$ and so $t\mu+(1-t)\nu \in \boldsymbol{K}$. It follows that for $0 \leqq t \leqq 1$ we have

$$p(f)^2 \leqq I[t\mu+(1-t)\nu] = t^2p(f)^2+(1-t)^2I(\nu)+2t(1-t)\nu(U\mu)$$
$$\Rightarrow (1+t)p(f)^2 \leqq 2t\nu(U\mu)+(1-t)I(\nu)\,.$$

Letting $t \uparrow 1$ we get

$$p(f)^2 \leqq \nu(U\mu) \Rightarrow \sigma(f) \leqq \sigma(U\mu)$$

which proves that $f \leqq U\mu$ C-a.e.

PROOF OF THEOREM 1.13. We first prove that $p(f) \leqq \|f\|$ for every function f in $\boldsymbol{H}$. Since $|f| \in \boldsymbol{B}(G)$ there is by lemma 1.14 a positive measure μ such that

$p(f)^2 = \mu(|f|) = I(\mu)$. Let ν denote a signed measure with the properties $\nu(f) = \mu(|f|)$ and $|\nu| = \mu$. By the Schwarz inequality we then get that

$$p(f)^2 = \nu(f) = (f, U\nu) \leqq \|f\| \|U\nu\| \leqq \|f\| \sqrt{I(\mu)} = \|f\| p(f).$$

From the inequality $p(f) \leqq \|f\|$ it follows that every Cauchy sequence $(f_n)_{n=1}^{\infty}$ in $\boldsymbol{H}$ has a limit f in $\boldsymbol{B}(G)$. The mapping $(f_n)_{n=1}^{\infty} \to f$ can be thought of as a mapping from $\bar{\boldsymbol{H}}$ to $\boldsymbol{B}(G)$. If we prove that this mapping is injective it follows that $\bar{\boldsymbol{H}}$ can be identified with a subset of $\boldsymbol{B}(G)$. Let $f_n = U\mu_n$ be a Cauchy sequence in $\boldsymbol{H}$ and suppose that $p(f_n) \to 0$ as $n \to \infty$. Since $\mu_n \in \boldsymbol{B}(G)^*$ we have that

$$\lim_{n\to\infty} \|f_n\|^2 = \lim_{n,m\to\infty} (f_n, f_m) = \lim_{n,m\to\infty} \mu_m(f_n) = 0$$

since $f_n \to 0$ in $\boldsymbol{B}(G)$. This proves that the mapping from $\bar{\boldsymbol{H}}$ to $\boldsymbol{B}(G)$ is injective.

We finally suppose that q is a monotone functional on $\boldsymbol{H}$ with $q(f) \leqq \|f\|$. If μ is the equilibrium measure for $|f|$ then the estimate $|f| \leqq U\mu$ implies that

$$q(f) \leqq q(U\mu) \leqq \|U\mu\| = \sqrt{I(\mu)} = p(f)$$

and the theorem is proved.

Let $\boldsymbol{H}^+$ denote the space of potentials of positive measures with finite energy. The interesting fact that $\boldsymbol{H}^+$ is a closed subset of $\bar{\boldsymbol{H}}$, although $\boldsymbol{H}$ in general is not, was proved by Cartan when G is a Newton kernel [5]. The next theorem shows that the completeness of $\boldsymbol{H}^+$ is a consequence of the imbedding theory of $\boldsymbol{H}$.

THEOREM 1.15. *Let $\boldsymbol{P}$ denote the cone of functions in $\bar{\boldsymbol{H}}$ which are non-negative C-a.e. It then follows that $\boldsymbol{H}^+ = \boldsymbol{P}^* =$ the dual cone of $\boldsymbol{P}$ in $\bar{\boldsymbol{H}}$.*

PROOF. That we have $\boldsymbol{H}^+ \subset \boldsymbol{P}^*$ follows from the formula $(f, U\mu) = \int f\, d\mu$ which holds for f in $\bar{\boldsymbol{H}}$ and $U\mu$ in $\boldsymbol{H}$. Now suppose that f belongs to $\bar{\boldsymbol{H}}$ and let σ be the equilibrium measure of $|f|$. We have that $-U\sigma \leqq f \leqq U\sigma$ C-a.e. and if u is a function in $\boldsymbol{P}^*$ it follows that $-(u, U\sigma) \leqq (u, f) \leqq (u, U\sigma)$. From this we get that

$$|(u, f)| \leqq (u, U\sigma) \leqq \|u\| \sqrt{I(\sigma)} = \|u\| p(f).$$

Since $\bar{\boldsymbol{H}}$ is a dense subset of $\boldsymbol{B}(G)$ it follows from theorem 1.12 that there is a unique measure μ which represents the functional $f \to (u, f)$, i.e. $(u, f) = \int f\, d\mu$ for all f in $\bar{\boldsymbol{H}}$ and $I(|\mu|) \leqq \|\mu\|^2$. It follows that $u = U\mu$ C-a.e. and it remains to prove that $\mu \geqq 0$. Since $u \leqq U_{|\mu|}$ C-a.e. we get that

$$\|u\|^2 \leqq (u, U_{|\mu|}) \leqq \|U_{|\mu|}\|^2 = I(|\mu|) \leqq \|u\|^2$$

and it follows that

$$I(\mu-|\mu|) = \|u-U_{|\mu|}\|^2 = \|u\|^2+\|U_{|\mu|}\|^2-2(u, U_{|\mu|}) = 0$$

which implies that $\mu=|\mu|$.

2. L^p-theory.

Let Ω denote a metric space as in the preceding chapter. An important example of a positive definite kernel on Ω is given by the formula

2.1
$$G(x,y) = \int K(x,z)K(y,z)\,d\omega(z)\,.$$

Here K is a positive symmetric and lower semicontinuous function on $\Omega\times\Omega$ and ω is a fixed positive measure on Ω. If μ is a measure on Ω then its energy with respect to the kernel G given by 2.1 can be written as

2.2
$$I(\mu) = \int |K\mu(z)|^2\,d\omega(z) = \|K\mu\|^2_{L^2(\omega)}$$

where the symbol $K\mu$ is used to denote the function $\int K(.,x)\,d\mu(x)$. If K has the property that $K\mu=0$ ω-a.e. implies that $\mu=0$, then the kernel G is clearly positive definite. From 2.2 it is seen that in this situation it is natural to study imbeddings of the space of potentials $Kf(x)=\int K(x,y)f(y)\,d\omega(y)$ with the norm defined as $\|Kf\|=\|f\|_{L^2(\omega)}$. We shall in fact consider the more general situation of imbeddings of the space $\boldsymbol{H}^p$ for $1<p<\infty$ consisting of potentials Kf of functions f in $L^p(\omega)$. The norm on $\boldsymbol{H}^p$ is defined as $\|Kf\|_p=\|f\|_{L^p(\omega)}$. The spaces $\boldsymbol{H}^p$ defined in this way are easily seen to be complete, and for a kernel G given by 2.1 the relation between $\boldsymbol{H}^2$ and the inner product space $\boldsymbol{H}$ of the preceding section is $\boldsymbol{H}^2=\bar{\boldsymbol{H}}$.

The properties of the kernel K we will need in order to construct an imbedding theory for $\boldsymbol{H}^p$ analogous to the Hilbert space theory of the preceding chapter is stated in the following definition.

DEFINITION 2.1. 1. K is a positive and lower semicontinuous function on $\Omega\times\Omega$ and $K(x,y)=K(y,x)$.

2. There is a positive measure ω on Ω such that $\omega(V)>0$ for every nonempty open set V in Ω.

3. The functions $K(x,.)$ belong to $L^1_{\text{loc}}(\omega)$ for all x in Ω and the mapping $x\to K(x,.)$ from Ω to $L^1_{\text{loc}}(\omega)$ is continuous. In case of a noncompact Ω it is also required that $\int_E K(x,y)\,d\omega(y)\to 0$ as $x\to\infty$ for every compact set E in Ω.

4. If μ is a measure on Ω and $K\mu=0$ ω-a.e. then $\mu=0$.

5. There is a constant A such that the following boundedness principle holds for the nonlinear potentials $u=Kf$ where the function f can be written as $f=(K\mu)^{q-1}$ for some positive measure μ with compact support $S\mu$ and a real number q in the interval $1<q<\infty$; $\sup_{\Omega} u \leqq A \sup_{S\mu} u$.

From 2.1.3 it follows that $Kf \in \boldsymbol{C}$ for every function f in $\boldsymbol{C}_0$. An important example of a kernel that satisfies the boundedness principle 2.1.5 is given by $K(x,y)=k(|x-y|)$ on $\mathbf{R}^n$ where k is a non negative and decreasing function on $(0,\infty)$. In [3, Theorem 2.3] Adams and Meyers prove that with a kernel of this type there is a constant A for which 2.1.5 holds, and A depends only on the number q and the dimension n.

In the following p and q will denote real numbers in $(1,\infty)$ which are conjugate i.e. such that $pq=p+q$. For a kernel K which satisfies the properties of definition 2.1 and p conjugate to the q of 2.1.5 we define the capacity $C_p(E)$ of a subset E of Ω as follows [13, def. 6]

2.3 $$C_p(E) = \inf\{\|f\|^p_{L^p(\omega)} : f \geqq 0 \text{ and } 1 \leqq Kf \text{ on } E\} .$$

In the next theorem we state a few properties of the capacity C_p analogous to those in theorem 1.3. For the proof of theorem 2.2 the reader is refered to [13].

THEOREM 2.2. 1. *For arbitrary subsets* E_n *of* Ω $C_p(\bigcup E_n) \leqq \sum C_p(E_n)$.

2. *If* $E_n \subset E_{n+1}$ *and* $E=\bigcup E_n$ *it follows that* $C_p(E)=\lim_{n\to\infty} C_p(E_n)$.

3. *If* E *is compact then* $C_p(E)<\infty$, *and for any set* E

$$C_p(E) = \inf\{C_p(V) : E \subset V,\ V \text{ open}\}$$

and for any Borel set E

$$C_p(E) = \sup\{C_p(F) : F \subset E,\ F \text{ compact}\} .$$

4. *To any set* E *with* $C_p(E)<\infty$ *there is a unique non negative function* f *in* $Lp(\omega)$ *such that* $C_p(E)=\|f\|^p_{L^p(\omega)}$ *and* $Kf \geqq 1$ C_p*-a.e. on* E. *If* E *is a Borel set there is a unique positive measure* μ *on* $\bar{E}$ *such that* $\mu(\bar{E})=C_p(E)$, $f=(K\mu)^{q-1}$ *and* $Kf \leqq 1$ *on* $S\mu$.

The largest monotone norm $p(.)$ on $\boldsymbol{H}^p$ such that $p(f) \leqq \|f\|_p$ for all f in $\boldsymbol{H}^p$ can now be defined in the following way (cf. [2]).

DEFINITION 2.3. $p(f)=\inf\{\|g\|_p : |f| \leqq g\ C_p\text{-a.e. and } g \in \boldsymbol{H}^p\}$.

The next theorem shows that in this case there are relations between the norm p and the capacity C_p analogous to those of the preceding chapter.

THEOREM 2.4. 1. *For any set E the equality* $p(\chi_E)=C_p(E)^{1/p}$ *holds.*

2. $p(f)\leqq \|f\|_p$ *for any function f in* $\boldsymbol{H}^p$.

3. $p(f)^p\leqq 4\int_0^\infty C_p(|f|>t)\,dt^p\leqq Bp(f)^p$ *for some constant B which only depends on the number p and the constant A in property* 2.1.5 *of the kernel K.*

PROOF. 1 and 2 are immediate and the left inequality of 3 follows from the inequality

2.4 $$p\Big(\sup_n f_n\Big)^p \leqq \sum_n p(f_n)^p$$

which is analogous to (and proved in the same way as) the first statement of theorem 2.2. Following Adams [2] we note that if $\alpha>1$ and $\int_0^\infty C_p(|f|>t)\,dt^p<\infty$ we have that $f=\sup_n f_n$ C_p-a.e. where $f_n=f\chi_{\{\alpha^n<|f|\leqq\alpha^{n+1}\}}$ and n is an integer. By 2.4.1 we also have

$$p(f_n)^p \leqq \alpha^{np+p}p(\chi_{\{\alpha^n<|f|\leqq\alpha^{n+1}\}})^p \leqq \alpha^{np+p}C_p(|f|>\alpha^n)\,,$$

and we get by 2.4 that

$$p(f)^p \leqq \alpha^p\sum_n \alpha^{np}C_p(|f|>\alpha^n) \leqq [\alpha^{2p}/(\alpha^p-1)]\int_0^\infty C_p(|f|>t)\,dt^p\,.$$

Since $\min_{\alpha>1}[\alpha^{2p}/(\alpha^p-1)]=4$ the inequality follows.

The right inequality will follow by a standard approximation argument if we show that it holds for functions of the form Kf where f is a continuous function of compact support. This is proved by a construction analogous to the one in theorem 1.6. Let again α be a real number >1. We have that $C_p(Kf>\alpha^n)<\infty$ for any integer n, and by theorem 2.2.4 there are measures μ_n on $\{Kf\geqq\alpha^n\}$ such that

$$C_p(Kf>\alpha^n) = \|\mu_n\| = \|K\mu_n\|^q_{L^q(\omega)}$$

and $K(K\mu_n)^{q-1}_{\cdot}\leqq 1$ on $S\mu_n$. With $\mu=\sum_n\alpha^{n(p-1)}\mu_n$ we have that

2.5 $$\sum_n \alpha^{np}C_p(Kf>\alpha^n) = \sum_n \alpha^{np}\int d\mu_n \leqq \sum_n\int Kf\alpha^{n(p-1)}\,d\mu_n = \int Kf\,d\mu$$
$$= \int f\,K\mu\,d\omega \leqq \|f\|_{L^p(\omega)}\|K\mu\|_{L^q(\omega)}$$

It remains to estimate the L^q-norm of $K\mu$. We note that the boundedness principle implies that $K(K\mu_n)^{q-1}\leqq A$ for all n. We also define $\nu_n=\alpha^{n(p-1)}\mu_n$ and $K\mu=\sum_n K\nu_n=\sum_n g_n$.

Let N denote the integral part of q. Since $(q-1)/N<1$ we have the following estimates

$$\Big(\sum_n g_n\Big)^q = \sum_m \Big(\sum_n g_n\Big)^{q-1} g_m = \sum_m \Big(\sum_{n_1 \dots n_N} g_{n_1} \cdots g_{n_N}\Big)^{(q-1)/N} g_m$$

$$\leqq N^{(q-1)/N} \sum_m \Big(\sum_{n_2 \dots n_N \leqq n_1} g_{n_1} \cdots g_{n_N}\Big)^{(q-1)/N} g_m$$

$$\leqq N^{(q-1)/N} \sum_m \Big(\sum_{\substack{n_2 \dots n_N \leqq n_1 \\ n_1 \leqq m}} g_{n_1} \cdots g_{n_N}\Big)^{(q-1)/N} g_m$$

$$+ N^{(q-1)/N} \sum_m \Big(\sum_{\substack{n_2 \dots n_N \leqq n_1 \\ m < n_1}} g_{n_1} \cdots g_{n_N}\Big)^{(q-1)/N} g_m$$

This implies that

$$\Big\|\sum_n g_n\Big\|^q_{L^p(\omega)} \leqq N^{(q-1)/N}(I_1 + I_2),$$

where

$$I_1 \leqq \sum_m \sum_{\substack{n_2 \dots n_N \leqq n_1 \\ n_1 \leqq m}} \int g_{n_1}^{(q-1)/N} \cdots g_{n_N}^{(q-1)/N} g_m \, d\omega .$$

Hölders inequality gives

$$\int g_{n_1}^{(q-1)/N} \cdots g_{n_N}^{(q-1)/N} g_m \, d\omega \leqq \Big(\int g_{n_1}^{q-1} g_m \, d\omega\Big)^{1/N} \cdots \Big(\int g_{n_N}^{q-1} g_m \, d\omega\Big)^{1/N}$$

and since we have that

$$\int g_{n_k}^{q-1} g_m \, d\omega = \int K(g_{n_k}^{q-1}) \, d\nu_m = \alpha^{n_k(p-1)(q-1)} \alpha^{m(p-1)} \int K(K\mu_{n_k})^{q-1} \, d\mu_m$$

$$\leqq \alpha^{n_k} \alpha^{m(p-1)} A \int d\mu_m = A \alpha^{n_k} \alpha^{m(p-1)} C_p(Kf > \alpha^m) .$$

It follows that

$$I_1 \leqq A \sum_m \alpha^{m(p-1)} C_p(Kf > \alpha^m) \sum_{\substack{n_2 \dots n_N \leqq n_1 \\ n_1 \leqq m}} \alpha^{(n_1 + \dots + n_N)/N}$$

$$\leqq A\alpha(\alpha^{1/N} - 1)^{-N} \sum_m \alpha^{mp} C_p(Kf > \alpha^m) .$$

The second integral I_2 is estimated in a similar way.

$$I_2 = \int \sum_m \Big(\sum_{\substack{n_2 \dots n_N \leqq n_1 \\ m < n_1}} g_{n_1} \cdots g_{n_N}\Big)^{(q-1)/N} g_m \, d\omega .$$

Writing $g_m = g_m^{N+1-q} g_m^{q-N}$ and using Hölder's inequality with respect to the measure $g_m^{q-N} d\omega$ we get that

$$I_2 \leqq \left[\int \sum_m \Big(\sum_{\substack{n_2 \dots n_N \leqq n_1 \\ m < n_1}} g_{n_1} \cdots g_{n_N} \Big) g_m^{q-N} \, d\omega \right]^{(q-1)/N} \cdot$$

$$\cdot \left[\int \sum_m g_m^N g_m^{q-N} \, d\omega \right]^{(N+1-q)/N}$$

$$\leqq \left[\sum_{m, n_2 \dots n_N \leqq n_1} \int g_{n_1} \cdots g_{n_N} g_m^{q-N} \, d\omega \right]^{(q-1)/N} \left[\sum_m \int g_m^q \, d\omega \right]^{(N+1-q)/N} .$$

Hölder's inequality again, now with respect to the measure $g_{n_1} d\omega$ then gives that

$$\int g_{n_1} \cdots g_{n_N} g_m^{q-N} \, d\omega \leqq \left(\int g_{n_2}^{q-1} g_{n_1} \, d\omega \right)^{1/(q-1)} \cdots \left(\int g_{n_N}^{q-1} g_{n_1} \, d\omega \right)^{1/(q-1)} \cdot$$

$$\cdot \left(\int g_m^{q-1} g_{n_1} \, d\omega \right)^{(q-N)/(q-1)}$$

$$\leqq A \alpha^{n_1(p-1)} C_p(Kf > \alpha^{n_1}) \alpha^{(n_2 + \dots + n_N)/(q-1)} \alpha^{m(q-N)/(q-1)} .$$

Since we also have that

$$\int g_m^q \, d\omega = \alpha^{m(p-1)q} \int (K\mu_m)^q \, d\omega = \alpha^{mp} C_p(Kf > \alpha^m) ,$$

it follows from the estimate for I_2 above that

$$I_2 \leqq B \sum_n \alpha^{np} C_p(Kf > \alpha^n)$$

for some constant B which depends on α, p and A. This together with the estimate for I_1 above implies that

$$\|K\mu\|_{L^q(\omega)}^q = \Big\| \sum_n g_n \Big\|_{L^q(\omega)}^q \leqq B \sum_n \alpha^{np} C_p(Kf > \alpha^n), \qquad B = B(\alpha, p, A) .$$

From 2.5 it then follows that

$$\sum_n \alpha^{np} C_p(Kf > \alpha^n) \leqq B^{p/q} \|f\|_{L^p(\omega)}^p ,$$

and finally that

$$\int_0^\infty C_p(Kf > t) \, dt^p \leqq (\alpha^p - 1) B^{p/q} \|f\|_{L^p(\omega)}^p .$$

In the argument above we have supposed that $q \neq N \neq 1$. The somewhat exceptional cases $q = N$ and $N = 1$ are technically simpler when treated with the same methods as above, and are left to the reader.

If we define the Banach space $\boldsymbol{B}^p$ as the closure of $\boldsymbol{C}_0$ in the space $\{f : p(f) < \infty\}/\{f : p(f) = 0\}$ we can with essentially the same methods as in chapter 1 prove the following imbedding result for $\boldsymbol{H}^p$.

THEOREM 2.5. 1. *$\boldsymbol{H}^p \subset \boldsymbol{B}^p$ in the sense of set inclusion, and $p(f) \leqq \|f\|_p$ for all f in $\boldsymbol{H}^p$.*

2. *A function f defines an element of $\boldsymbol{B}^p$ if and only if f is quasicontinuous with respect to the capacity C_p and*

$$\int_0^\infty C_p(|f| > t)\, dt^p < \infty .$$

3. *If $q(.)$ is a monotone norm on $\boldsymbol{H}^p$ such that $q(f) \leqq \|f\|_p$ for all f in $\boldsymbol{H}^p$, then $q(f) \leqq p(f)$ for all f in $\boldsymbol{H}^p$.*

4. *The dual space $\boldsymbol{B}^{p*}$ of $\boldsymbol{B}^p$ consists of all signed measures μ such that $K(|\mu|) \in L^q(\omega)$ and the norm of μ as a functional on $\boldsymbol{B}^p$ is equal to $\|K(|\mu|)\|_{L^q(\omega)}$.*

PROOF. The first statement follows if we prove that Kf when f is in $L^p(\omega)$ can be approximated in $\boldsymbol{B}^p$ by functions in $\boldsymbol{C}_0$. This can be done in exactly the same way as in the proof of lemma 1.8 if we note that it is enough to consider the case when $0 \leqq f \in \boldsymbol{C}_0$, and that in this case we have by property 3 in definition 2.1 that $Kf \in \boldsymbol{C}$.

The second statement is proved in the same way as theorem 1.11, and the third statement is an immediate consequence of the definition of the norm $p(.)$.

To prove the fourth statement we note that if the measure μ represents an element of $\boldsymbol{B}^{p*}$ on $\boldsymbol{C}_0$, i.e. if $|\mu(f)| \leqq Ap(f)$ for all f in $\boldsymbol{C}_0$, then it follows that $|\mu|(f) \leqq Ap(f)$ for all f in $\boldsymbol{C}_0$. By approximation from below we then get that

$$|\mu|(Kf) \leqq Ap(Kf) \leqq A\|f\|_{L^p(\omega)} ,$$

and since $|\mu|(Kf) = \int K(|\mu|) f\, d\omega$ the inequality $\|K(|\mu|)\|_{L^q(\omega)} \leqq A$ follows from the converse of Hölder's inequality. The inequality $\|\mu\| = A \leqq \|K(|\mu|)\|_{L^q(\omega)}$ is however a direct consequence of Hölder's inequality, and the theorem is proved.

3. Some examples and applications.

In this section we show how some classical imbedding theorems which go back to Sobolev [15] relate to the imbedding theory developed in the preceding chapters. We will consider the case when Ω is either $\mathbf{R}^n$ or a ball $\{x : |x| \leqq R\}$. The measure ω is the Lebesgue measure, and the kernel $K(x,y)$ is supposed to be of the form $k(|x-y|)$ where $k(r)$ is a positive and non increasing function on $0<r<\infty$ which satisfy the conditions in definition 2.1. We also suppose that k satisfies the following growth condition

3.1 $$\int_0^r k(\varrho)\varrho^{n-1}\,d\varrho \leqq ak(r)r^n$$

for some constant a which does not depend on r. In the case when $\Omega = \mathbf{R}^n$ the further restriction is imposed

3.2 $$\int_r^\infty k(\varrho)^q \varrho^{n-1}\,d\varrho < \infty \qquad \text{for all } r>0 \text{ and some } q>1\,.$$

In the case of a Riesz kernel, $k(r)=r^{\alpha-n}$, 3.1 is satisfied when $0<\alpha<n$ and for 3.2 to hold we must have $\alpha p<n$ where $p+q=pq$. The motivation for 3.2 is that we want the capacity of every ball with positive radius to be positive.

By replacing k with the equivalent kernel $\bar{k}(r)=r^{-n}\int_0^r k(\varrho)\varrho^{n-1}\,d\varrho$ if necessary, it is clear from 3.1 that we can assume that

3.3 $$k(sr) \leqq s^{-n}k(r) \qquad \text{for } 0<r \text{ and } 0<s\leqq 1\,.$$

The spaces which will be our main interest in this section consist of Lebesgue measurable functions. They are supposed to be partially ordered and complete in the sense that if $|f| \leqq g$ a.e. and g belong to the space, then so does f. Another property which we assume is the so called Fatou property; If $\|\cdot\|$ is a norm for the space, and if $(f_n)_1^\infty$ is a monotone sequence of non negative functions in the space such that $\|f_n\|$ is bounded, then $f=\lim f_n$ belong to the space and $\|f\| = \lim \|f_n\|$. Spaces of the above type have many properties in common with the usual L^p-spaces [17, Ch. 15]. In particular they are Banach spaces and if the sequence $(f_n)_1^\infty$ converges to f in the space, then a subsequence of $(f_n)_1^\infty$ converges to f a.e. When we in the following speak of a partially ordered Banach space of Lebesgue measurable functions we always mean a space with the properties described above.

Definition 3.1. A partially ordered Banach space $\boldsymbol{B}$ of Lebesgue measurable functions on Ω is rearrangement invariant if $f \in \boldsymbol{B}$ implies that $g \in \boldsymbol{B}$ whenever f and g have the same distribution function. I.e. whenever

$$m(\{x : |f(x)|>t\}) = m(\{x : |g(x)|>t\}) \qquad \text{for all } t>0\,.$$

For the theory of rearrangement invariant spaces see [11]. Our aim will be to construct the smallest rearrangement invariant space which contains $\boldsymbol{H}^p$, and to compare it with the usual spaces into which H^p is imbeddable when $k(r)$ is the Riesz kernel.

As our notation for different rearrangements of a function f we will use the following: The nonincreasing rearrangement on $0<t<\infty$ will be denoted by f^* and the nonincreasing spherical symmetrization by $\bar{f}$.

The following relation between f^* and $\bar{f}$ then holds with $t_0=m(\{x : |x|<1\}.)$.

3.4 $$\bar{f}(x) = f^*(t_0|x|^n) .$$

The construction of the rearrangement invariant target space for H^p is based on the following isoperimetric estimate for the capacity C_p.

LEMMA 3.2. *If E is a subset of Ω and if $\bar{E}$ denotes the ball with center at the origin which has the same measure, then the inequality*

$$C_p(\bar{E}) \leqq bC_p(E)$$

holds for all E in Ω with a constant b which only depends on the dimension n and the constant a of 3.1.

PROOF. We use the following symmetrization inequality which in the one dimensional case are due to F. Riesz. The n-dimensional case was proved by Sobolev [15].

3.5 $$\iint k(|x-y|)f(x)g(y)\,dx\,dy \leqq \iint k(|x-y|)\bar{f}(x)\bar{g}(y)\,dx\,dy$$

Suppose that $C_p(E)<\infty$ and $0<m(E)<\infty$. Let f be a nonnegative function in $L^p(\Omega)$ such that $\|f\|_{L^p}^p<\varepsilon+C_p(E)$ for some $\varepsilon>0$, and $Kf(y)=\int k(|x-y|)f(x)\,dx \geqq 1$ for y in E. Choose a set $F\subset E$ with $2^n m(F)=m(E)$ and define $g=m(F)^{-1}\chi_F$. By 3.5 we have the inequality

3.6 $$1 \leqq \iint k(|x-y|)f(x)g(y)\,dx\,dy \leqq \int\left[\int k(|x-y|)\bar{g}(y)\,dy\right]\bar{f}(x)\,dx .$$

If we use the symbol $⨍_E f\,dx$ to denote the mean value $\int_E f\,dx/\int_E dx$, and if r is the radius of $\bar{E}$, it follows by the argument below that

3.7 $$\int k(|x-y|)\bar{g}(y)\,dy = ⨍_{|y|\leqq r/2} k(|x-y|)\,dy \leqq na4^n k(|x|+r) .$$

If $|x|\leqq r$ we have that

$$⨍_{|y|\leqq r/2} k(|x-y|)\,dy \leqq ⨍_{|y|\leqq r/2} k(|y|)\,dy$$
$$\leqq nak(r/2) \leqq na4^n k(2r) \leqq na4^n k(|x|+r)$$

by 3.1 and 3.3.

In the case when $|x|>r$ we have that

$$\int_{|y|\leqq r/2} k(|x-y|)\,dy \leqq k(|x|-r/2)$$

$$\leqq k(|x|/2) \leqq 4^n k(2|x|) \leqq 4^n k(|x|+r) < na4^n k(|x|+r)$$

and 3.7 follows.

Combining 3.6 and 3.7 we get that

$$1 \leqq \int k(|x|+r)na4^n\bar{f}(x)\,dx \leqq \int k(|x-y|)na4^n\bar{f}(x)\,dx$$

for all y such that $|y|\leqq r$. This implies that

$$C_p(\bar{E}) \leqq (na4^n)^p\|\bar{f}\|_{L^p}^p = (na4^n)^p\|f\|_{L^p}^p < (na4^n)^p[\varepsilon + C_p(E)]$$

for all $\varepsilon>0$, and the statement of the lemma follows with $b=(na4^n)^p$ at least in the case when $0<m(E)<\infty$. The general case then follows by a standard argument using for example theorem 2.2.

If we define the function $\varphi(t)$ for $0\leqq t<\infty$ by

$$\varphi(t) = C_p(\{x : |x|^n \leqq t/t_0\}) = \text{the capacity of a ball of measure } t\,,$$

then the following imbedding theorem can be proved.

THEOREM 3.3. *Every function f in $\boldsymbol{B}^p$ has the property that*

3.8 $$\int_0^\infty f^*(t)^p\,d\varphi(t) < \infty\,,$$

and if $\boldsymbol{B}$ is a rearrangement invariant space such that $\boldsymbol{H}^p\subset \boldsymbol{B}$ then every function f satisfying 3.8 belongs to $\boldsymbol{B}$.

PROOF. By lemma 3.2 we have the inequality $\varphi(m(\{|f|>s\}))=C_p(\{\bar{f}>s\}) \leqq bC_p(\{|f|>s\})$ for $s>0$, and if $f\in \boldsymbol{B}^p$ the first statement follows from theorem 2.5 and the formula

3.9 $$\int_0^\infty f^*(t)^p\,d\varphi(t) = \int_0^\infty \varphi(m(\{|f|>s\}))\,ds^p\,.$$

To prove the second statement we first observe that if $C_p(E)=0$, then by lemma 3.2 $C_p(\bar{E})=0$, which implies (by property 3.2 of k) that $\bar{E}=\{0\}$ and so that $m(E)=0$. This implies by a standard argument that if $\boldsymbol{H}^p\subset\boldsymbol{B}$ then the natural inclusion mapping from $\boldsymbol{H}^p$ to $\boldsymbol{B}$ is closed, and by the closed graph theorem there is a constant A such that $\|f\|\leqq A\|f\|_p$ holds for every f in $\boldsymbol{H}^p$ if $\|\cdot\|$ is a norm for $\boldsymbol{B}$. By theorem 2.5.3 and the fact that $\boldsymbol{B}$ is a partially ordered space it then follows that $\|f\| \leqq Ap(f)$ holds for f in $\boldsymbol{H}^p$, and by an approximation argument using the Fatou property the last inequality can be extended to any measurable function f. Now suppose that $\int_0^\infty f^*(t)^p\, d\varphi(t) < \infty$. From 3.9 again and theorem 2.4.3 we have that

$$\int_0^\infty f^*(t)^p\, d\varphi(t) = \int_0^\infty \varphi(m(\{\bar{f} > s\}))\, ds^p$$

$$= \int_0^\infty C_p(\bar{f} > s)\, ds^p \geqq B^{-1} p(\bar{f})^p \geqq B^{-1} A^{-p} \|\bar{f}\|^p ,$$

and it follows that $\bar{f} \in \boldsymbol{B}$ which since $\boldsymbol{B}$ is rearrangement invariant implies that $f \in \boldsymbol{B}$.

By theorem 3.3 it is seen that a minimal rearrangement invariant space containing $\boldsymbol{H}^p$ should be defined in the following way.

DEFINITION 3.4. The space $L(\varphi, p)$ consists of all measurable functions f on Ω such that f^* exist and $\int_0^\infty f^*(t)^p\, d\varphi(t) < \infty$.

The space $L(\varphi, p)$ is analogous to the Lorentz $L(p, q)$-spaces which are used in real interpolation theory. A norm for $L(\varphi, p)$ can in the case when $1 < p < \infty$ be defined as (cf. [16, ch. V 3])

3.10 $$\|f\|_{L(\varphi, p)} = \left[\int_0^\infty f^{**}(t)^p\, d\varphi(t)\right]^{1/p}, \quad f^{**}(t) = t^{-1} \int_0^t f^*(s)\, ds .$$

The triangle inequality follows from the fact that $(f+g)^{**} \leqq f^{**} + g^{**}$, and that 3.10 defines a norm on $L(\varphi, p)$ is a consequence of the inequality

3.11 $$\left[\int_0^\infty f^*(t)^p\, d\varphi(t)\right]^{1/p} \leqq \|f\|_{L(\varphi, p)} \leqq q\left[\int_0^\infty f^*(t)^p\, d\varphi(t)\right]^{1/p} .$$

The left inequality is immediate since $f^* \leqq f^{**}$. To prove the right inequality we observe that 3.3 implies that $s\varphi(t) \leqq \varphi(st)$ for $0 < t$ and $0 < s \leqq 1$. Since we have $f^{**}(t) = \int_0^1 f^*(st)\, ds = \int_0^1 f_s^*(t)\, ds$, it follows from Minkowski's inequality for integrals that

3.12 $$\|f\|_{L(\varphi, p)} = \left\|\int_0^1 f_s^*\, ds\right\|_{L^p(d\varphi)} \leqq \int_0^1 \|f_s^*\|_{L^p(d\varphi)}\, ds .$$

Since $f^*(t)^p$ is non increasing it follows that $\int_0^\infty f^*(t)^p\, d\alpha(t) \leqq \int_0^\infty f^*(t)^p\, d\beta(t)$ for every pair of non decreasing functions α and β such that $\alpha(t) \leqq \beta(t)$. If we insert $\alpha(t) = s\varphi(t)$ and $\beta(t) = \varphi(st)$ it follows that

$$\|f_s^*\|_{L^p(d\varphi)} = s^{-1/p}\left[\int_0^\infty f_s^*(t)^p\,d\alpha(t)\right]^{1/p} \leqq s^{-1/p}\left[\int_0^\infty f_s^*(t)^p\,d\beta(t)\right]^{1/p}$$
$$= s^{-1/p}\|f^*\|_{L^p(d\varphi)}\,,$$

which in combination with 3.12 proves the right inequality.

In the case of a Riesz kernel $k(r)=r^{\alpha-n}$ it follows by homogeneity that

$$C_p(\{|x|\leqq r\}) = C_p(\{|x|\leqq 1\})r^{\alpha-n} \quad \text{when } \alpha p < n\,.$$

This implies that $\varphi(t)\sim t^{1-p/n}$, where the $\sim$-sign means that the quotient of the members are bounded from above and below by positive numbers which does not depend on r. By the definition of $L(p,q)$-spaces [16, ch. V 3] it then follows that $L(\varphi,p)=L(p^*,p)$ where $p^*=np/(n-\alpha p)$. This is an imbedding theorem of O'Neil [14], and it follows that his result is optimal in our sense.

In the case when $\alpha p=n$ we take $\Omega=\{|x|\leqq R\}$, and for $0<r<R$ we have

$$C_p(\{|x|\leqq r\})\sim[\log(2R/r)]^{1-p}\,,$$

which gives

$$\varphi(t) \sim [\log(2^nT/t)]^{1-p} \quad \text{for } 0<t<T=m(\Omega)\,.$$

It follows that $L(\varphi,p)$ in this case consist of those functions f for which

3.13
$$\int_0^T\left[\frac{f^*(t)}{\log(2^nT/t)}\right]^p\frac{dt}{t} < \infty\,.$$

It is easy to see that 3.13 implies that

3.14
$$\int_{|x|\leqq R}\exp[f(x)^q]\,dx < \infty\,,$$

which is a result of Trudinger, Strichartz and Hedberg (see [9]). The example

$$f^*(t) = [\log(2^nT/t)]^{1/q}[\log\log(2^nT/t)]^{-1/p}, \quad p+q = pq$$

shows however that the integral 3.14 may be finite in cases when 3.13 is not.

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REFERENCES

1. D. R. Adams, *On the existence of capacitary strong type estimates in* $\mathbf{R}^n$, Ark. Mat. 14 (1976), 125–140.
2. D. R. Adams, *Set and functions of finite* L^p*-capacity*, Indiana Univ. Math. J. 27 (1978), 611–627.
3. D. R. Adams and N. G. Meyers, *Bessel potentials. Inclusion relations among classes of exceptional sets*, Indiana Univ. Math. J. 22 (1973), 873–905.
4. M. Brelot, *Lectures on potential theory*, Tata institute of fundamental research, Bombay, 1960.

5. H. Cartan, *Théorie du potentiel Newtonien*, Bull. Soc. Math. France 73 (1945), 74–106.
6. B. Dahlberg, *Regularity properties of Riesz potentials*, Preprint, Chalmers Univ. of Technology and Univ. of Göteborg, 1977.
7. B. Fuglede, *On the theory of potentials in locally compact spaces*, Acta Math. 103 (1960), 139–215.
8. K. Hansson, *On a maximal imbedding theorem of Sobolev type and spectra of Schrödinger operators*, Linköping Studies in Science and Technology. Dissertations, 26 (1978).
9. L. I. Hedberg, *On certain convolution inequalities*, Proc. Amer. Math. Soc. 36 (1972), 505–510.
10. K. Hildén, *Symmetrization of functions in Sobolev spaces and the isoperimetric inequality*, Manuscripta Math. 18 (1976), 215–235.
11. W. A. I. Luxemburg, *Rearrangement invariant Banach function spaces*, Queen's Papers in Pure and Applied Mathematics, 10, Kingston, 1967.
12. V. G. Mazýa, *On capacitary estimates of strong type for fractional norms*, (Russian), Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 70 (1977), 161–168.
13. N. G. Meyers, *A theory of capacities for potentials of functions in Lebesgue classes*, Math. Scand. 26 (1970), 255–292.
14. R. O'Neil, *Convolution operators and $L(p,q)$ spaces*, Duke Math. J. 30 (1963), 129–142.
15. S. L. Sobolev, *On a theorem in functional analysis*, (Russian), Math. Sb. 46 (1938), 471–497.
16. E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces* (Princeton Mathematical Series 32), Princeton University Press, Princeton, 1971.
17. A. C. Zaanen, *Integration*, North Holland Publ. Co., Amsterdam, 1967.

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