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UNIVERSALLY WEAKLY INNER ONE-PARAMETER AUTOMORPHISM GROUPS OF SEPARABLE C*-ALGEBRAS

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Abstract.

A characterization of such automorphism groups in terms of approximation by inner groups is obtained, and applied to show that such an automorphism group can be lifted from a quotient of a separable C*-algebra. These results are analogous to those obtained for smaller classes of automorphism groups by Pedersen in [5] (the case of bounded spectrum) and by Olesen and Pedersen in [3] (the case of semibounded spectrum).

1. Introduction.

It was shown by Sakai in [7], using results of Kadison and Kaplansky, that if a unitary group determining a one-parameter automorphism group of a C*-algebra of operators can be chosen to have a bounded generator then it can be chosen to lie in the weak closure of the C*-algebra. Using this result, Borchers showed in [1] that "bounded" may be replaced by "semibounded". This is sometimes expressed by saying that a one-parameter automorphism group with semibounded spectrum is weakly inner.

In [3], Olesen and Pedersen showed that if a one-parameter automorphism group of a separable C*-algebra has semibounded spectrum universally, that is, in the universal representation, then it is a limit of inner automorphism groups. By Borcher's theorem, such an automorphism group is universally weakly inner.

The main purpose of this paper is to show that an arbitrary universally weakly inner one-parameter automorphism group of a separable C*-algebra is a limit of inner automorphism groups.

While the basic strategy of the proof is similar to that followed in [3], the monotone functional calculus so successfully used in [3] must here be replaced by other techniques. Order-theoretic methods are still needed at one point in the proof, however — see 2.3.

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Of course, not every limit of inner automorphism groups of a C*-algebra is weakly inner. In [6], Powers and Sakai even suggest that any (continuous) one-parameter automorphism group of an approximately finite-dimensional separable C*-algebra may be such a limit. On the other hand, if the unitary groups in the C*-algebra can be chosen so as to converge to a unitary group in the weak closure, then the limit of the automorphism groups they determine will clearly be weakly inner.

Since in this paper (just as in [3]) unitary groups are constructed which converge strongly in the universal representation, in fact a characterization is obtained of universally weakly inner one-parameter automorphism groups of separable C*-algebras (2.1).

Much as in [3], this characterization leads to a solution of a lifting problem for automorphism groups in this class—the problem of showing that such an automorphism group of a quotient of a separable C*-algebra is induced by an automorphism group of the C*-algebra itself. There are some differences here, however. In [3], a characterization is given of the subclass of automorphism groups with universally semibounded spectrum which permits a lifting within the same class. The present construction, on the other hand, does not seem to yield a universally weakly inner lifting. This is connected with the fact that a slightly different construction is used in the characterization and in solving the lifting problem (compare 2.1 and 2.5).

2. Approximation by inner automorphism groups.

2.1. THEOREM. *Let A be a separable C*-algebra, and let α be a one-parameter automorphism group of A. Then α is universally weakly inner if and only if there exists a net (h_γ) of selfadjoint elements of A such that:*

(i) *$\|\alpha_t(a)-\exp(ith_\gamma)a\exp(-ith_\gamma)\| \to 0$ for each $a \in A$, uniformly for t in each compact subset of $\mathbf{R}$;*

(ii) *the net $((h_\gamma+i)^{-1})$ is Cauchy in the *-ultrastrong topology on $A \subset A^{**}$.*

2.2. LEMMA (cf. 3.1 of [3]). *Let h be a selfadjoint operator, and let a be a bounded operator leaving the domain of h invariant and such that the operator $[h,a]$, defined on the domain of h, is bounded. Then the family of operators $[h(1+(\varepsilon h)^2)^{-1},a]$, $\varepsilon>0$, is uniformly bounded, and*

$$[h,a]-[h(1+(\varepsilon h)^2)^{-1},a] \to 0 \textit{ strongly as } \varepsilon \to 0\,.$$

PROOF. The conclusion follows immediately from the identity

$$\begin{aligned}&[h,a]-[h(1+k^2)^{-1},a]\\ &\quad= [h,a]k^2(1+k^2)^{-1}+h(1+k^2)^{-1}(k[k,a]+[k,a]k)(1+k^2)^{-1}\,,\end{aligned}$$

valid for k a polynomial in h, by setting $k=\varepsilon h$.

2.3. LEMMA. *Let A be a C*-algebra, α an ultraweakly continuous one-parameter automorphism group of A^{**}, k a selfadjoint element of A^{**}, and $f_1, \ldots, f_p$ continuous complex-valued functions on* R *with compact support. Then there exists a bounded net (k_j) of selfadjoint elements of A such that*

$$k - k_j, \int \alpha_t(k-k_j) f_1(t)\, dt, \ldots, \int \alpha_t(k-k_j) f_p(t)\, dt$$

all converge ultrastrongly to 0.

PROOF. We may suppose that $f_1, \ldots, f_p \geqq 0$. Then by Dini's theorem the set of all selfadjoint elements of A^{**} satisfying the conclusion in place of k is monotone closed. Hence by work of Kadison and Pedersen (see, for example, [4, page 956, the first two paragraphs]), this is the set of all selfadjoint elements of A^{**}.

2.4. PROOF OF 2.1. By Lemma 2 of [2], condition (ii) is equivalent to the ultrastrong convergence of the net $(\exp(ith_\gamma))$ in A^{**}, uniformly on compact subsets of R. Therefore conditions (i) and (ii) are sufficient for α to be universally weakly inner.

Suppose that α is universally weakly inner, and choose a selfadjoint operator h affiliated with A^{**} such that $\alpha_t = \operatorname{Ad}\exp(ith)$, $t \in$ R.

If $a = \int \alpha_t(b) f(t)\, dt$ with $b \in A$ and f a smooth complex-valued function with compact support, then it is well known that a and h satisfy the hypotheses of 2.2, and that the closure of $[ih, a]$ is $-\int \alpha_t(b) f'(t)\, dt$. (This can be verified by showing that if ξ is in the domain of h then the derivative of $s \mapsto \exp(ish)a\xi$ at 0 is $\int \alpha_t(b)(ih\xi f(t) - \xi f'(t))\, dt$.)

Each $b \in A$ lies in the norm closure of the elements $\int \alpha_t(b) f(t)\, dt$, f a smooth complex-valued function with compact support. (Although $t \mapsto \alpha_t(b)$ is not explicitly assumed to be norm-continuous, it is ultraweakly continuous in A^{**} and thus weakly continuous in A. This implies that b lies in the weak closure in A of the above elements, which is equal to the norm closure as they form a subspace.) Hence we may choose a norm-dense sequence (a_p) in A consisting of elements of this form.

Fix a smooth real-valued function f with compact support and with integral 1. We shall construct a sequence (k_n) of selfadjoint elements of A such that with $h_n = \int \alpha_t(k_n) f(t)\, dt$ we have for each $n = 1, 2, \ldots$:

(a) $\|[h - h_n, a_p]\| \leqq n^{-1}, \quad p = 1, \ldots, n;$
(b) $\|[h - h_n, h_p]\| \leqq n^{-1}, \quad p = 1, \ldots, n-1;$
(c) $\|[h, h_n]\| \leqq n^{-1}.$

For each $\varepsilon>0$, by 2.3 with $k=h(1+(\varepsilon h)^2)^{-1}$, $f_1=f$ and $f_2=f'$ we obtain a net (k_j^ε) of selfadjoint elements of A such that in the ultrastrong topology of A^{**}:

$$\int \alpha_t(k_j^\varepsilon)f(t)\,dt \to h(1+(\varepsilon h)^2)^{-1};$$

$$\int \alpha_t(k_j^\varepsilon)f'(t)\,dt \to 0\,.$$

Hence by 2.2 with $a=a_1$, there exists a net $(k_j^{(1)})$ of selfadjoint elements of A such that in the weak topology of A:

$$\left[\int \alpha_t(k_j^{(1)})f(t)\,dt, a_1\right] \to \text{closure}\,[h,a_1];$$

$$\text{closure}\left[h, \int \alpha_t(k_j^{(1)})f(t)\,dt\right] = -i\int \alpha_t(k_j^{(1)})f'(t)\,dt \to 0\,.$$

Setting a suitable convex combination of $k_j^{(1)}$'s equal to k_1, and $\int\alpha_t(k_1)f(t)\,dt$ equal to h_1, we have (a) and (c) with $n=1$.

Similarly, by 2.2 with $a=a_1, a_2$ and h_1, there exists a net $(k_j^{(2)})$ of selfadjoint element of A such that, in the weak topology of A:

$$\left[\int \alpha_t(k_j^{(2)})f(t)\,dt, a\right] \to \text{closure}\,[h,a], \qquad a=a_1,a_2,h_1;$$

$$\text{closure}\left[h, \int \alpha_t(k_j^{(2)})f(t)\,dt\right] \to 0\,.$$

Setting a suitable convex combination of $k_j^{(2)}$'s equal to k_2, and $\int\alpha_t(k_2)f(t)\,dt$ equal to h_2, we have (a), (b) and (c) with $n=2$.

It is possible to continue in this way to obtain the desired sequence (k_n).

By [3, Lemma 5], (a) and (c) imply (i). To ensure (ii) we must now instead of the sequence (h_n) construct a net.

Let V be a fixed $*$-ultrastrong neighbourhood of $(h+i)^{-1}$ in A^{**}. We shall now revise slightly the choice of the sequence (k_n) so that all $(h_n+i)^{-1}$ belong to $(h+i)^{-1}+V$.

Choose a $*$-ultrastrong neighbourhood U of 0 in A^{**} such that $U+U\subset V$. We may restrict attention to $\varepsilon>0$ sufficiently small that $(h(1+(\varepsilon h)^2)^{-1}+i)^{-1}\in(h+i)^{-1}+U$. For each $\varepsilon>0$, we may restrict attention to a final subnet of (k_j^ε) such that $\int\alpha_t(k)f(t)\,dt$ is ultrastrongly close to $h(1+(\varepsilon h)^2)^{-1}$ for all k in the convex hull of the k_j^ε's—close enough that

$$\left(\int \alpha_t(k)f(t)\,dt+i\right)^{-1} \in ((h(1+(\varepsilon h)^2)^{-1}+i)^{-1}+U$$

(see [2, Lemma 2]). Now if (k_n) is chosen as above, then for all $n=1,2,\ldots,$

(d) $(h_n+i)^{-1} \in (h+i)^{-1}+V$.

For each *-ultrastrong neighbourhood V of 0 in A^{**}, choose a sequence (h_n) of selfadjoint elements of A as above satisfying (a), (b) and (c) and, moreover, for all n, $(h_n+i)^{-1} \in (h+i)^{-1}+V$. Denote this sequence by $(h_{n,V})$.

With $(m, U) \leqq (n, V)$ understood to mean that $m \leqq n$ and $U \supset V$, we now have a net $(h_{n,V})$.

From (a) and (c) (which hold for $h_n = h_{n,V}$ with V arbitrary) we deduce (i) (by [3, Lemma 5]). From (d) (which holds for $h_n = h_{n,V}$ with n arbitrary) we deduce (ii). (b) we shall use in 2.5 which follows.

2.5. THEOREM. *Let A be a separable C*-algebra, and let α be a universally weakly inner one-parameter automorphism group of A. Denote by δ the generator of α—the closed densely defined derivation of A such that $\alpha_t = \exp t\delta$, $t \in \mathbf{R}$. Let (a_p) be a dense sequence in the domain of δ. Then there exists a sequence (h_n) of selfadjoint elements of A such that*:

(i) $[ih_n, a_p] \to \delta(a_p)$, $\quad p=1,2,\ldots;$

(ii) $[h_m, h_n] \to 0$ $\quad$ (*as* $m,n \to \infty$).

PROOF. Choose h_n as in the proof of 2.1 satisfying (a), (b) and (c), $n=1,2,\ldots.$ Then (a) implies (i), and by the triangle inequality (b) and (c) imply (ii).

3. A lifting theorem.

3.1. THEOREM. *Let A be a separable C*-algebra, let I be a closed two-sided ideal of A, and let α be a continuous one-parameter automorphism group of the quotient C*-algebra A/I such that for some dense sequence (a_p) in the domain of the generator δ of α, the conclusion of* 2.5 (*with A/I in place of A*) *holds for δ. Then there exists a continuous one-parameter automorphism group of A leaving I invariant and inducing α in the quotient.*

PROOF. Since the quotient map $A \to A/I$ is open there is a dense sequence (b_p) in A such that $a_p = b_p + I$, $p=1,2,\ldots.$

Let (h_n) be a sequence of selfadjoint elements of A/I such that 2.5 (i) and 2.5 (ii) hold. Passing to a subsequence we may suppose that:

(i)′ $\|[h_{n+1}-h_n, a_p]\| < 2^{-n}$, $\quad p=1,\ldots,n;$

(ii)′ $\|[h_{n+1}-h_n, h_p]\| < 2^{-n}$, $\quad p=1,\ldots,n.$

By [5] there exists an approximate unit (u_i) for I which is also a central sequence for A, that is, $[u_i, a] \to 0$ for all $a \in A$. Let (g_n) be any sequence in A

such that $g_n + I = h_n$, $n = 1, 2, \ldots$. In much the same way as in [5] and [3] we shall replace the sequence (g_n) by a sequence (k_n), changing each g_n only by an element of I, so that (i)′ and (ii)′ hold with (k_n) in place of (h_n) and (b_p) in place of (a_p).

Let $g_1 = k_1$, and, in view of (i)′ for $n = 1$, set

$$k_1 + (1 - u_{i_1})(g_2 - g_1) = k_2 ,$$

where i_1 is large enough that for $b = b_1, k_1$,

$$\|[1 - u_{i_1}, b](g_2 - g_1)\| + \|(1 - u_{i_1})[g_2 - g_1, b]\| < 2^{-1} ,$$

so that $\|[k_2 - k_1, b]\| < 2^{-1}$. Continue in this way, setting

$$k_n + (1 - u_{i_n})(g_{n+1} - g_n) = k_{n+1} ,$$

where i_n is large enough that (i)′ and (ii)′ hold with (k_n) in place of (h_n) and (b_p) in place of (a_p).

It follows by the triangle inequality that for each $p = 1, 2, \ldots$, as $m, n \to \infty$,

$$[k_m - k_n, a_p] \to 0 ,$$

$$[k_m, k_n] \to 0 .$$

Repeating the above construction with the modification that k_{n+1} is defined to be $k_n + (1 - u_{i_n})(g_{n+1} - g_n)(1 - u_{i_n})^*$ (for suitably large i_n), we have that each k_n is selfadjoint.

Hence by [3, Lemma 5], the sequence

$$(\exp(itk_n) b \exp(-itk_n))$$

is Cauchy, first for $b = b_p$, $p = 1, 2, \ldots$, and therefore for any $b \in A$, uniformly for t in any compact subset of $\mathbf{R}$. The limit defines a continuous one-parameter automorphism group which clearly leaves any closed two-sided ideal of A invariant, and in the quotient by I induces the given one-parameter group α (since, again by [3, Lemma 5], α is the simple limit of the inner groups determined by $\exp ith_n = \exp itk_n + I$, $n = 1, 2, \ldots$).

3.2. Remark. While 3.1 shows that a universally weakly inner one-parameter automorphism group of a quotient of a separable C*-algebra can be lifted to a continuous one-parameter group, it does not show that it can be lifted to be universally weakly inner. (In [3], an automorphism group of the class considered was lifted to another group in the same class.)

A slight modification of the proof of 3.1 shows, though, that if the given group is norm-continuous then it can be lifted to a norm-continuous group. This gives an elementary proof of the lifting theorem in [5]. First, by [7] such a one-parameter group α is universally weakly inner, and if the C*-algebra is separable then Kaplansky's density theorem together with the Hahn–Banach theorem yields a bounded sequence (h_n) in the C*-algebra such that for every element a,

$$[ih_n, a] \to \delta(a) ,$$

where $\alpha_t = \exp t\delta$, $t \in \mathbf{R}$. (One shows this first for all a in a dense sequence (a_p), by taking convex combinations of elements h such that $[ih, a_p]$ converges weakly to $\delta(a_p)$ for finite sets of p. Hence by boundedness, convergence in norm holds for all elements a of the C*-algebra.) If this holds in the quotient A/I of the separable C*-algebra A by the closed two-sided ideal I, then choose a dense sequence (a_p) in A, choose a subsequence of (h_n) which in place of (h_n) satisfies

$$\|[h_{n+1} - h_n, a_p + I]\| < 2^{-n}, \quad p = 1,\ldots,n ,$$

and choose a sequence (g_n) in A with $(g_n + I) = (h_n)$. As in the proof of 3.1, set $g_1 = k_1$, and set

$$k_1 + (1 - u_{i_1})(g_2 - g_1) = k_2$$

where (u_i) is an approximate unit for I which is also a central sequence for A, and i_1 is sufficiently large that

$$\|[(1 - u_{i_1})(g_2 - g_1), a_1]\| < 2^{-1} .$$

Continuing in this way, set

$$k_n + (1 - u_{i_n})(g_{n+1} - g_n) = k_{n+1} ,$$

where i_n is large enough that

$$\|[(1 - u_{i_n})(g_{n+1} - g_n), a_p]\| < 2^{-n}, \quad p = 1,\ldots,n .$$

Then

$$\|[k_{n+1} - k_n, a_p]\| < 2^{-n}, \quad p = 1,\ldots,n ,$$

so for every $p = 1, 2,\ldots$, the sequence $([k_n, a_p])$ is Cauchy.

To show that $\operatorname{ad} ik_n$ converges simply to a bounded derivation of A lifting δ, it is now enough to show that the sequence (k_n) is bounded. For this purpose, we assume that $0 \leqq u_i \leqq 1$ for all i. We suppose also that h_n and g_n are selfadjoint for all n, and that $\|g_n\| = \|h_n\|$ so that the sequence (g_n) is bounded, say by 1. We may choose i_n so that $u_{i_n}^{\frac{1}{2}}$ almost commutes with k_n and $(1 - u_{i_n})^{\frac{1}{2}}$ almost commutes with $g_{n+1} - k_n$ (and also

$$\|[(1 - u_{i_n})(g_{n+1} - k_n), a_p]\| < 2^{-n}, \quad p = 1,\ldots,n) ,$$

so that if, say, $\|k_n\| < 2$ then also $k_n + (1 - u_{i_n})(g_{n+1} - k_n)$ has norm < 2, since it is almost equal to

$$u_{i_n}^{\frac{1}{2}} k_n u_{i_n}^{\frac{1}{2}} + (1 - u_{i_n})^{\frac{1}{2}} g_{n+1} (1 - u_{i_n})^{\frac{1}{2}} ,$$

which lies strictly between $\pm 2(u_{i_n} + (1 - u_{i_n})) = \pm 2$.

3.3. PROBLEM. Let A be a C*-algebra and let δ be a densely defined derivation in A. Suppose that (cf. 2.5, 3.1) there exists a sequence (h_n) of selfadjoint element of A such that:

$$\begin{array}{lll} \text{(i)} & [ih_n, a] \to \delta(a), & a \in \operatorname{domain} \delta; \\ \text{(ii)} & [h_m, h_n] \to 0 & (\text{as } m, n \to \infty). \end{array}$$

Then by [3, Lemma 5], $\operatorname{Ad} \exp ith_n$ converges to a continuous one-parameter automorphism group α of A. Is the generator of α the closure of δ?

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