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ČEBYŠEV SUBSPACES OF C*-ALGEBRAS

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A subspace V of a Banach space A is called a Čebyšev subspace if each vector in A admits a unique closest point in V. The story of Čebyšev subspaces is long and the monograph [7] tells it in painful detail. The highlight, however, is the theorem of Haar [2] characterizing an n-dimensional Čebyšev subspace of $C(X)$, X compact, by the condition that no non-zero function in the subspace has more than $n-1$ zeros.

In the first section of the paper we shall consider the extension of the Haar's theorem to non-commutative C*-algebras. The second section contains a negative result about the existence of Čebyšev C*-subalgebras — they are almost all trivial.

The study of Čebyšev subspaces in operator algebras was initiated by Guyan Robertson in [5], later joined by Yost in [6]. Most of what we shall do is directly inspired by their work. Thus the characterization of one-dimensional Čebyšev subspaces was conjectured in [6] and the necessity of the condition, i.e. the implication (iii) $\Rightarrow$ (i) in our Theorem 3 is Proposition 1.6 of [6]. Looking for Čebyšev C*-subalgebras B of a C*-algebra A, Guyan Robertson and Yost find the trivial examples $B=A$ and $B=\mathbf{C}1$; and in the case $A=\mathbf{M}_2$ they also find $B=\mathbf{C}^2$ (the diagonal matrices). They then ask whether these examples are the only Čebyšev C*-subalgebras and prove in [6, Theorem 1.5] that such is the case if A is a von Neumann algebra of type I. As we shall see, their result is actually true for general C*-algebras.

We use the notation from [4]. Thus A will denote a C*-algebra, A'' its enveloping von Neumann algebra. Moreover, regarding A as a subalgebra of A'' we let $\tilde{A}=A+\mathbf{C}1$ and denote by $M(A)$ the C*-algebra of multipliers of A in A''. When $1 \in A$ we have, of course, $A=\tilde{A}=M(A)$.

1. Finite-dimensional Čebyšev subspaces.

Our first result is a straightforward adaptation of known procedures (cf. [7, p. 29]) to the case of C*-algebras.

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LEMMA 1. *Let V be a finite-dimensional subspace of a C*-algebra A and assume that V is not a Čebyšev subspace. There are then elements y in A, x_0 in $V \setminus \{0\}$, and a state φ of A such that*

(i) $\varphi(yy^*) = \|y\| = \|\varphi\| = 1$;
(ii) $\varphi(x_0^*x_0) = \varphi(yx_0x_0^*y^*) = 0$;
(iii) $\varphi(yV) = 0$.

PROOF. By assumption there is an element y in A such that y^* has at least two nearest points in V. We may assume that zero is one and denote the other by x_0. Normalizing y we may take $\|y\| = 1$. Thus $\|y^*\| = \|y^* - x_0\| = 1$ and $\|y^* - x\| \geqq 1$ for all x in V. The functional ψ_0 on $\mathbf{C}y^* + V$ defined by $\psi_0(\lambda y^* + x) = \lambda$ is bounded by 1 and extends by Hahn–Banach's theorem to a functional ψ on A with $\|\psi\| = 1$. Regarding A as operators on its universal Hilbert space H (cf. [4, 3.7]) there are then unit vectors ξ, η in H such that $\psi(a) = (a\xi \mid \eta)$ for all a in A. Since $(y^*\xi \mid \eta) = 1$ and $\|y\| = 1$ we conclude that $y^*\xi = \eta$ and $y\eta = \xi$. Since moreover $((y^* - x_0)\xi \mid \eta) = 1$ and $\|y^* - x_0\| = 1$ we have $x_0\xi = x_0^*\eta = 0$. Finally, $(V\xi \mid \eta) = 0$. Put $\varphi(a) = (a\xi \mid \xi)$, $a \in A$. It is straightforward to verify that the triple y, x_0, φ satisfies the conditions (i), (ii) and (iii).

LEMMA 2. *Let V be a finite dimensional subspace of a C*-algebra A and assume that V is not a Čebyšev subspace of A. There is then a unitary operator u in $\tilde{A}$, a non-zero element x_0 in V and a non-empty, finite set $\{\varphi_i\}$ of orthogonal pure states of A (i.e. $\|\varphi_i - \varphi_j\| = 2$) such that $\varphi_i(x_0^*x_0) = \varphi_i(ux_0x_0^*u^*) = 0$ for all i. If there is a finite upper bound for the possible cardinalities of sets $\{\varphi_i\}$ we can further find a convex combination $\varphi = \sum \lambda_i\varphi_i$ such that $\varphi(uV) = 0$.*

PROOF. Choose y, x_0 and φ as in Lemma 1, and let p denote the support of φ, i.e. p is the smallest projection in the enveloping von Neumann algebra A'' such that $\varphi(p) = 1$. Since p is a closed projection, hence universally measurable, and the atomic representation of A (extended to A'') is isometric on the space of universally measurable operators (cf. 4.3.13 and 4.3.15 of [4]) we may consider p as a projection on the atomic Hilbert space H_a of A. Take any finite subset $\{\xi_i\}$ of an orthonormal basis for pH_α, chosen such that each basis vector belongs to only one irreducible space. It is immediate from Kadison's transitivity theorem [1, Corollary 7] (or [4, 2.7.5] which covers also the case $1 \notin A$) that if $\varphi_i(a) = (a\xi_i \mid \xi_i)$, $a \in A$, then $\{\varphi_i\}$ is a set of orthogonal pure states of A.

Since $\varphi(yy^*) = \|y\| = 1$ we have $p \leqq yy^*$, whence $yy^*\xi_i = \xi_i$ for all i. It follows that $\{y^*\xi_i\}$ is an orthogonal set of unit vectors, so by the transitivity theorem quoted above there is a unitary u in $\tilde{A}$ such that $u^*\xi_i = y^*\xi_i$. As $\varphi(x_0^*x_0) = 0$ implies $x_0^*x_0 \leqq 1 - p$ we get $\varphi_i(x_0^*x_0) = 0$ for all i. Likewise

$$0 = \varphi_i(yx_0x_0^*y^*) = \varphi_i(ux_0x_0^*u^*)$$

for all i.

The number of φ_i's is arbitrary unless p has finite dimension. In that case φ is an atomic functional and can be regarded as a functional on the matrix algebra $pA''p$. We can therefore express φ as a convex combination $\varphi=\sum\lambda_i\varphi_i$ of orthogonal pure states $\{\varphi_i\}$ (see [3, Proposition 4]).

For every x in V we have

$$\varphi(ux) = \sum \lambda_i(ux\xi_i \mid \xi_i) = \sum \lambda_i(yx\xi_i \mid \xi_i) = \varphi(yx) = 0\,,$$

as desired.

LEMMA 3. *Let V be a subspace of a* C*-*algebra A and assume that there are elements y in A, x_0 in $V\setminus\{0\}$, and a state φ of A satisfying the three conditions in Lemma* 1. *Let $y=bv$ be a factorization with b in A_+, v in $M(A)$ and $\varphi(b)=\|b\|=1$. Then neither vV nor Vv are Čebyšev subspaces of A.*

PROOF. Put $x_1=vx_0$ and note that $\varphi(x_1^*x_1)=\varphi(x_1x_1^*)=0$ by condition (ii). Further, set

$$a = x_1^*x_1+(1-x_1^*x_1)^{\frac{1}{2}}b(1-x_1^*x_1)^{\frac{1}{2}}\,.$$

Then $\varphi(a)=\|a\|=1$, and since $x_1^*x_1\leqq a$ there is by [4, 1.4.5] an element x_2 in A such that $x_1=x_2a^{1/3}$. In fact, x_2 is the norm limit of elements of the form $x_1(n^{-1}+a)^{-1/3}$. It follows from this that $\varphi(x_2^*x_2)=\varphi(x_2x_2^*)=0$.

Put $x_2=h+ik$ with h,k in A_{sa} and define

$$z = (\|h\|-|h|+\|k\|-|k|)a^{1/3}\,.$$

Since $x_2^*x_2+x_2x_2^*=2(h^2+k^2)$ we have $\varphi(|h|+|k|)=0$. Therefore

$$\|z-vx\| \geqq |\varphi(z-vx)| = \|h\|+\|k\|$$

for every x in V, as $\varphi(vV)=\varphi(bvV)=0$. On the other hand, if $|\lambda|\leqq 1$,

$$\|z-\lambda x_1\| = \|(\|h\|-|h|+\|k\|-|k|)a^{1/3}-\lambda x_2a^{1/3}\|$$

$$\leqq \|\,\|h\|-|h|-\lambda h\|+\|\,\|k\|-|k|-i\lambda k\| \leqq \|h\|+\|k\|\,,$$

since for every complex homomorphism ω of $C^*(h)$ we have

$$|\,\|h\|-\omega(|h|)-\lambda\omega(h)| \leqq \|h\|-\omega(|h|)+|\lambda\omega(h)| \leqq \|h\|\,.$$

Since $x_1 \in vV$ this set is not a Čebyšev subspace of A.

The result for Vv follows by considering the state $\psi=\varphi(v\cdot v^*)$, which vanishes on Vv, and reason on the element x_0v as above.

THEOREM 1. *Let V be an n-dimensional subspace of a C*-algebra A and assume that there is a unitary u in $M(A)$ and a non-zero element x_0 in V such that $\varphi_i(x_0^*x_0)=\varphi_i(ux_0x_0^*u^*)=0$ for at least n orthogonal pure states $\varphi_1,\ldots,\varphi_n$ of A. Then V is not a Čebyšev subspace of A.*

PROOF. Since uV has the Čebyšev property if and only if V has it, we may assume that $u=1$ replacing otherwise x_0 and V by ux_0 and uV.

Let $x_1,\ldots,x_n$ be a basis for V and consider the matrix with elements $\alpha_{ij}=\varphi_i(x_j)$. If $x_0=\sum\lambda_jx_j$ we have $\sum\alpha_{ij}\lambda_j=0$ for all i. Thus the determinant of (α_{ij}) is zero. Consequently there is a non-trivial solution $\mu_1,\ldots,\mu_n$ to the system of equations $\sum\alpha_{ij}\mu_i=0$, $1\leqq j\leqq n$. Normalizing such that $\sum|\mu_i|=1$ we put $\varphi=\sum|\mu_i|\varphi_i$.

Realizing A as operators on its atomic Hilbert space (cf. [4, 4.3.7]) there are orthogonal unit vectors $\xi_1,\ldots,\xi_n$ such that $\varphi_i(a)=(a\xi_i\,|\,\xi_i)$. By Kadison's transitivity theorem ([1, 7] or [4, 2.7.5]) there is an element b in A_+ with $\|b\|=1$ such that $b\xi_i=\xi_i$ for all i and also a unitary v in $\tilde{A}$ with $v\xi_i=\mu_i|\mu_i|^{-1}\xi_i$ if $\mu_i\neq0$, $v\xi_i=\xi_i$ otherwise. Put $y=bv$ and note that $\varphi(b)=1$ and $\varphi(v\cdot v^*)=\varphi$.

For each x_j we have

$$\varphi(yx_j) = \varphi(vx_j) = \sum \mu_i\varphi_i(x_j) = 0\,,$$

whence $\varphi(yV)=0$. Furthermore,

$$\varphi(x_0^*x_0) = 0 = \varphi(x_0x_0^*) = \varphi(vx_0x_0^*v^*) = \varphi(yx_0x_0^*y^*)$$

so that the triple y, x_0 and φ satisfies the three conditions in Lemma 1. It follows from Lemma 3 that vV is not a Čebyšev subspace and since v is unitary, neither is V.

THEOREM 2. *Let V be an n-dimensional subspace of a C*-algebra A. The following conditions are equivalent:*

(i) *V is not a Čebyšev subspace;*

(ii) *There is unitary operator u in $\tilde{A}$, a non-zero element x_0 in V and an atomic state φ, which is a convex combination of m orthogonal pure states, such that*

$$\varphi(x_0^*x_0) = \varphi(ux_0x_0^*u^*) = 0\,.$$

If $m<n$ we further have $\varphi(uV)=0$.

PROOF. (ii) ⇒ (i) follows from Theorem 1 and (if $m<n$) from Lemma 3 by taking $y=bu$, where $b\in A_+$ with $\varphi(b)=\|b\|=1$. The existence of such a b is assured by Kadison's transitivity theorem.

(i) ⇒ (ii): If we cannot find n orthogonal pure states $\{\varphi_i\}$ such that $\varphi_i(x_0^*x_0)=\varphi_i(ux_0x_0^*u^*)=0$ for some unitary u in $\tilde{A}$ and some non-zero x_0 in V then the result follows from Lemma 2. If we can find n such pure states the assumptions in Theorem 1 are satisfied, and the proof of the theorem shows the existence of a state $\varphi = \sum |\mu_i| \varphi_i$ and a unitary v in $\tilde{A}$ such that $\varphi(v \cdot v^*) = \varphi$ and $\varphi(v(uV)) = 0$. Moreover,

$$\varphi(x_0^* x_0) = 0 = \varphi(u x_0 x_0^* u^*) = \varphi(v u x_0 x_0^* u^* v^*) ,$$

so that (ii) is satisfied with vu in place of u.

REMARK 1. In the commutative case there is a converse to Theorem 1, viz. Haar's theorem, [2] or [7, p. 215]. This theorem is easily obtained from Theorem 2 by linear algebra plus the (commutative) fact that $|\varphi(x)| = \varphi(|x|)$ for any pure state. In the non-commutative case this is not so, and Theorem 2 seems to be the best we can do.

A specific counterexample to the converse of Theorem 1 is obtained by taking $A = \mathsf{M}_3$ and $V = \mathbf{C}e_{11} + \mathbf{C}e_{22} + \mathbf{C}e_{33}$, i.e. V is the 3-dimensional space of diagonal 3×3-matrices. Since V is a C*-subalgebra of A it follows from Theorem 5 that V is not a Čebyšev subspace of A. This can also be verified by direct computation, since the distance from e_{12} to an element $x = \alpha e_{11} + \beta e_{22} + \gamma e_{33}$ in V does not depend on γ if $|\gamma| \leqq 1$. However, if $\varphi_1, \varphi_2, \varphi_3$ are three orthogonal pure states of A then $\varphi_1 + \varphi_2 + \varphi_3 = \mathrm{Tr}$. Therefore we cannot have $\varphi_i(x^* x + w x x^* w^*) = 0$ for all i if x is a non-zero element of V.

The one-dimensional case can, however, be expressed in the classical form.

THEOREM 3. *Let x_0 be a non-zero element in a* C*-*algebra A. The following conditions are equivalent*:

(i) $\mathbf{C}x_0$ *is a* Čebyšev subspace of A;

(ii) $x_0^* x_0 + u x_0 x_0^* u^*$ *is strictly positive in A for every unitary u in $M(A)$*;

(iii) *In no irreducible representation (π, H) of A do the operators $\pi(x_0)$ and $\pi(x_0^*)$ both have zero as an eigenvalue.*

PROOF. (i) $\Leftrightarrow$ (ii) is a special case of Theorem 2. Since the unitary operators in $M(A)$ act transitively on the set of unit vectors in any irreducible representation it is straightforward to show that (ii) $\Leftrightarrow$ (iii).

REMARK 2. Note that when $1 \in A$, condition (ii) says that $x^* x + u x x^* u^*$ is invertible for every unitary u in A.

PROPOSITION 1. *Let x_0 be an element in a* C*-*algebra A with unit and assume that x_0 is not proportional to* 1. *The following conditions are equivalent*:

(i) *The 2-dimensional space $V = \mathbf{C}1 + \mathbf{C}x_0$ is a Čebyšev subspace of A*;

(ii) *For a given complex number λ there is at most one irreducible representation (π, H) of A (up to equivalence) in which x_0 and x_0^* have the eigenvalues λ and $\bar{\lambda}$, respectively. Moreover, none of the multiplicities of λ and $\bar{\lambda}$ in H exceed 1 and the corresponding eigenvectors are not orthogonal.*

PROOF. (i) $\Rightarrow$ (ii): Suppose that (π_1, H_1) and (π_2, H_2) are inequivalent irreducible representations such that

$$\pi_i(x_0-\lambda)\xi_i = \pi_i(x_0-\bar{\lambda})\eta_i = 0$$

for some unit vectors ξ_i, η_i in H_i, $i=1,2$. Choose by Kadison's transitivity theorem a unitary u in A such that $\pi_i(u)\eta_i=\xi_i$, $i=1,2$, and define two orthogonal pure states φ_1 and φ_2 by $\varphi_i(y)=(\pi_i(y)\xi_i \mid \xi_i)$, $y \in A$. Then

$$\begin{aligned}\varphi_i((x_0-\lambda)^*(x_0-\lambda)+u(x_0-\lambda)(x_0-\lambda)^*u^*)\\ = \|\pi_i(x_0-\lambda)\xi_i\|^2+\|\pi_i(x_0^*-\bar{\lambda})\eta_i\|^2 = 0\end{aligned}$$

for $i=1,2$, which by Theorem 1 implies that V is not a Čebyšev subspace of A.

If (π, H) is an irreducible representation of A in which x_0 and x_0^* have eigenvalues λ and $\bar{\lambda}$ we must exclude the cases where the eigenspaces of λ or $\bar{\lambda}$ have dimension higher than 1 and where the eigenvectors corresponding to λ and $\bar{\lambda}$ are orthogonal. Assume that

$$\pi(x_0-\lambda)\xi = \pi(x_0^*-\bar{\lambda})\eta = 0$$

for some unit vectors ξ, η in H. If $\pi(x_0-\lambda)\xi'=0$ for some unit vector ξ' not proportional to ξ then we can find a linear combination ξ'' of ξ and ξ' such that $(\xi'' \mid \eta)=0$. Similarly if $\pi(x_0^*-\bar{\lambda})\eta'=0$ we find η'' such that $(\xi \mid \eta'')=0$. We may therefore concentrate on the last case and assume that $(\xi \mid \eta)=0$.

Choose a unitary u in A such that $\pi(u)\eta=\xi$ and define $\varphi(y)=(\pi(y)\xi \mid \xi)$, $y \in A$. Then as before

$$\begin{aligned}\varphi((x_0-\lambda)^*(x_0-\lambda)+u(x_0-\lambda)(x_0-\lambda)^*u^*)\\ = \|\pi(x_0-\lambda)\xi\|^2+\|\pi(x_0^*-\bar{\lambda})\eta\|^2 = 0\,.\end{aligned}$$

Moreover

$$\varphi(u1) = (\pi(u)\xi \mid \xi) = 0\,,$$

$$\varphi(ux_0) = (\pi(ux_0)\xi \mid \xi) = \lambda(\pi(u)\xi \mid \xi) = 0\,,$$

so that $\varphi(u\cdot)$ annihilates the subspace V. It follows from Theorem 2 that V is not a Čebyšev subspace of A.

(ii) $\Rightarrow$ (i): If V is not a Čebyšev subspace of A then by Theorem 2 there is either a pure state φ of A, a complex number λ and a unitary u in A such that $\varphi(uV)=0$ and

$$\varphi((x_0-\lambda)^*(x_0-\lambda)+u(x_0-\lambda)(x_0-\lambda)^*u^*) = 0;$$

or else there are two orthogonal pure states φ_1 and φ_2 of A, a complex number λ and a unitary u in A such that

$$\varphi_i((x_0-\lambda)^*(x_0-\lambda)+u(x_0-\lambda)(x_0-\lambda)^*u^*) = 0$$

for $i=1,2$.

It is straightforward to check that the first case corresponds to x_0 and x_0^* having eigenvalues λ and $\bar{\lambda}$, respectively, with orthogonal eigenvectors (since $\varphi(u1)=0$); whereas the second case corresponds either to x_0 and x_0^* having eigenvalues λ and $\bar{\lambda}$ in two (classes of) irreducible representations (if φ_1 and φ_2 are inequaivalent) or to x_0 and x_0^* having eigenvalues λ and $\bar{\lambda}$ in one irreducible representation, both with multiplicity 2 (if φ_1 and φ_2 are equivalent). But these are precisely the situations prohibited by (ii).

REMARK 3. Let $A=\mathbf{M}_3$ and consider the matrix $x_0=e_{22}+2e_{33}$. The eigenvalues of x_0 have multiplicity 1, and $x_0=x_0^*$. Thus x_0 satisfies the conditions in (ii) so that $\mathbf{C}1+\mathbf{C}x_0$ is a 2-dimensional Čebyšev subspace of A. However, the 3-dimensional subspace $\mathbf{C}1+\mathbf{C}x_0+\mathbf{C}x_0^2$ is equal to the set of diagonal elements in $\mathbf{M}_3$ and therefore not a Čebyšev subspace, cf. Remark 1.

Given Theorem 2 it should in principle be possible to find necessary and sufficient conditions on an element x_0 in a C*-algebra under which each subspace of polynomials in x_0 with a given degree is a Čebyšev subspace. Proposition 1 may serve as an example of the difficulties one must expect in order to generalize Čebyšev's original theorem [8].

2. Čebyšev subalgebras.

Throughout this section B will denote a non-zero C*-subalgebra of A which at the same time is a Čebyšev subspace. For each x in A we shall denote by $\alpha(x)$ the unique best approximant to x in B. Note that $\alpha(x+b)=\alpha(x)+b$ for every b in B.

As proved in [6, Theorem 1.3], if A has a unit 1 then $1 \in B$; and if B has a unit this is also a unit for A.

LEMMA 4. *For each selfadjoint x in B with 0 in* Sp (x) *we have $xAx \subset B$.*

PROOF. Assume first that $1 \in A$ (whence $1 \in B$). Given $\varepsilon>0$ there are continuous functions f, g and h on Sp (x) such that f, g vanish in an neighbourhood of 0, $h(0)=1$; and such that $fg=f$, $gh=0$ and $|f(t)-t| \leqq \varepsilon$ for all t in Sp (x). Applying these functions to x we obtain elements y, e and z in B

such that $ye=y$, $ez=0$ and $\|y-x\| \leqq \varepsilon$. Since $0 \in \mathrm{Sp}(x)$ we have $z \neq 0$ and may assume therefore that $\|e\| \leqq \|z\|=1$. If $yAy \not\subset B$ take a in $yAy \setminus B$ and let $b = \alpha(a)$. Then

$$\|a-ebe\| = \|e(a-b)e\| \leqq \|a-b\| ,$$

whence $ebe=b$ since b is unique. But for each λ in $\mathbf{C}$ we have

$$\|a-(b+\lambda z)\| = \|e(a-b)e-\lambda z\| = \|a-b\| \vee \|\lambda z\| ,$$

so that $b+\lambda z=\alpha(a)$ for all λ with $|\lambda| \leqq \|a-b\|$, a contradiction. Consequently $yAy \subset B$. With $\varepsilon = n^{-1}$ we find a sequence $\{y_n\}$ in B such that $y_n \to x$ and $y_nAy_n \subset B$ for all n. Since B is closed it follows that $xAx \subset B$.

If $1 \notin A$ there are two cases: either 0 is an isolated point in $\mathrm{Sp}(x)$ or not. If not, we can use essentially the same argument as above; taking now $h(0)=0$ (in order that $z=h(x) \in B$), but $h(\lambda)=1$ for some λ in $\mathrm{Sp}(x) \cap [-\varepsilon/2, \varepsilon/2]$.

If 0 is isolated in $\mathrm{Sp}(x)$ there is by spectral theory a projection e in B with $ex=x$. Since $1 \notin A$, e is not a unit for B, so there is a non-zero element z in $(1-e)B(1-e)$. If now $xAx \not\subset B$ take a in $xAx \setminus B$ and put $b=\alpha(a)$. As we saw above this implies that $b=ebe$, and again we reach a contradiction by showing that $b+\lambda z=\alpha(a)$ for all small λ.

THEOREM 4. *If A is a C^*-algebra without unit and B a Čebyšev C^*-subalgebra then $B=A$.*

PROOF. Since $1 \notin A$, $0 \in \mathrm{Sp}(x)$ for every x in B_{sa}, whence $xAx \subset B$ by Lemma 4. It follows from the polarization identity that $xAy \subset B$ for all x,y in B.

We claim that $xa \in B$ for all x in B and a in A. Indeed, take y in B_{sa} and put $u = \exp(\mathrm{i}ty)$ in $\tilde{A}$. Then $1-u \in B$ so $\alpha(xa)u \in B$. Since u is unitary it follows that $\alpha(xau)=\alpha(xa)u$, so the distance of xa and xau to B is the same. On the other hand, $xa(1-u) \in B$ from what we proved above. Since B is a Čebyšev subspace this implies that $\alpha(xau)=\alpha(xa)-xa(1-u)$.

Combining these equations gives $\alpha(xa)u=\alpha(xa)-xa(1-u)$ or

$$0 = (xa-\alpha(xa))(1-u) = (xa-\alpha(xa))(1-\exp(\mathrm{i}ty)) .$$

Differentiating at $t=0$ we conclude that $(xa-\alpha(xa))y=0$ for every y in B_{sa}. Put $z=xa-\alpha(xa)$ and note that $\alpha(z)=0$. However, z is orthogonal to B, so $\|z-b\| = \|z\| \vee \|b\|$ for every b in B. Since $B \neq 0$ this leads to a contradiction unless $z=0$, whence $xa=\alpha(xa) \in B$.

Now take a in A and x in B_{sa} and put $v=\exp(\mathrm{i}tx)$ in $\tilde{A}$. Then $1-v \in B$ whence $(1-v)a \in B$ from the argument above. Consequently

$$v\alpha(a) = \alpha(va) = \alpha(a-(1-v)a) = \alpha(a)-(1-v)a ,$$

or, equivalently,

$$0 = (1-v)(a-\alpha(a)) = (1-\exp(\mathrm{i}tx))(a-\alpha(a)).$$

Differentiating a $t=0$ gives $x(a-\alpha(a))=0$ for every x in B_{sa} so that $a-\alpha(a)$ is orthogonal to B. As above this forces $a-\alpha(a)=0$, and therefore $A=B$.

LEMMA 5. *If* $1 \in A$ *and* $x \in B$ *with* $0 \leqq x \leqq 1$, *such that* $\{0,1\} \subset \mathrm{Sp}(x)$ *then* $x(1-x)A \subset B$.

PROOF. Applying Lemma 4 to x and $1-x$ we obtain $xAx+(1-x)A(1-x) \subset B$, whence

$$x(1-x)A = x(1-x)Ax+x(1-x)A(1-x) \subset B.$$

LEMMA 6. *If* $1 \in A$ *and if* $\mathrm{Sp}(x)$ *contains more than two points for some* x *in* B_{sa} *then* $B=A$.

PROOF. Take a point λ_0 in $\mathrm{Sp}(x)$. By assumption there are at least two more points λ_1, λ_2 in $\mathrm{Sp}(x)$. Let f be a continuous function on $\mathrm{Sp}(x)$ with $f(\lambda_0)=\frac{1}{2}$, $f(\lambda_1)=0$, $f(\lambda_2)=1$ and $0 \leqq f \leqq 1$. Put $g=f(1-f)$ and note that $g(\lambda_0)=1/4$. Moreover, by Lemma 5 we have $g(x)A \subset B$, since $\{0,1\} \subset \mathrm{Sp}(f(x))$. Applying this argument to every point in $\mathrm{Sp}(x)$ together with a standard compactness argument produce a finite set $\{g_\varkappa\}$ of continuous functions on $\mathrm{Sp}(x)$ with $\sum g_\varkappa(\lambda)>0$ for all λ in $\mathrm{Sp}(x)$ and such that $g_\varkappa(x)A \subset B$ for every $\varkappa$. Then $y=\sum g_\varkappa(x)$ is an invertible element in B with $yA \subset B$, whence $A=B$.

THEOREM 5. *If* A *is a* C**-algebra with unit,* B *a Čebyšev* C**-subalgebra of* A *then either* $B=A$, $B=\mathbf{C}1$ *or else* $A=\mathbf{M}_2$ *and* B *is isomorphic to the algebra of diagonal matrices.*

PROOF. Assume that B is neither A nor $\mathbf{C}1$. The latter condition shows that B_{sa} contains an element e with at least two points in its spectrum. Using spectral theory we may assume that $0 \leqq e \leqq 1$ and $\{0,1\} \subset \mathrm{Sp}(e)$. Since $B \neq A$ we see from Lemma 6 that $\{0,1\}=\mathrm{Sp}(e)$, i.e. e is a non-trivial projection. By Lemma 4, $eAe \subset B$. Moreover, each self-adjoint element y in eAe must be proportional to e, since otherwise we can find y such that $\mathrm{Sp}(y+1-e)$ contains more than two points, in contradiction with Lemma 6. Thus $eAe=\mathbf{C}e$ and likewise $(1-e)A(1-e)=\mathbf{C}(1-e)$. This is only possible for $A=\mathbf{M}_2$ or $A=\mathbf{C}^2$. The latter possibility is ruled out since $B \neq A$, so we must have $A=\mathbf{M}_2$. Now $eAe+(1-e)A(1-e) \subset B$ and since B is a proper subalgebra of $\mathbf{M}_2$ we must have equality above; i.e. B is the algebra $\mathbf{C}e+\mathbf{C}(1-e)$ of diagonal matrices.

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