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POSITIVE PROJECTIONS AND JORDAN STRUCTURE IN OPERATOR ALGEBRAS

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Introduction.

Suppose A is a unital C*-algebra and that $P: A \to A$ is a positive unital projection, i.e. $P \geqq 0$, $P(1)=1$, and $P^2=P$. Simple matrix examples show that the range $P(A)$ need not be a C*-subalgebra of A. Nonetheless it was shown in [3] that if P is completely positive then $P(A)$ becomes a C*-algebra when provided with the given Banach space and *-operations, and the new product $(r,s) \to P(rs)$. Letting A_h denote the self-adjoint elements in A, we have that A_h is a Jordan algebra under the product

$$a \circ b = \tfrac{1}{2}(ab+ba) . \tag{1}$$

In this paper we will show that if P is only assumed to be positive, then $P(A_h)$ is itself a Jordan algebra, and in fact a "JC-algebra" when provided with the given Banach space operations and the new multiplication $(r,s) \to P(r \circ s)$. Since the natural setting for this theorem is that of Jordan algebras we shall prove it when A_h is replaced by an arbitrary unital JC-algebra. A consequence of the theorem is that if A is a von Neumann algebra (or a JW-algebra) and Φ is a normal unital positive map of A into itself then the set of $a \in A_h$ such that $\Phi(a) = a$ has a natural multiplication making it into a JW-algebra. As a converse to the theorem we prove that every simple JC-algebra and every JW-factor is of the form $P(A_h)$ with A a C*-algebra and P as above.

This paper may be regarded as an attempt to place the recent monograph of Arazy and Friedman [2] in a general setting. In a technical *tour de force* the latter authors characterized the ranges of contractive projections in the algebra of compact operators on a separable Hilbert space. A closer inspection of their results seems to indicate that what they are doing is classifying certain Jordan and Lie algebras of operators. Our approach might explain the unexpected occurrence of Jordan algebras. The corresponding Lie algebra theory must apparently await the development of non-positive forms of the Kadison–Schwarz inequality [6].

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We recall that a JC-*algebra* is a norm closed real vector space of bounded self-adjoint operators on a complex Hilbert space closed under the Jordan product (1), [12]. We shall also employ the abuse of notation of calling a normed Jordan algebra a JC-algebra if it has an isometric Jordan representation as a JC-algebra. A JC-algebra A is called a JW-*algebra* if it is closed in the weak topology. The *center* of A is the set $Z = A \cap A'$, where A' is the commutant of A. A is said to be JW-*factor* if Z consists of scalar operators. If e is a projection in A its central carrier is the smallest projection in Z majorizing e. e is said to be abelian if $eAe = Ze$. A is said to be of *type* I_n if there exist n orthogonal abelian projections in A with central carriers the identity and with sum 1. Finally, a *Jordan ideal* J in a Jordan algebra A is a linear subspace such that $a \in A$, $b \in J$ imply $a \circ b \in J$.

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1. Projective images of JC-algebras.

In this section we show that the image of a unital positive projection of a JC-algebra into itself is a JC-algebra. Since the Kadison–Schwarz inequality $P(a^2) \geqq P(a)^2$, [6], only depends on the self-adjoint part of the unital C*-algebra generated by a and 1 it is clear that the inequality can be extended to JB-algebras as defined in [1]. Furthermore, slight modifications of our proof show that our theorem holds for JB-algebras as well. The proof is divided into some lemmas. The completely positive analogue of the first was first proved in [3]. The argument that we use was suggested by the more recent proof of Hamana [5].

LEMMA 1.1. *Suppose that A is a JC-algebra and that $P: A \to A$ is a unital positive projection. Then for any $a, b \in A$ we have*

$$P(P(a) \circ P(b)) = P(a \circ P(b)) .$$

PROOF. It suffices to prove that if ϱ is a state on A and $\omega = \varrho \circ P$ then

$$\omega(P(a) \circ P(b)) = \omega(a \circ P(b)), \qquad a, b \in A .$$

ω determines a real scalar product on A via $(a, b) = \omega(a \circ b)$. We let

$$N_\omega = \{a \in A : \omega(a^2) = 0\}$$

and H_ω be the real Hilbert space completion of $A/N_\omega = \{[a] : a \in A\}$ where $[a] = a + N_\omega$. We define a map $Q: A/N_\omega \to A/N_\omega$ by $Q([a]) = [P(a)]$. That this is well defined, and in fact a contraction is a consequence of the Kadison–Schwarz inequality, since by the latter $P(P(a)^2) \leqq P^2(a^2) = P(a^2)$ hence

$$\|Q([a])\|^2 = \|[P(a)]\|^2 = \omega(P(a)^2) = \varrho(P(P(a)^2)) \leqq \varrho(P(a^2)) = \|[a]\|^2 .$$

Q thus has an extension to H_ω which we also denote by Q. Since Q is then a contractive map satisfying $Q^2 = Q$, it follows that $Q = Q^*$, i.e.

$$(Qx, y) = (x, Qy) \quad \text{for } x, y \in H_\omega .$$

To see this, note that if $Qx = x$ and $Qy = 0$, then for all $\alpha \in \mathbf{R}$,

$$\|x\|^2 = \|Q(x + \alpha y)\|^2 \leqq \|x + \alpha y\|^2 .$$

It follows that the function

$$f(\alpha) = \|x + \alpha y\|^2 = \|x\|^2 + 2\alpha(x, y) + \alpha^2 \|y\|^2$$

assumes a minimum value at $\alpha = 0$. Thus

$$0 = f'(0) = 2(x, y) ,$$

and the range and the kernel of Q are orthogonal. Q must therefore coincide with the orthogonal projection onto $Q(H_\omega)$.

Finally we have

$$\begin{aligned} \omega(P(a) \circ P(b)) &= ([P(a)], [P(b)]) = (Q[a], Q[b]) \\ &= ([a], Q^2[b]) = ([a], Q[b]) \\ &= ([a], [P(b)]) = \omega(a \circ P(b)) . \end{aligned}$$

Given a JW-algebra M on a Hilbert space H and a normal (i.e. ultraweakly continuous (unital positive projection $P: M \to M$, we define the *support projection* e of P to be the complement of the maximal projection f for which $P(f) = 0$. As in the case of normal states we have that $P(a) = P(eae)$ for all $a \in M$, and if $a \in M^+$, then $P(a) = 0$ if and only if $eae = 0$ (one way to verify this is to use the corresponding facts for the states $\varrho \circ P$, ϱ a normal state on M). We let $[P(M)]$ denote the JW-algebra generated by $P(M)$.

LEMMA 1.2. *Suppose M is a JW-algebra and that $P: M \to M$ is a normal unital positive projection with support projection e. Then for all $a \in M$, $r \in P(M)$, $x, y \in [P(M)]$ we have*

(1) $P(r \circ a) = P(ere \circ eae)$,
(2) $er = re$,
(3) $eP(x)e = exe$,
(4) $P(x \circ y) = P(P(x) \circ P(y))$.

PROOF. Since $r = P(r)$ we have from Lemma 2.1

$$P(r \circ a) = P(r \circ P(a)) = P(r \circ P(eae)) = P(r \circ eae)$$
$$= P(e(r \circ eae)e) = P(ere \circ eae).$$

In particular it follows that

$$P(r^2) = P(r \circ r) = P((ere)^2) = P(rer),$$

i.e. $P(r^2 - rer) = 0$, and since $r^2 - rer \geqq 0$,

$$0 = e(r^2 - rer)e = ((1-e)re)^*(1-e)re).$$

We conclude that $(1-e)re = 0$, hence $re = ere$, and taking adjoints, $re = er$.

Turning to (3) let $A_1 = P(M)$, and for each $n \geqq 1$ let $A_{n+1} = A_n \circ A_n$ (=span $\{a \circ b : a, b \in A_n\}$). Then $\bigcup A_n$ is the Jordan algebra generated by A_1. (3) is trivially satisfied by elements in A_1. Suppose it is true for A_n. Then given $x \in A_n$ we have from (2) that

$$P(e(P(x^2) - xex)e) = P(x^2 - xex) = 0,$$

and from the Kadison–Schwarz inequality together with (3) for A_n,

$$e(P(x^2) - xex)e \geqq eP(x)^2 e - exexe = ex^2 e - exexe = 0,$$

hence we have $e(P(x^2) - xex)e = 0$, and $eP(x^2)e = ex^2e$. Using the identity $2x \circ y = (x+y)^2 - x^2 - y^2$, (3) follows for all $x, y \in A_{n+1}$, and therefore by induction for all x, y in the Jordan algebra generated by $P(M)$. Using that P is normal the general statement follows.

Finally, since (2) also holds for elements in $[P(M)]$, (4) follows since

$$P(x \circ y) = P(exe \circ eye) = P(eP(x)e \circ eP(y)e) = P(P(x) \circ P(y)).$$

LEMMA 1.3. *Suppose that M is a* JW-*algebra and that $P \colon M \to M$ is a normal unital positive projection. Then $P(M)$ is a Jordan algebra under the given vector operations and the product*

$$r * s = P(r \circ s), \qquad r, s \in P(M).$$

PROOF. We have that $1 * r = P(1 \circ r) = P(r) = r$, and the Jordan identity for $*$ follows from that for $\circ$ and (4) of Lemma 1.2:

$$\begin{aligned}(r*r)*(s*r) &= P(P(r \circ r) \circ P(s \circ r)) \\ &= P((r \circ r) \circ (s \circ r)) \\ &= P(((r \circ r) \circ s) \circ r)\end{aligned}$$

$$= P(P((r\circ r)\circ s)\circ r)$$
$$= P(P(P(r\circ r)\circ s)\circ r)$$
$$= ((r*r)*s)*r) .$$

We are now in position to prove the main result of this section. The completely positive version of the theorem can be found in [3] and [5].

THEOREM 1.4. *Suppose A is a unital* JC*-algebra and that $P\colon A \to A$ is a unital positive projection. Let $N=\{n \in A\colon P(n^2)=0\}$. Then we have*

(1) *$P(A)$ is a* JC*-algebra under the given vector operation and the product $r*s = P(r\circ s)$.*
(2) *$P(A)+N$ is a* JC*-subalgebra of A.*
(3) *P restricts to a Jordan homomorphism of $P(A)+N$ onto $P(A)$ with kernel N.*
(4) *$P(A)+N$ consists of all $a \in A$ for which $P(a^2)=P(P(a)^2)$.*

PROOF. Letting B be the C*-algebra generated by A in some faithful representation, we may identify the second dual A^{**} with the ultraweak (=weak*) closure of A in the von Neumann algebra B^{**}. A^{**} will thus be a JW-algebra with dense subalgebra A (see [4]). The second adjoint of P provides a unique extension of P to a normal unital positive projection $P\colon A^{**} \to A^{**}$. The fact that $P(A)$ is a Jordan algebra is thus a consequence of Lemma 1.3. To show $P(A)$ is a JC-algebra we must first show (2)–(4).

To show (4) we note that if $r \in P(A)$ and $n \in N$ then $0 \leqq P(n)^2 \leqq P(n^2)=0$, so by Lemma 1.1 $P(r\circ n)=P(P(r)\circ P(n))=0$. Thus we have $P((r+n)^2)= P(P(r+n)^2)$. Conversely, if $x \in A$ and $P(x^2)=P(P(x)^2)$, then let $n=x-P(x)$. We have by Lemma 1.1

$$P(n^2) = P(x^2)-2P(x\circ P(x))+P(P(x)^2) = 2P(P(x)^2)-2P(P(x)^2) = 0 ,$$

hence $x=P(x)+n \in P(A)+N$, and (4) follows.

To show (2) let $r \in P(A)$. Then $r^2-P(r^2) \in N$, since Lemma 1.2 (4) applied to $x=y=r^2$ yields

$$\begin{aligned} P((r^2-P(r^2))^2) &= P(r^4-2r^2\circ P(r^2)+P(r^2)^2) \\ &= P(P(r^2)^2-2P(r^2)\circ P(r^2)+P(r^2)^2) \\ &= 0 . \end{aligned}$$

Thus we have that $r^2 \in P(A)+N$.

If $n \in N$, then $P(n^2)=0$, so that $en^2e=0$, where e is the support of P, i.e., $ne =0$. If $r \in P(A)$ then $er=re$ by Lemma 1, 2, so $e(r\circ n)=0$, and $e(r\circ n)^2e=0$.

Thus $r \circ n \in N$. Since $(n^2)^2 \leqq \|n\|^2 n^2$ we conclude that $n^2 \in N$, and therefore

$$(r+n)^2 = r^2 + 2r \circ n + n^2 \in P(A)+N .$$

This implies that $P(A)+N$ is a Jordan subalgebra of A. From (4) it follows that $P(A)+N$ is norm closed and thus is a JC-algebra.

From (4) we have that $P(a^2) = P(a) * P(a)$ for $a \in P(A)+N$, i.e. P is a Jordan homomorphism of $P(A)+N$ onto $P(A)$. Since $P(r+n) = r$ for $r \in P(A)$, $n \in N$, N is the kernel of this Jordan homomorphism. Thus (3) follows. The induced map

$$P(A)+N \quad /N \to P(A)$$

by $a+N \to P(a)$ is an isometry, since if $a = r+n$, $r \in P(A)$, $n \in N$, then

$$\|r+n\| \geqq \|P(r+n)\| = \|r\| \geqq \|r+N\| .$$

It follows from [4] that $P(A)$ is a JC-algebra.

The weakly closed analogue of Theorem 1.4 is the following.

COROLLARY 1.5. *Suppose that M is a JW-algebra and P: $M \to M$ is a normal unital positive projection with support projection e. Let $N = \{a \in M : P(a^2) = 0\}$. Then we have*

(1) *$P(M)$ is a JW-algebra under the given vector operations and the product $r * s = P(r \circ s)$.*
(2) *$P(M)+N = eP(M)e + (1-e)M(1-e)$.*
(3) *$P(M)+N$ is a JW-subalgebra of M.*
(4) *P restricts to a normal Jordan homomorphism of $P(M)+N$ onto $P(M)$ with kernel N.*

PROOF. Let $f = 1-e$ and $R = eP(M)e + fMf$. Since clearly $N = fMf$, $f \in N$, so $e, f \in P(M)+N$. Thus we have the inclusion $R \subset P(M)+N$. Conversely it is clear that $N \subset R$. Let $a \in P(M)$, then by Lemma 1.2

$$a = ae + af = eae + faf \in eP(M)e + fMf = R ,$$

so that $P(M)+N \subset R$, and (2) follows. Since $P(M)$ is weakly closed, since P is ultraweakly continuous, R is weakly closed, hence by Theorem 1.4, $P(M)+N$ is a JW-subalgebra of M. Similarly (1) and (4) follow from Theorem 1.4.

The above corollary can be extended to positive linear maps of JW-algebras. This was pointed out to us by A. Connes.

COROLLARY 1.6. *Let M be a JW-algebra and Φ: $M \to M$ a normal unital positive map. Then the set of $a \in M$ for which $\Phi(a) = a$ has a natural structure as a JW-algebra.*

PROOF. For each positive integer n let $\Phi_n = n^{-1}(\Phi + \Phi^2 + \ldots + \Phi^n)$. Since M is weakly closed there is a unital positive map P of M into itself such that a subnet (Φ_{n_α}) of (Φ_n) converges to P in the point-ultraweak topology [7]. Note that if $a \in M$, if the limits are taken in the ultraweak topology, we have

$$\begin{aligned}\Phi^n(P(a)) &= \Phi^n\left(\lim_\alpha n_\alpha^{-1} \sum_1^{n_\alpha} \Phi^k(a)\right) \\ &= \lim_\alpha n_\alpha^{-1} \sum_1^{n_\alpha} \Phi^{n+k}(a) \\ &= \lim_\alpha n_\alpha^{-1}\left(\sum_1^{n_\alpha} \Phi^k(a) - \sum_1^{n} \Phi^k(a) + \sum_1^{n} \Phi^{k+n_\alpha}(a)\right) \\ &= \lim_\alpha n_\alpha^{-1}\left(\sum_1^{n_\alpha} \Phi^k(a)\right) \\ &= P(a)\,.\end{aligned}$$

In particular, $\Phi_n \circ P = P$, and we have $P^2(a) = P(P(a)) = \lim_\alpha \Phi_{n_\alpha}(P(a)) = P(a)$, so that P is a projection. Clearly $\Phi(a) = a$ implies $P(a) = a$. Conversely, if $P(a) = a$ then by the above, $a = P(a) = \Phi(P(a)) = \Phi(a)$, so that if $M_\Phi = \{a \in M : \Phi(a) = a\}$ then $M_\Phi = P(M)$. By Theorem 1.4 if $A = P(M)$ with the Jordan product $a * b = P(a \circ b)$ then A is a JC-algebra. Since Φ is ultraweakly continuous being normal, M_Φ is weakly closed. Thus A is a JW-algebra. To show that this Jordan structure on M_Φ is in a natural sense unique we consider another point-ultraweak limit point P' of the sequence (Φ_n), and we let B denote the JW-algebra obtained by giving M_Φ the Jordan product defined by P'. By Theorem 1.4 (3) the identity map ι of M_Φ onto itself defines an order-isomorphism of A onto B. But then by the Kadison–Schwarz inequality applied to ι and its inverse, ι is a Jordan isomorphism, see [6], and A can be identified with B.

2. Existence of projections.

A JC-algebra is said to be *simple* if it has no norm closed proper nonzero Jordan ideals. In this section we shall prove the following two results.

THEOREM 2.1. *Let B be a simple unital* JC-*algebra. Let M denote the* C*-*algebra generated by B, and let A be the self-adjoint part of M. Then there exists a unital positive projection P: $M \to M$ such that $P(A) = B$.*

THEOREM 2.2. *Let B be a* JW-*factor. Let M denote the von Neumann algebra generated by B, and let A be the self-adjoint part of M. Then there exists a unital positive projection P: $M \to M$ such that $P(A) = B$.*

The bulk of the proofs consists in verifying the theorems when B is a spin factor, or equivalently, a JW-factor of type I_2 [10]. For the reader's convenience we recall the construction of these factors [1, 10, 11, 12, 13]. We begin with a *spin system* $\mathscr{P}$, i.e. a collection of nontrivial symmetries, i.e. operators s for which $s^2=1$, $s=s^*$, $s\neq\pm 1$, on a Hilbert space H, which anticommute: $s\circ t=0$ for $s,t\in\mathscr{P}$, $s\neq t$. Each element of the linear span $(\mathscr{P})$ of $\mathscr{P}$ must again be a multiple of a symmetry:

$$(\sum \alpha_i s_i)^2 = \left(\sum_i \alpha_i^2\right)1+2\sum_{i<j}\alpha_i\alpha_j s_i\circ s_j = (\sum \alpha_i^2)1 .$$

The weak closure $(\mathscr{P})^-$ also consists of such multiples. To see this note that $(\mathscr{P})^-$ coincides with the strong closure of $(\mathscr{P})$, since $(\mathscr{P})$ is convex. If $b_\nu=\beta_\nu s_\nu\in(\mathscr{P})$ with $\beta_\nu\in\mathbf{R}$, s_ν a symmetry, and $b_\nu\to b$ strongly, then $\beta_\nu^2 1=b_\nu^2\to b^2$ weakly, and $b^2=(\lim\beta_\nu^2)1$. Fixing $s_0\in\mathscr{P}$ we have $s_0=e_0-f_0$ for nonzero projections e_0,f_0 with $e_0+f_0=1$. We then fix unit vectors $\xi\in e_0H$, $\eta\in f_0H$, and we define a normal state ω on $B(H)$ by $\omega(b)=\frac{1}{2}((b\xi,\xi)+(b\eta,\eta))$. We claim that $\omega(s)=0$ for all $s\in\mathscr{P}$. This is evident if $s=s_0$. If $s\neq s_0$, then since $s_0ss_0=-s$,

$$\begin{aligned}\omega(s) &= -\omega(s_0ss_0)\\ &= -\tfrac{1}{2}((ss_0\xi,s_0\xi)+(ss_0\eta,s_0\eta))\\ &= -\tfrac{1}{2}((s\xi,\xi)+(s(-\eta),(-\eta)))\\ &= -\omega(s) .\end{aligned}$$

It follows that $\omega\,|\,(\mathscr{P})^-=0$.

The *spin factor B* defined by $\mathscr{P}$ is the linear space $\mathbf{R}1+(\mathscr{P})^-$. It is a simple matter to verify that $\mathbf{R}1+(\mathscr{P})^-$ is a JW-factor of type I_2. The restriction $\tau=\omega\,|\,B$ is a trace, i.e. for all symmetries $s\in B$, $\tau(sbs)=\tau(b)$. τ determines a real pre-Hilbert space norm on B by $\|b\|_2=\tau(b^2)^{\frac{1}{2}}$. If $b=\beta 1+\sigma s$, $s=e-f\in(\mathscr{P})^-$, we have

$$\|b\| = \|\beta 1+\sigma s\| = \|(\beta+\sigma)e+(\beta-\sigma)f\| = \max|\beta\pm\sigma| = |\beta|+|\sigma| ,$$

$$\|b\|_2 = \tau((\beta^2+\sigma^2)1+2\beta\sigma s)^{\frac{1}{2}} = (\beta^2+\sigma^2)^{\frac{1}{2}} ,$$

hence the uniform norm and the Hilbert space norm are equivalent. In particular, B is complete in the $\|\cdot\|_2$-norm. On the other hand, if $b_\nu\to b$ strongly in B, then

$$\|b_\nu-b\|_2^2 = \|(b_\nu-b)\xi\|^2+\|(b_\nu-b)\eta\|^2\to 0 ,$$

and we conclude that the uniform and strong topologies coincide on B. In particular B may also be described as the smallest unital JC-algebra containing the spin system $\mathscr{P}$.

LEMMA 2.3. *Let B be a spin factor acting on the complex Hilbert space H. Then there exists a unital positive projection P of $B(H)$ into itself such that $P(B(H)_h)=B$.*

PROOF. The proof is divided into three steps.

(1). Assume H is finite dimensional. Let Tr be the normalized trace on $B(H)$. Then τ is the restriction of Tr to B. Since B is identified with the real Hilbert space defined by τ, B equals its own dual under this identification. Thus if $x \in B(H)_h$, there is an element $P(x) \in B$ such that

$$\mathrm{Tr}\,(xa) = (P(x), a) \quad (=\tau(P(x)\circ a))$$

for all $a \in B$. P so defined is clearly linear, unital and idempotent. Let $x \geqq 0$. Then

$$(P(x), a) \geqq 0 \quad \text{for all } a \in B^+ .$$

If $P(x)$ were not positive, by spectral theory there would exist real numbers α, β with $\beta>0$ and nonzero projections $e, f \in B$ with $e+f=1$ such that $P(x)=\alpha e-\beta f$. But then

$$0 \leqq (P(x), f) = -\beta(f, f) < 0 ,$$

a contradiction. Thus $P(x) \geqq 0$, and P is a positive projection.

(2). Assume H is infinite dimensional but B finite dimensional. Then there exists a finite spin system $\mathscr{P}=\{s_1, \ldots, s_n\}$ in B such that $B=\mathbf{R}1+(\mathscr{P})$. Thus the identity representation of B considered as a real Hilbert space, into $B(H)$, is a representation of the canonical anticommutation relations. Hence the C*-algebra generated by B is finite dimensional [8]. In particular there exists a finite dimensional projection e with central carrier 1 in the commutant of B. By case (1) there exists a unital positive projection P_e of $B(H)_e$ into itself such that $P_e((B(H)_e)_h)=Be$. If M is the von Neumann algebra generated by B we denote by α the isomorphism $\alpha\colon Me \to M$ by $ae \to a$, and we define P on $B(H)$ by $P(x)=\alpha(P_e(exe))$. Then clearly P is unital and positive. If $a \in B$ then $eae=ae \in eB$, so $P_e(eae)=ae$. Thus $P(a)=a$, and P is the desired projection of $B(H)_h$ onto B.

(3). Assume the dimensions of B and H are arbitrary. Let $\mathscr{P}$ be a spin system so that $B=\mathbf{R}1+(\mathscr{P})^-$. Since each finite subset of $\mathscr{P}$ is also a spin system on H the linear span of the symmetries $\{1, s : s \in J\}$ forms a finite dimensional spin factor B_J for each finite subset J of $\mathscr{P}$. By part (2) there is a unital positive projection P_J of $B(H)_h$ onto B_J. Order the finite subsets of $\mathscr{P}$ by inclusion. Then the P_J form a net, which by [7] has a subnet (P_{J_α}) which converges in the point-ultraweak topology to a unital positive map P of $B(H)$ into itself. If $x=x^* \in B(H)$ then $P_{J_\alpha}(x) \to P(x)$ ultraweakly. Since $P_{J_\alpha}(x) \in B$, and B is ultraweakly closed, $P(x) \in B$. In order to show P is idempotent let $a \in B$. Since $\{1\} \cup \mathscr{P}$ is an orthonormal base for B considered as a real Hilbert space

$$a = \tau(a)1 + \sum_{s \in \mathscr{P}} (a,s)s .$$

Let $b \in B$, $\|b\| \leqq 1$. Let $\varepsilon > 0$. Then there is a finite subset K of $\mathscr{P}$ such that if

$$a_K = \tau(a)1 + \sum_{s \in K} (a,s)s ,$$

then $\|a - a_K\| < \varepsilon$. Since $\|P\| \leqq 1$,

$$\|P(a) - P(a_K)\| < \varepsilon .$$

Since (P_{J_α}) is a subnet of $(P_J)_{J \subset \mathscr{P}}$, for arbitrary large finite J there exists J_α containing J, hence there is $J_\alpha \supset K$ such that

$$|(P(a_K) - P_{J_\alpha}(a_K), b)| < \varepsilon .$$

But then we have

$$\begin{aligned}|(P(a) - a, b)| &\leqq |(P(a) - P(a_K), b)| + |(P(a_K) - P_{J_\alpha}(a_K), b)| + |(a_K - a, b)| \\ &< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon .\end{aligned}$$

Since ε is arbitrary, $(P(a) - a, b) = 0$, and since b is arbitrary in B, $P(a) = a$.

LEMMA 2.4. *Let B be a* JW-*algebra of type* I_n, $n < \infty$. *Let π be a Jordan representation of B. Then the weak closure $\pi(B)^-$ of $\pi(B)$ is of type* I_n.

PROOF. If $e_1, \ldots, e_n$ are orthogonal abelian projections in B with central carriers 1 and sum 1, then so are the projections $\pi(e_1), \ldots, \pi(e_n)$ in $\pi(B)^-$. Thus $\pi(B)^-$ is of type I_n.

Recall that a JC-algebra B is said to be *reversible* if products of the form $\prod_{i=1}^m a_i + \prod_{i=m}^1 a_i \in B$ whenever $a_i \in B$. If $R(B)$ denotes the norm closed algebra over the reals generated by products of the form $\prod_{i=1}^m a_i$, $a_i \in B$, then $R(B)$ is a real Banach *-algebra, and B is reversible if and only if $B = R(B)_h$, see [9].

LEMMA 2.5. *Let B be a* JC-*algebra such that $\pi(B)^-$ is reversible for all *-representations π of the* C*-*algebra generated by B. Then B is reversible.*

PROOF. Let $C = R(B)_h$, and M be the C*-algebra generated by B. Then C is a reversible JC-subalgebra of M_h, and $B \subset C \subset M_h$. If $B \neq C$ there exist two states ϱ and ω on M such that $\varrho|C \neq \omega|C$ while $\varrho|B = \omega|B$. Let π be the GNS representation of M defined by the state $\frac{1}{2}(\varrho+\omega)$, and let $\bar{\varrho}$ and $\bar{\omega}$ be normal states on $\pi(M)^-$ such that $\bar{\varrho}\circ\pi=\varrho$ and $\bar{\omega}\circ\pi=\omega$ on M. Since $\bar{\varrho}$ and $\bar{\omega}$ are ultraweakly continuous they coincide on $\pi(B)^-$. However, $\pi(B)^-$ being reversible, contains $\pi(R(B)_h)=\pi(C)$. Thus $\bar{\varrho}\,|\,\pi(C)=\bar{\omega}\,|\,\pi(C)$, or $\varrho\,|\,C=\omega\,|\,C$, a contradiction. Therefore $B=C$, and B is reversible.

LEMMA 2.6. *Let B be a simple unital* JC-*algebra which is not reversible. Then B is a spin factor.*

PROOF. By Lemma 2.5 there exists a *-representation π of the C*-algebra M generated by B such that $\pi(B)^-$ is not reversible. By [10,Theorems 6.4 and 6.6] there is a central projection e in $\pi(B)^-$ such that $e\pi(B)^-$ is of type I_2 and not reversible. Considering $e\pi$ instead of π we may thus assume $\pi(B)^-$ is of type I_2. Let ϱ be a pure state of $\pi(B)^-$ and $\bar{\varrho}$ a pure state extension of ϱ to the C*-algebra generated by $\pi(B)^-$. Let π_ϱ denote the GNS-representation defined by $\bar{\varrho}$. Then $\pi_\varrho(\pi(B)^-)$ is irreducible with weak closure of type I_2 by Lemma 2.4. Since it contains $\pi_\varrho(\pi(B))$ which is not reversible, it is a nonreversible spin factor by [10, Theorem 7.1]. Since B is simple $\pi_\varrho(\pi(B))$ is isomorphic to B, hence $\pi_\varrho(\pi(B))$ is a simple JC-subalgebra of the spin factor $\pi_\varrho(\pi(B)^-)^-$. Since the strong and the norm topologies coincide in a spin factor $\pi_\varrho(\pi(B))$ is itself a spin factor, being simple. But then B has the structure of a spin factor. If we choose the trace on B to arise from a normal state on $B(H)$, then since the topology defined by the trace is the same as the norm topology, B is seen to be strongly hence weakly closed. Thus B is a spin factor.

PROOF OF THEOREM 2.1. Let B be a simple unital JC-algebra. If B is not reversible, B is a spin factor by Lemma 2.6, hence the theorem follows from Lemma 2.3 in this case. If B is reversible let

$$J = R(B) \cap iR(B), \qquad \text{where } iR(B) = \{ib : b \in R(B)\}\ .$$

Then a straightforward computation shows that J_h is a norm closed Jordan ideal in B. Since B is simple, either $J_h(0)$ or $J_h=B$. In the latter case B is already the self-adjoint part of the C*-algebra M it generates [9], so the theorem is trivial in this case. If $J_h=(0)$, $R(B)+iR(B)=M$ [11, Theorem 2.1]. Then let $\alpha(x+iy)=x^*+iy^*$ when $x,y\in R(B)$. The map α is a *-antiautomorphism of order 2 of M, and the map $P=\frac{1}{2}(\iota+\alpha)$, where ι is the identity automorphism of M, is the desired positive projection of M_h onto B.

PROOF OF THEOREM 2.2. If B is a JW-factor then by [10, Theorems 6.4, 6.6, 7.1] B is either a spin factor, the self-adjoint part of the von Neumann algebra M it generates, or B is reversible with

$$R(B)^- \cap iR(B)^- = (0)\,.$$

In the latter case $M = R(B)^- + iR(B)^-$ by [10, Theorem 2.4]. Thus the same proof as in the simple case applies.

REMARK 2.7. While the conclusion of Theorem 2.1 does not seem to extend to general JC-algebras, we expect Theorem 2.2 to extend to arbitrary JW-algebras. By [10, Theorem 6.4] a trivial extension of the above proof proves it for reversible JW-algebras. By [10, Theorem 6.6] we have thus reduced the problem to the case when the JW-algebra is of type I_2. In this case it seems apparent that global techniques should finish the proof.

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