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SPECTRAL THEORY FOR FACIALLY HOMOGENEOUS SYMMETRIC SELF-DUAL CONES

JEAN BELLISSARD and BRUNO IOCHUM

I. Introduction.

It is well known that measure theory is strongly connected with the order structure in linear spaces. Such a connection has been pointed out by A. Connes [9] in the non commutative integration theory defined by the data of normal states in von Neumann algebras: it is possible to characterize any von Neumann algebra $\mathfrak{M}$ by means of a cone in the Hilbert space in which $\mathfrak{M}$ acts. This cone is the analog of the cone of positive functions belonging to $L^2(\Omega,\mu)$ in the commutative case.

From a geometric point of view, these cones have remarkable properties: they are self-dual, facially homogeneous and orientable. If H is a real Hilbert space, a cone H^+ is self-dual if it coincides with

$$\{\xi \in H \mid \langle \xi,\eta\rangle \geqq 0,\ \forall \eta \in H^+\}\ .$$

Such a cone is automatically convex and weakly closed. It is orientable if the quotient of the Lie algebra of its derivations by its center [9], is complex. H^+ is facially homogeneous if for any face F, the operator $P_F - P_{F^\perp}$ is a derivation, P_F (respectively $P_{F^\perp}$) being the projector onto the closed linear subspace generated by F (respectively $F^\perp$: the orthogonal face of F).

This last property has recently been used by the authors and R. Lima [6] and by M. Ajlani [1] to understand the connections with the notion of transitive homogeneity if H is finite dimensional. In that case the two properties coincide [6], and more generally if the cone is of finite type ([5] and [8]). Recall that a cone is of finite type if it contains a trace vector, i.e. a vector ξ such that the orthogonal $F_\xi^\perp$ of the face F_ξ that it generates is reduced to zero and which is invariant by the unitary group of the cone. In the previous work we prove a theorem analogue to A. Connes's:

H^+ is a self-dual cone which is facially homogeneous and of finite type if and only if one can find a Jordan J.B.W. algebra $\mathfrak{M}$ ([4] and [10]), with a trace state φ such that H^+ is isomorphic to the completion of the cone $\mathfrak{M}^+$ of positive elements of $\mathfrak{M}$, with respect to the Hilbert structure defined by φ.

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Roughly speaking, to suppress orientability condition means to replace von Neumann algebra by Jordan–Banach algebra.

In the proof of this theorem, one of the main tools was a spectral theory for derivations ([5], [8]). The facial derivation $\delta_F = \frac{1}{2}(\mathbf{1} + P_F - P_{F^\perp})$, where F is a face, plays the role of spectral projections in this theory.

The object of the present paper is to extend the spectral theory for homogeneous self-dual cone without trace. It is in fact a crucial step in proving that this class of cone is in a one-to-one correspondance with the category of Jordan–Banach algebras which are Banach dual spaces [11]. In this work we will need an additional property: the symmetry. A self-dual cone is symmetric if it is invariant under the subspace generated by F and $F^\perp$, F being a face. It has been proven by the authors that any facially homogeneous self-dual cone is symmetric [11]. The main result consists in giving a spectral decomposition of any self-adjoint derivation in term of the facial derivations δ_F. As a consequence, the extremal points of the order interval $[0, \mathbf{1}]$ in the set of self-adjoint derivations are precisely the facial derivations.

II. Main theorem.

In what follows H will be a real Hilbert space and H^+ a self-dual cone:

$$H^+ = \{\xi \in H \mid \langle \xi, \eta \rangle \geqq 0, \ \forall \eta \in H^+\} .$$

Every ξ in H has a unique decomposition (called the Jordan decomposition) $\xi = \xi^+ - \xi^-$ such that ξ^+ and ξ^- are in H^+ and $\langle \xi^+, \xi^- \rangle = 0$. A face F is a subcone of H^+ such that if $0 \leqq \eta \leqq \xi \in F$ then $\eta \in F$. There $\leqq$ denotes the partial order defined by H^+. The set

$$F^\perp = \{\xi \in H^+ \mid \langle \xi, \eta \rangle = 0, \ \forall \eta \in F\}$$

is a closed face [9]. The group $\mathrm{GL}\,(H^+)$ of the cone is the group of bounded operators A on H, with bounded inverse such that A and A^{-1} leave H^+ invariant. A derivation δ is a bounded operator on H such that $e^{t\delta} \in \mathrm{GL}\,(H^+)$ $\forall t \in \mathbf{R}$. The set of derivations $\mathscr{D}(H^+)$ is a weakly closed Lie algebra [9]. The self-dual cone H^+ is facially homogeneous (for short homogeneous) if for any face F,

$$\delta_F = \tfrac{1}{2}(\mathbf{1} + P_F - P_{F^\perp})$$

is a derivation. In any such cone, $P_F = P_{F^{\perp\perp}}$ and $F^{\perp\perp} = P_F H^+$ ([5] and [8]). This property allows us to restrict ourself to complete faces (i.e. $F = F^{\perp\perp}$) as far as derivations are concerned. Denote by $\mathscr{F}(H^+)$ the set of complete faces of H^+. H^+ is symmetric if $U_F = 2(P_F + P_{F^\perp}) - \mathbf{1}$ belongs to $\mathrm{GL}\,(H^+)$. Note that this operator is nothing but the symmetry with respect to the closed linear space generated by F and $F^\perp$. In [11] it is proved that every facially homogeneous self-dual cone is symmetric.

LEMMA II.1. *Let H^+ be a self-dual homogeneous cone. Then H^+ is symmetric if and only if*

$$(1) \qquad |\langle \tfrac{1}{2}(\mathbf{1}-U_F)\xi,\eta\rangle|^2 \leqq 4\langle P_F\xi,\eta\rangle\langle P_{F^\perp}\xi,\eta\rangle$$

where $F \in \mathscr{F}(H^+)$, ξ and η are in H^+.

PROOF. Suppose H^+ is symmetric. Since $U_F P_F = P_F$, $U_F P_{F^\perp} = P_{F^\perp}$ and $U_F^2 = \mathbf{1}$, one gets $\forall \xi,\eta \in H^+$, $\forall t \in \mathbf{R}$ and $n=0,1$,

$$0 \leqq \langle \eta, e^{t(P_F-P_{F^\perp})}U_F^n\xi\rangle = e^t\langle\eta,P_F\xi\rangle + e^{-t}\langle\eta,P_{F^\perp}\xi\rangle \pm \langle\eta,\tfrac{1}{2}(\mathbf{1}-U_F)\xi\rangle\,.$$

Therefore (1) holds.

Conversely if (1) holds then for any ξ and η in H^+

$$\begin{aligned}\langle U_F\xi,\eta\rangle &= \langle P_F\xi,\eta\rangle + \langle P_{F^\perp}\xi,\eta\rangle + \tfrac{1}{2}\langle(U_F-\mathbf{1})\xi,\eta\rangle\\ &\geqq (\langle P_F\xi,\eta\rangle^{\frac{1}{2}} - \langle P_{F^\perp}\xi,\eta\rangle^{\frac{1}{2}})^2\\ &\geqq 0\,.\end{aligned}$$

Since H^+ is self-dual, U_F leaves H^+ invariant.

In the sequel $\mathfrak{M}$ will be the space of self-adjoint derivations and

$$\mathfrak{M}_1^+ = \{\delta \in \mathfrak{M} \mid 0 \leqq \delta \leqq \mathbf{1}\}\,.$$

Note that $\mathfrak{M}_1^+$ is a weakly compact convex set.

MAIN THEOREM. *Let H be a real Hilbert space and H^+ be a self-dual, facially homogeneous and symmetric cone in H.*

Then for any $\delta \in \mathfrak{M}$, there exists a unique increasing family of faces: $\lambda \in \mathbf{R} \to F(\lambda)$ such that

i) $F(\lambda) = F(\lambda)^{\perp\perp}$ $\quad\forall \lambda$.
ii) $F(a-\varepsilon) = F(b+\varepsilon)^\perp = 0$ $\quad\forall \varepsilon > 0$ *if spectrum* $(\delta) \subset [a,b]$.
iii) $\bigcap_{\varepsilon>0} F(\lambda+\varepsilon) = F(\lambda)$.
iv) $\delta = \int_{a^-}^{b^+} \lambda\, d\delta_{F(\lambda)}$ (Lebesgue–Stieltjes weak integral).

COROLLARY. *The set $\{\delta_F \mid F \in \mathscr{F}(H^+)\}$ is exactly the set of the extremal points of $\mathfrak{M}_1^+$.*

This last result was conjectured by A. Connes a long time ago.

III. Towards the proof of the theorem: Construction of spectral faces.

Let δ in $\mathfrak{M}_1^+$. By the spectral theorem one can construct an increasing family $\lambda \in \mathsf{R} \to \pi(\lambda)$ of projectors in H which is right weakly semi-continuous and satisfies:

$$\pi(\lambda) = 0 \text{ if } \lambda < 0 \quad \text{and} \quad \pi(\lambda) = \mathbf{1} \text{ if } \lambda \geqq 1$$

$$\delta = \int_{0^-}^{1^+} \lambda \, d\pi(\lambda) \quad \text{(in the sense of weak topology)}\,.$$

Let

$$F(\lambda) = \{\xi \in H^+ \mid \pi(\lambda)\xi = \xi\}\,.$$

PROPOSITION III.1. *$F(\lambda)$ is a complete face and $F(\lambda)^\perp = \{\xi \geqq 0 \mid \pi(\lambda)\xi = 0\}$.*

The proof of this proposition requires the following steps.

LEMMA III.2. *Let F be a face of H^+, and ϱ be a derivation of H^+ commuting with P_F. Then ϱ commutes with $P_{F^\perp}$.*

PROOF. We remark that $e^{t\varrho}P_{F^\perp}$ preserves H^+ if $t \in \mathsf{R}$. Moreover $P_F e^{t\varrho} P_{F^\perp} = e^{t\varrho}P_F P_{F^\perp} = 0$. Then $e^{t\varrho}P_{F^\perp}H^+ \subset F^\perp$ (cf. [5, Lemma 1.2]). Since $H = H^+ - H^+$ this means $P_{F^\perp}e^{t\varrho}P_{F^\perp} = e^{t\varrho}P_{F^\perp}$. The same argument applied to ϱ^* shows that ϱ commutes with $P_{F^\perp}$.

The next lemma will be left to the reader.

LEMMA III.3. *Let μ, ν be finite Borel positive measures on R such that*

$$\int e^{t\lambda}\, d\mu(\lambda) \leqq \int e^{t\lambda}\, d\nu(\lambda), \qquad \forall t \in \mathsf{R}\,.$$

If ν is concentrated on the (not necessarily closed) interval I, then μ is also concentrated on I.

PROOF OF THE PROPOSITION III.1. For ξ in H^+ let $d\nu_\xi(\lambda)$ be the spectral measure $d\langle \xi, \pi(\lambda)\xi\rangle$. Clearly $\xi \in F(\lambda)$ means that $d\nu_\xi$ is concentrated on $]-\infty, \lambda[$, and $\pi(\lambda)\xi = 0$ means that $d\nu_\xi$ is concentrated on $]\lambda, \infty[$. By construction $F(\lambda) = \pi(\lambda)H \cap H^+$ is a closed convex cone in H^+. Let η be such that $0 \leqq \eta \leqq \xi \in F(\lambda)$. Then:

$$0 \leqq \langle e^{t\delta}\eta, \eta\rangle \leqq \langle e^{t\delta}\xi, \xi\rangle\,.$$

By the spectral theorem and Lemma III.3, $d\nu_\eta$ is concentrated on $]-\infty,\lambda[$, which means that $\eta \in F(\lambda)$ and $F(\lambda)$ is a face.

Since $F(\lambda)\subset\pi(\lambda)H$, clearly $P_{F(\lambda)}\leqq\pi(\lambda)$; moreover $P_{F(\lambda)}=P_{F(\lambda)^{\perp\perp}}$ because H^+ is facially homogeneous ([5, Lemma 2.2]). Therefore

$$F(\lambda) = P_{F(\lambda)}F(\lambda)=P_{F(\lambda)^{\perp\perp}}(\pi(\lambda)H\cap H^+) = \pi(\lambda)[(P_{F(\lambda)^{\perp\perp}}H)\cap H^+]$$

$$= \pi(\lambda)F(\lambda)^{\perp\perp} = F(\lambda)^{\perp\perp}$$

which proves that $F(\lambda)$ is a complete face.

The same argument works for $G(\lambda)=\{\xi\geqq 0 \mid \pi(\lambda)\xi=0\}$. Clearly $G(\lambda)\subset F(\lambda)^\perp$; thus $[P_{F(\lambda)}, P_{G(\lambda)^\perp}]=0$ and $P_{G(\lambda)^\perp}$ commute with $P_{F(\lambda)^\perp}$ ([5, Lemma 2.5]). If we prove that $K=F(\lambda)^\perp\cap G(\lambda)^\perp$ is reduced to zero, then $F(\lambda)^\perp=G(\lambda)$ because $F(\lambda)^\perp$ is itself homogeneous and self-dual. K is a self-dual cone in P_KH ([5, Lemma 2.4]) and $P_K=P_{F(\lambda)^\perp}P_{G(\lambda)^\perp}$ commutes with $\pi(\lambda)$. Indeed, by definition, δ leaves $F(\lambda)$ and $G(\lambda)$ invariant, and therefore commutes with $P_{F(\lambda)^\perp}$ and $P_{G(\lambda)^\perp}$ by Lemma III.2. Let L be the subspace $L=(1-\pi(\lambda))P_KH$ of P_KH. We get $L^\perp=\pi(\lambda)P_KH$ where $L^\perp$ is the subspace of P_KH orthogonal to L. By definition $L\cap K=\{0\}=L^\perp\cap K$. But this is impossible except if $K=\{0\}$ because of the following result.

LEMMA III.4. *Let H^+ be a self-dual cone in H and L a closed subspace of H. Then either $L\cap H^+ \neq \{0\}$ or $L^\perp\cap H^+\neq\{0\}$.*

PROOF. One can assume $L\neq\{0\}$ and $L^\perp\neq\{0\}$. If $L\cap H^+=\{0\}$, by the Hahn–Banach theorem there exists $\xi\in H$, $\xi\neq 0$ and $t\in \mathbf{R}$ such that $\langle\xi,\varrho\rangle\geqq t$, $\forall\varrho\in H^+$ and $\langle\xi,\eta\rangle\leqq t$, $\forall\eta\in L$. Using the dilatation invariance of H^+ and L, t will have to be zero. Therefore $\langle\xi,\eta\rangle=0$, $\forall\eta\in L$ and $\xi\in H^+$.

The following corollary is an immediate consequence of the previous proposition (cf. [5, Lemma 2.5]).

COROLLARY III.5. *$(P_{F(\lambda)})_{\lambda\in\mathbf{R}}$ and $(1-P_{F(\lambda)^\perp})_{\lambda\in\mathbf{R}}$ are two spectral families of projectors with supports in $[0,1]$ which mutually commute and*

$$P_{F(\lambda)} \leqq \pi(\lambda) \leqq 1-P_{F(\lambda)^\perp} \quad \forall\lambda\in\mathbf{R}\,. \tag{2}$$

IV. Reconstruction of spectral projectors.

The aim of this section is to prove the following reconstruction result.

PROPOSITION IV.1. *The family* $(Q(\alpha,\beta))_{(\alpha,\beta)\in\mathbf{R}^2}$ *where* $Q(\alpha,\beta)=P_{F(\alpha)}(\mathbf{1}-P_{F(\beta)^\perp})$ *defines a spectral measure on* $\mathbf{R}\times\mathbf{R}$ *with support in* $[0,1]\times[0,1]$ *and such that*

$$\pi(\lambda) = \int_{\alpha+\beta\leqq 2\lambda} dQ(\alpha,\beta)\,. \tag{3}$$

The equation (3) means that if

$$A(\lambda) = \{(\alpha,\beta)\in\mathbf{R}^2 \mid \alpha+\beta\leqq 2\lambda\}$$

and $\chi_{A(\lambda)}$ is the characteristic function of $A(\lambda)$, then the Lebesgue–Stieltjes integral $\int_{\mathbf{R}^2}\chi_{A(\lambda)}(\alpha,\beta)\,dP_{F(\alpha)}(\mathbf{1}-P_{F(\beta)^\perp})$ is equal to $\pi(\lambda)$. The first part of this proposition is standard [7]. To prove (3), one needs some elementary steps.

LEMMA IV.2. *If* H^+ *is a self-dual, homogeneous and symmetric cone, then*

$$P_{F(\alpha)}(\mathbf{1}-P_{F(\beta)^\perp}) \leqq \pi(\tfrac{1}{2}(\alpha+\beta)) \tag{4}$$

$$P_{F(\beta)^\perp}(\mathbf{1}-P_{F(\alpha)}) \leqq \mathbf{1}-\pi(\tfrac{1}{2}(\alpha+\beta))\,. \tag{5}$$

PROOF. If $\alpha\leqq\beta$ then $P_{F(\alpha)}\leqq P_{F(\beta)}\leqq\mathbf{1}-P_{F(\beta)^\perp}$. Since $\alpha\leqq\frac{1}{2}(\alpha+\beta)$ then $P_{F(\alpha)}\leqq \pi(\alpha)\leqq\pi(\frac{1}{2}(\alpha+\beta))$. This case is trivial.

Assume now that $\alpha>\beta$. Then $P_{F(\beta)}\leqq\pi(\frac{1}{2}(\alpha+\beta))$ and $P_{F(\beta)}$ is orthogonal to $P_{F(\alpha)}(\mathbf{1}-P_{F(\beta)}-P_{F(\beta)^\perp})$. If we can prove that this last projector is less than $\pi(\frac{1}{2}(\alpha+\beta))$, we have equation (4) because

$$P_{F(\alpha)}(\mathbf{1}-P_{F(\beta)^\perp}) = P_{F(\beta)}+P_{F(\alpha)}(\mathbf{1}-P_{F(\beta)}-P_{F(\beta)^\perp}) \leqq \pi(\tfrac{1}{2}(\alpha+\beta))\,.$$

Let ξ be in $F(\alpha)$. Then $\xi\in H^+$ and $\pi(\alpha)\xi=\xi$. We define η by

$$\eta = P_{F(\alpha)}(\mathbf{1}-P_{F(\beta)}-P_{F(\beta)^\perp})\xi$$

then $\eta=\pi(\alpha)\eta$.

Since H^+ is homogeneous and symmetric, we can apply Lemma II.1: $\forall t\in\mathbf{R}^+$ and $\forall\varepsilon>0$

$$\begin{aligned}
0 &\leqq e^{t\varepsilon}\|(\mathbf{1}-\pi(\tfrac{1}{2}(\alpha+\beta)+\varepsilon))\eta\|^2 \leqq \int_{\frac{1}{2}(\alpha+\beta)+\varepsilon}^{\alpha} e^{t(\lambda-\frac{1}{2}(\alpha+\beta))}\,d\langle\eta,\pi(\lambda)\eta\rangle \\
&\leqq e^{-t\frac{1}{2}(\alpha+\beta)}\langle\eta,e^{t\delta}\eta\rangle = e^{-t\frac{1}{2}(\alpha+\beta)}\langle(\mathbf{1}-P_{F(\beta)}-P_{F(\beta)^\perp})P_{F(\alpha)}\xi,e^{t\delta}\xi\rangle \\
&\leqq 2\langle P_{F(\beta)}\xi,e^{t(\delta-\beta)}\xi\rangle^{\frac{1}{2}}\langle P_{F(\beta)^\perp}P_{F(\alpha)}\xi,e^{t(\delta-\alpha)}\xi\rangle^{\frac{1}{2}} \\
&\leqq 2\|\xi\|^2
\end{aligned}$$

using the spectral theorem. Hence if $t\to\infty$ and $\varepsilon\to 0$, $\eta=\pi(\frac{1}{2}(\alpha+\beta))\eta$.

The equation (4) is proved since $H = H^+ - H^+$. The other equation is similarly obtained.

PROOF OF PROPOSITION IV.1: For $(\alpha,\beta) \in [0,1]\times[0,1]$ define the set $C(\alpha,\beta)$ by

$$
\begin{aligned}
C(\alpha,\beta) &= [0,\alpha]\times[0,\alpha] && \text{if } \alpha \leqq \beta \\
C(\alpha,\beta) &= [0,\alpha]\times[0,\alpha]-[\beta,1]\times[\beta,1] && \text{if } \beta<\alpha .
\end{aligned}
$$

By usual properties on a spectral measure ([7, page 12]), we have

$$\int_{C(\alpha,\beta)} dQ(\alpha',\beta') = P_{F(\alpha)}(1-P_{F(\beta)^\perp}) .$$

Let U be an open subset of $[0,1]\times[0,1]$ containing $A(\lambda)$; then there exists a $\varepsilon > 0$ such that $A(\lambda+\varepsilon) \subset U$, and it is possible to choose a finite number of points $(\alpha_k,\beta_k)_{k\in\{1,\dots,N\}}$ such that

$$A(\lambda) \subset A(\lambda+\tfrac{1}{2}\varepsilon) \subset \bigcup_{k=1}^{N} C(\alpha_k,\beta_k) \subset A(\lambda+\varepsilon) .$$

By Lemma IV.2, (4), we have

$$Q(A(\lambda)) \leqq \bigvee_k Q(\alpha_k,\beta_k) \leqq \pi(\lambda+\varepsilon) .$$

Using (5) we obtain by a symmetric argument:

$$\pi(\lambda) \leqq \pi(\lambda+\tfrac{1}{2}\varepsilon) \leqq \bigvee_k Q(\alpha_k,\beta_k) \leqq Q(A(\lambda+\varepsilon)) \leqq Q(U) .$$

The regularity of the spectral measure Q shows that

$$Q(A(\lambda)) = \bigwedge_{U\supset A(\lambda)} Q(U) \leqq \pi(\lambda) .$$

On the other hand

$$\pi(\lambda) \leqq \bigwedge_{U\supset A(\lambda)} Q(U) = Q(A(\lambda))$$

and the proof is achieved.

V. Proof of theorem and its corollary.

Using a property of integration for an image measure, we have

$$
\begin{aligned}
\delta &= \int_{\mathbf{R}\times\mathbf{R}} \tfrac{1}{2}(\alpha+\beta)\, dP_{F(\alpha)}\, d(1-P_{F(\beta)^\perp}) \\
&= \tfrac{1}{2}\int_{0^-}^{1^+} \alpha\, dP_{F(\alpha)} \int_{0^-}^{1^+} d(1-P_{F(\beta)^\perp}) + \tfrac{1}{2}\int_{0^-}^{1^+} \beta\, d(1-P_{F(\beta)^\perp}) \int_{0^-}^{1^+} dP_{F(\alpha)} .
\end{aligned}
$$

In particular

$$\delta = \frac{1}{2}\int_{0^-}^{1^+} \lambda \, d(P_{F(\lambda)} + \mathbf{1} - P_{F(\lambda)^\perp})$$
$$= \int_{0^-}^{1^+} \lambda \, d\delta_{F(\lambda)} \, .$$

Since $\lambda \to \delta_{F(\lambda)}$ is an increasing family of derivations, the theorem is proved for $\delta \in \mathfrak{M}_1^+$. If $\delta \in \mathfrak{M}$ there exists $(a,b) \in \mathrm{R}^+ \times \mathrm{R}$ such that $a\delta + b \in \mathfrak{M}_1^+$, and the proof is complete because the uniqueness of this spectral resolution follows from the uniqueness of the usual spectral resolution.

Now we come to the proof of the Corollary. Suppose that F is a complete face and δ_F admits a convex decomposition in $\mathfrak{M}_1^+$:

$$\delta_F = \lambda\delta_1 + (1-\lambda)\delta_2, \qquad 0 \leqq \lambda \leqq 1 \, .$$

Since $P_{F^\perp}\delta_F = 0$ we have $P_{F^\perp}\delta_1 = 0 = \delta_1 P_{F^\perp}$ because δ_1 is positive. This implies that $F^\perp \subset F_1(0)$ and by the theorem: $\delta_1 \leqq \mathbf{1} - \delta_{F_1(0)} \leqq \mathbf{1} - \delta_{F^\perp} = \delta_F$. Therefore $\delta_1 = \delta_2 = \delta_F$.

Conversely suppose that δ is an extremal point in $\mathfrak{M}_1^+$. The theorem gives the following convex decomposition in $\mathfrak{M}_1^+$:

$$\delta = \frac{1}{2}\int_{0^-}^{1^+} \lambda^2 \, d\delta_{F(\lambda)} + \frac{1}{2}\int_{0^-}^{1^+} (2\lambda - \lambda^2) \, d\delta_{F(\lambda)} \, .$$

Hence $\delta = \int_{0^-}^{1^+} \lambda^2 \, d\delta_{F(\lambda)}$ or equivalently $\int_{0^-}^{1^+} \lambda(1-\lambda) \, d\delta_{F(\lambda)} = 0$. It follows that $\lambda = 0$ or $\lambda = 1$, almost everywhere with respect to $d\delta_{F(\lambda)}$. Therefore there exists F such that $\delta = \delta_F$.

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REFERENCES

1. M. Ajlani, *Les cônes autopolaires en dimension finie*, Séminaire Choquet, 1974/75, n° 18.
2. E. M. Alfsen and F. W. Shultz, *Non commutative spectral theory for affine function spaces on convex sets*, Memoirs Amer. Math. Soc. 172, Providence R.I., 1976.
3. E. M. Alfsen and F. W. Shultz, *State spaces of Jordan algebras*, Acta Math. 140 (1978), 155–190.
4. E. M. Alfsen, F. W. Shultz and E. Størmer, *A Gelfand–Neumark theorem for Jordan algebras*, Advances in Math. 28 (1978), 11–56.
5. J. Bellissard and B. Iochum, *Homogeneous self-dual cones, versus Jordan algebras: The theory revisited*, Ann. Inst. Fourier, Grenoble, 28 (1978), 27–67.

6. J. Bellissard, B. Iochum and R. Lima, *Homogeneous and facially homogeneous self-dual cones*, Linear Algebra and Appl. 19 (1978), 1–16.
7. S. K. Berberian, *Notes on spectral theory* (Van Nostrand Mathematical Studies 5), D. Van Nostrand Co., Princeton, 1966.
8. W. Bös, *The structure of finite homogeneous cones and Jordan algebras*, Preprint, Osnabrück 1976.
9. A. Connes, *Caractérisation des espaces vectoriels ordonnés sousjacents aux algèbres de von Neumann*, Ann. Inst. Fourier, Grenoble, 24 (1974), 121–155.
10. F. W. Shultz, *On normed Jordan algebras which are Banach dual spaces*, J. Functional Analysis 31 (1979), 360–376.
11. J. Bellissard and B. Iochum, *L'algèbre de Jordan d'un cône autopolaire facialement homogène*, C. R. Acad. Sci. Paris, Sér. A 288 (1979), 229–232.

UNIVERSITE DE PROVENCE. MARSEILLE

AND

CENTRE DE PHYSIQUE THEORIQUE
C.N.R.S. LUMINY CASE 907
F-13288 MARSEILLE
FRANCE