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A CLASSIFICATION OF IDEALS IN CROSSED PRODUCTS

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Abstract.

Given a C*-dynamical system (A,G,α) and a closed subgroup H of the locally compact abelian group G we show that the crossed product $G \times_\alpha A$ is prime if and only if two conditions are satisfied: (a) the sub-crossed product $H \times_\alpha A$ is G-prime (any two non-zero G-invariant ideals of $H \times_\alpha A$ have non-zero intersection) and (b) the Connes spectrum $\Gamma(\alpha)$ contains the annihilator $H^\perp$ of H in the dual group $\hat{G}$ of G. When G/H is discrete, we obtain that the crossed product is simple if and only if (a) $H \times_\alpha A$ is G-simple (has no non-trivial G-invariant ideals) and (b) $\Gamma(\alpha)$ contains $H^\perp$.

Introduction.

Let (A,G,α) be a C*-dynamical system, $(G \times_\alpha A, \hat{G}, \hat{\alpha})$ its dual system. In an earlier paper [8] it was shown that the existence of G-invariant ideals in A is equivalent to the existence of $\hat{G}$-invariant ideals in $G \times_\alpha A$. Here we pursue this line of thought to show that given a closed subgroup H of G, the existence of G-invariant ideals in the "sub"-crossed product $H \times_\alpha A$ is equivalent to the existence of $H^\perp$-invariant ideals in $G \times_\alpha A$. The result is based on the characterization of $H \times_\alpha A$ obtained in [6], and indicates that it is of interest to classify the ideals in $G \times_\alpha A$ by the largest closed subgroup of $\hat{G}$ under which they are invariant.

Using the established correspondence between ideals in $H \times_\alpha A$ and $G \times_\alpha A$ we get new criteria for the simplicity (and primeness) of $G \times_\alpha A$, formulated in terms of properties of $H \times_\alpha A$. Our method also yields new results for W*-dynamical systems, extending results in [4].

1. Notation.

This paper constitutes a sequel of [6] and [8], to which we refer the reader for terminology. For more general reference on C*- and W*-dynamical systems one may consult [9]. Let (A,G,α) be a C*-dynamical system, H a

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closed subgroup of the locally compact abelian group G. The action α^H of G on the space $K(H,A)$ of continuous functions from H into A with compact support is defined by

$$(\alpha_t^H y)(s) = \alpha_t(y(s))$$

and this is easily seen to coincide with the action $\operatorname{Ad}\lambda$ (as defined in [8]) on $M(G \times_\alpha A)$ when restricted to $K(H,A)$.

By α^H we shall henceforth mean the canonical extension of the action on $K(H,A)$ to all of $H \times_\alpha A$, and whenever we speak of G acting on $H \times_\alpha A$ we think of $(H \times_\alpha A, G, \alpha^H)$.

2. An ideal correspondence.

LEMMA 2.1. *Let (A,G,α) be a C*-dynamical system, $(G \times_\alpha A, \hat{G}, \hat{\alpha})$ its dual system. Let H be a closed subgroup of $G, H^\perp$ its annihilator in $\hat{G}$. For every non-trivial $H^\perp$-invariant ideal J in $G \times_\alpha A$ let $I_H(J)$ denote the closed linear span of*

$$\{x \in H \times_\alpha A \mid \exists y \in J : x = I_H(y)\}.$$

Then $I_H(J)$ is a non-trivial G-invariant ideal in $H \times_\alpha A$.

PROOF. By [6, Lemma 2.6] $I_H(J)$ is non-zero when J is non-zero since $\lambda_f y \lambda_g \in J$ when $y \in J$ and f,g are in $K(G)$. Furthermore, $I_H(J) \neq H \times_\alpha A$ when $J \neq G \times_\alpha A$. Indeed, let φ be a non-zero functional on $G \times_\alpha A$ which annihilates J. By the $H^\perp$-invariance of J, φ annihilates $I_H(J)$, but φ cannot annihilate all of $H \times_\alpha A$ since this contains an approximate unit for $G \times_\alpha A$. That $I_H(J)$ is a closed *-subspace of $H \times_\alpha A$ is obvious, and that it is an ideal follows easily from the pointwise $H^\perp$-invariance of the elements of $H \times_\alpha A$ which ensures that

$$x_1 I_H(y) = I_H(x_1 y)$$

whenever $x_1 \in H \times_\alpha A$, $y \in B_0^* B_0 \cap J$. The G-invariance follows from

$$I_H(\lambda_t y \lambda_t^*) = \lambda_t I_H(y) \lambda_t^*$$

for all y in $B_0^* B_0$ and t in G.

LEMMA 2.2. *Let (A,G,α), $(G \times_\alpha A, \hat{G}, \hat{\alpha})$ and $H^\perp$ be as in 2.1. Let N be a non-trivial G-invariant ideal in $H \times_\alpha A$. Denote by* $\operatorname{Ext} N$ *the closed linear span of elements*

$$\{xy, yx \mid x \in N, y \in G \times_\alpha A\}.$$

Then $\operatorname{Ext} N$ *is a non-trival $H^\perp$-invariant ideal in $G \times_\alpha A$.*

PROOF. We regard $H \times_\alpha A$ as the C*-subalgebra of $M(G \times_\alpha A)$ satisfying the conditions of [6, Theorem 2.1]. By the G-invariance of N, one easily verifies that $\operatorname{Ext} N$ is an ideal in $G \times_\alpha A$. The $H^\perp$-invariance follows from

$$\hat{\alpha}_\gamma(xy) = x\hat{\alpha}_\gamma(y), \qquad \gamma \in H^\perp ,$$

whenever $x \in N$ and $y \in G \times_\alpha A$. When N is non-zero, so is $\operatorname{Ext} N$—if not, some non-zero x in N would be orthogonal to $G \times_\alpha A$, which is impossible. When N is not all of $H \times_\alpha A$ we want to conclude $\operatorname{Ext} N \neq G \times_\alpha A$. Let $\tilde{\pi}^0$ be a representation of $H \times_\alpha A$ on a Hilbert space $\mathscr{H}^0$, or equivalently a pair (π^0, u^0), where π^0 is a representation of A on $\mathscr{H}^0$ and u^0 is a unitary representation of H on $\mathscr{H}^0$ such that for s in H and x in A

$$\pi^0(\alpha_s(x)) = u_s^0 \pi^0(x) u_{-s}^0 .$$

Let $\tilde{\pi} = (\pi, u)$ be the induced representation of $G \times_\alpha A$ (or covariant representation of (A, G, α)) on the Hilbert space $\mathscr{H}$ of measurable functions $\xi\colon G \to \mathscr{H}^0$ satisfying that for all s in H and t in G $\xi(t+s) = u_{-s}^0 \xi(t)$ and that $\int_{G/H} \|\xi\|^2 < \infty$, as defined in [11, Section 3]. Clearly (π, u) restricts to a covariant representation of (A, H, α), and hence

$$\tilde{\pi}''(x) = \int_H \pi(x(s)) u_s \, ds$$

extends from a representation of $K(H, A)$ to $H \times_\alpha A$, cf. [9]. Since $(u_s\xi)(t) = \xi(t-s) = (u_s^0\xi)(t)$ for s in H we have

$$\tilde{\pi}(x)(t) = \tilde{\pi}^0(\alpha_{-t}(x))(t)$$

for all t in G. This shows that by choosing a non-trivial $\tilde{\pi}^0$ of $H \times_\alpha A$ such that $N \subset \ker \tilde{\pi}^0$ one obtains from the G-invariance of N that $N \subset \ker \tilde{\pi}''$, hence $\operatorname{Ext} N \subset \ker \tilde{\pi}$, and the conclusion follows.

REMARK 2.3. The above direct proof that $N \neq H \times_\alpha A$ implies $\operatorname{Ext} N \neq G \times_\alpha A$ was pointed out by the referee. Originally, a different proof of this was given, using the complicated machinery of [5]. For the benefit of the reader acquainted with [5] it might be worthwhile to point out that one can deduce from [5, Proposition 12] that the quotient of $G \times_\alpha A$ by $\operatorname{Ext} N$ is isomorphic to the twisted crossed product of G with the quotient of $H \times_\alpha A$ by N. Hence if the latter is non-trivial, so is the former.

REMARK 2.4. In [8], we worked with the case $H = \{0\}$ and established the correspondence from a $\hat{G}$-invariant ideal in $G \times_\alpha A$ to a G-invariant ideal in A exactly as in 2.1 above—i.e. by using the conditional expectation $I = I_{\{0\}}$. To go back, however, from a G-invariant ideal N in A to a $\hat{G}$-invariant J in $G \times_\alpha A$

we simply referred to the Takai duality (cf. [10]), using that $N \otimes C(L^2(G))$ naturally identifies with a G-invariant ideal in the double crossed product, and thus maps onto a $\hat{G}$-invariant ideal in $G \times_\alpha A$.

This construction is in fact equivalent to the Ext-operation used in 2.2. One way of seeing this is to note that when N is a G-invariant ideal in A, $\operatorname{Ext} N$ as defined in 2.2 identifies with $G \times_\alpha N$. Indeed, observe that $K(G, A)$ is dense in $G \times_\alpha A$ and for y in $K(G, A)$, $xy \in K(G, N)$ when $x \in N$. Now $N \otimes C(L^2(G))$ identifies with the double crossed product $\hat{G} \times_{\hat{\alpha}} G \times_\alpha N$, thus the canonical conditional expectation maps $N \otimes C(L^2(G))$ onto $G \times_\alpha N$, and we are back.

PROPOSITION 2.5. *Let (A, G, α) be a C*-dynamical system, $(G \times_\alpha A, \hat{G}, \hat{\alpha})$ its dual system, H a closed subgroup of G with annihilator $H^\perp$ in G. Then $H \times_\alpha A$ is G-prime (resp. G-simple) if and only if $G \times_\alpha A$ is $H^\perp$-prime (resp. $H^\perp$-simple).*

PROOF. (i) Prime version: assume J_1 and J_2 are orthogonal non-zero $H^\perp$-invariant ideals in $G \times_\alpha A$. The associated G-invariant ideals $I_H(J_1)$ and $I_H(J_2)$ obtained from 2.1 are then also orthogonal since

$$x_1 x_2 = I_H(y_1) I_H(y_2) = \int_{H^\perp} \int_{H^\perp} \hat{\alpha}_\tau(y_1) \hat{\alpha}_\gamma(y_2) \, d\tau \, d\gamma = 0$$

when $y_1 \in J_1$ and $y_2 \in J_2$.

Assume conversely that N_1 and N_2 are non-zero orthogonal G-invariant ideals in $H \times_\alpha A$, and let $\operatorname{Ext} N_1$ and $\operatorname{Ext} N_2$ be the associated $H^\perp$-invariant ideals in $G \times_\alpha A$ obtained from 2.2. Take the four combinations $x_1 y_1 y_2 x_2$, $y_1 x_1 x_2 y_2$, $y_1 x_1 y_2 x_2$ and $x_1 y_1 x_2 y_2$ where $x_1 \in N_1$, $x_2 \in N_2$ and y_1 and y_2 are in $G \times_\alpha A$. All such elements must be zero. Indeed, for $y_1 x_1 x_2 y_2$ this is obvious, since already $x_1 x_2 = 0$. To prove it for the other three cases note that whenever $y \in K(G, A)$

$$x_1 y x_2 = x_1 \left(\left(\int \iota(y(s)) \lambda_s \, ds \right) x_2 \right.$$

$$= \int \lambda_s \lambda_s^* x_1 \lambda_s \iota(\alpha_{-s}(y(s)) x_2) \, ds = 0 ,$$

since $\lambda_s^* x_1 \lambda_s \in N_1$ by G-invariance, and $\iota(\alpha_{-s}(y(s)))$ belongs to the multiplier algebra of $H \times_\alpha A$, thus $\iota(\alpha_{-s}(y(s))) x_2 \in N_2$.

(ii) Simple version: this is obvious from lemmas 2.1 and 2.2.

PROPOSITION 2.6. *Let (A, G, α), $(G \times_\alpha A, \hat{G}, \hat{\alpha})$, H and $H^\perp$,* Ext *and I_H be as in* 2.1 *and* 2.2. *Let J be an $H^\perp$-invariant ideal of $G \times_\alpha A$, N a G-invariant ideal of $H \times_\alpha A$. Then*

(i) $\operatorname{Ext}(I_H(J)) \subseteq J$, *and*
(ii) $I_H(\operatorname{Ext} N) \supseteq N$.

PROOF. (i) Take $G \underset{\alpha}{\times} A$ in its universal representation. By the $H^{\perp}$-invariance, $I_H(J)$ is contained in the weak closure J'' of J. Thus Ext $(I_H(J))$ is a closed ideal in $J'' \cap (G \underset{\alpha}{\times} A) = J$.

(ii) Take $A \subset B(\mathscr{H})$, $G \underset{\alpha}{\times} A \subset B(L^2(G, \mathscr{H}))$. Let $\mathscr{D}(I_H)$ denote the dense subset of $G \underset{\alpha}{\times} A$ of elements that are in the domain of I_H and map onto elements of $H \underset{\alpha}{\times} A$ ([6, Lemmas 2.5 and 2.6]). Note that the range of such elements under I_H is dense in $H \underset{\alpha}{\times} A$ by the proof of [6, Lemma 2.6]. Thus the generating set for $\operatorname{Ext} N$ has the dense subset $\{xy, yx \mid x \in N,\ y \in \mathscr{D}(I_H)\}$ which maps onto a dense subset of N, and so $I_H(\operatorname{Ext} N)$ must contain N.

When G is discrete, the crossed product $G \underset{\alpha}{\times} A$ identifies with the C*-algebra generated by $\iota(A)$ and λ_G on $L^2(G, \mathscr{H})$. In this case we have the simple inclusion chain

$$\iota(A) \subset H \underset{\alpha}{\times} A \subset G \underset{\alpha}{\times} A$$

for every subgroup H of G. As a further simplification the dual system is now based on a compact group $\hat{G}$ thus the map I_H becomes everywhere defined. For this special setting, we obtain more transparent results concerning the ideal correspondence.

PROPOSITION 2.7. *Let (A, G, α), $(G \underset{\alpha}{\times} A, \hat{G}, \hat{\alpha})$, H and $H^{\perp}$ be as in 2.1, and assume G to be discrete. Let N be a G-invariant ideal in $H \underset{\alpha}{\times} A$. The extension of N is the smallest ideal in $G \underset{\alpha}{\times} A$ containing N and*

$$I_H(\operatorname{Ext} N) = N .$$

PROOF. N is now a subspace of $G \underset{\alpha}{\times} A$, thus

$$\{xy, yx \mid y \in G \underset{\alpha}{\times} A,\ x \in N\}$$

spans the smallest ideal containing N. Note that

$$\operatorname{Ext} N = \operatorname{cl.span} \bigcup_{t \in G \setminus H} N\lambda_t .$$

When applying $I_H = \int_{H^{\perp}} \hat{\alpha}_\gamma \, d\gamma$ to elements in this closed linear span, we obviously get the elements from N back.

PROPOSITION 2.8. *Let (A, G, α), $(G \underset{\alpha}{\times} A, \hat{G}, \hat{\alpha})$, H and $H^{\perp}$ be as in 2.1, and assume G to be discrete. Let J be an $H^{\perp}$-invariant in $G \underset{\alpha}{\times} A$, then $I_H(J) = J \cap (H \underset{\alpha}{\times} A)$ and*

$$\operatorname{Ext}(I_H(J)) = J .$$

PROOF. $H^{\perp}$ is compact, thus I_H is a projection of norm one from $G \underset{\alpha}{\times} A$ onto $H \underset{\alpha}{\times} A$. Now take an element x in J, then $\hat{\alpha}_\gamma(x) \in J$ for every γ in $H^{\perp}$, thus

$I_H(x) \in J$, and at the same time $I_H(x)$ is fixed under $\hat{\alpha}_\gamma, \gamma$ in $H^\perp$, so $I_H(x) \in H \underset{\alpha}{\times} A$. Conversely, every x in J which belongs to $H \underset{\alpha}{\times} A$ satisfies that $x = I_H(x)$.

Take (u_λ) to be an approximate unit for J. By the $H^\perp$-invariance of J and compactness of $H^\perp$ it follows that $(I_H(u_\lambda))$ is an approximate unit for J. However, $(I_H(u_\lambda))$ also belongs to $H \underset{\alpha}{\times} A$. Thus the smallest ideal in $G \underset{\alpha}{\times} A$ which contains $J \cap (H \underset{\alpha}{\times} A)$ contains an approximate unit for J and is contained in J, therefore equals J.

3. Prime and simple crossed products.

In this section we obtain the natural generalizations of [8, Theorems 5.8 and 6.5].

THEOREM 3.1. *Let (A, G, α) be a C*-dynamical system, $(G \underset{\alpha}{\times} A, \hat{G}, \hat{\alpha})$ its dual system, H a closed subgroup of G with annihilator $H^\perp$ in $\hat{G}$. Then the following conditions are equivalent*

(i) *$G \underset{\alpha}{\times} A$ is prime*;
(ii) (a) *$H \underset{\alpha}{\times} A$ is G-prime and* (b) $\Gamma(\alpha) \supseteq H^\perp$.

PROOF. That (i) implies (ii) (a) follows directly from proposition 2.5. From [8, Theorem 5.8] we know that (i) implies $\Gamma(\alpha) = \hat{G}$, so obviously (ii) (b) holds. For the reverse, note that by 2.4 above, (ii) (a) implies that $G \underset{\alpha}{\times} A$ is $H^\perp$-prime. Assume that J_1 and J_2 were non-zero orthogonal ideals in $G \underset{\alpha}{\times} A$. Since $H^\perp \subseteq \Gamma(\alpha)$ we then have

$$J_1 \cap \hat{\alpha}_\gamma(J_2) = \{0\}$$

for each γ in $H^\perp$, reasoning as in the proof of [8, Theorem 3.4]. But then J_1 has zero intersection with the span J_3 of all $\hat{\alpha}_\gamma(J_2)$, $\gamma \in H^\perp$, and repeating the argument with J_3, J_1 in place of J_1, J_2 we obtain orthogonal $H^\perp$-invariant ideals, in contradiction with $G \underset{\alpha}{\times} A$ being $H^\perp$-prime.

THEOREM 3.2. *Let (A, G, α) be a C*-dynamical system, $(G \underset{\alpha}{\times} A, \hat{G}, \hat{\alpha})$ its dual system. Let H be a closed subgroup of G such that the quotient group G/H is discrete, and denote by $H^\perp$ the* (*compact*) *annihilator of H in $\hat{G}$. Then the following conditions are equivalent*

(i) *$G \underset{\alpha}{\times} A$ is simple*
(ii) (a) *$H \underset{\alpha}{\times} A$ is G-simple and* (b) $\Gamma(\alpha) \supseteq H^\perp$.

PROOF. That (i) implies (ii) (a) follows from 2.5 (simple version), and that (i) implies $\Gamma(\alpha) = \hat{G}$, a fortiori that $\Gamma(\alpha) \supseteq H^\perp$ follows from [8, Theorem 6.5]. That

(ii) (a) and (b) imply that $G \times_\alpha A$ is prime follows from 3.1, and that $G \times_\alpha A$ is $H^\perp$-simple from 2.5. To complete the proof we need just the following generalization of [8, Lemma 6.4].

LEMMA 3.3. *Let (B, K, β) be a C*-dynamical system and N a compact subgroup of K. If B is prime and N-simple, then it is simple.*

PROOF. A verbatim repetition of the proof of [8, Lemma 6.4] shows that given a non-zero ideal I in B there exists a non-zero y_0 which belongs to $\beta_n(I)$ for all n in N, thus the intersection over these form a non-zero N-invariant ideal.

A natural question at this point concerns the exact relationship between $\Gamma(\alpha^H)$ and $\Gamma(\alpha)$. We obtain the following, using notation as above.

PROPOSITION 3.4. $\Gamma(\alpha^H) = \Gamma(\alpha) \cap H^\perp$.

PROOF. (i) It follows from the proof of [7, 4.2] that the annihilator $\Gamma(\alpha^H)^\perp$ must contain all t in G for which α_t^H is multiplier-inner in the fixed-point algebra of the bitransposed action $(\alpha^H)''$. Obviously, α_h^H is inner in this sense for all h in H, whence $\Gamma(\alpha^H)^\perp \supseteq H$, thus $\Gamma(\alpha^H) \subseteq H^\perp$.

(ii) To see that $\Gamma(\alpha^H) \subseteq \Gamma(\alpha)$ note that whenever $B \in \mathscr{H}^\alpha(A)$, $H \times_\alpha B \in \mathscr{H}^{\alpha^H}(H \times_\alpha A)$. It is easily checked that the Arveson spectrum $\mathrm{Sp}(\alpha \mid B)$ equals $\mathrm{Sp}(\alpha^H \mid H \times_\alpha B)$. From this we conclude that

$$\Gamma(\alpha^H) \subseteq \bigcap \mathrm{Sp}(\alpha^H \mid H \times_\alpha B) = \bigcap \mathrm{Sp}(\alpha \mid B) = \Gamma(\alpha)\,.$$

(iii) To prove $\Gamma(\alpha^H) \supseteq \Gamma(\alpha) \cap H^\perp$ we appeal to results in [5]. Indeed, let $\gamma \in \Gamma(\alpha) \cap H^\perp$, then we want to see that for every non-zero ideal J in $G \times_{\alpha^H} (H \times_\alpha A)$,

$$\hat{\alpha}_\gamma^H(J) \cap J \neq 0\,.$$

By [5, Proposition 1] the crossed product $G \times_\alpha A$ identifies with the quotient

$$G \times_{\alpha^H} (H \times_\alpha A)/I_H$$

where I_H is the intersection of $G \times_{\alpha^H} (H \times_\alpha A)$ with the ideal in $M(G \times_{\alpha^H} (H \times_\alpha A))$ generated by

$$\{\alpha_h^H(y) - \lambda_h y \mid h \in H,\ y \in H \times_\alpha A\}\,,$$

cf. [5, p. 196]. Using that by [6, 2.1] $\hat{\alpha}_\gamma^H$ leaves $H \times_\alpha A$ pointwise invariant we have that for $\gamma \in H^\perp$

$$\hat{\alpha}_\gamma^H(\alpha_h^H(y) - \lambda_h y) = \alpha_h^H(\hat{\alpha}_\gamma^H(y) - (h, \gamma)\lambda_h \hat{\alpha}_\gamma^H(y)) = \alpha_h^H(y) - \lambda_h y\,.$$

This means that I_H is pointwise invariant under $\widehat{\alpha^H_\gamma}$. Hence if $J \cap I_H \neq \{0\}$, we immediately obtain the desired conclusion. If $J \cap I_H = \{0\}$, the quotient $J = J/I_H$ is a non-zero ideal in $G \times_\alpha A$, and furthermore $\widehat{\alpha^H_\gamma}(J)/I_H = \alpha_\gamma(J)$, hence it follows from $\gamma \in \Gamma(\alpha)$ that $\hat{\alpha}_\gamma(J) \neq \{0\}$, so $J \cap \widehat{\alpha^H_\gamma}(J) \neq \{0\}$.

4. The von Neumann algebra case.

Let (M, G, α) be a W*-dynamical system with M σ-finite and G separable, $\Gamma(\alpha)$ its Connes spectrum as defined in [3, 2.2.1]. Using the above ideas and the characterization in [4, III, Theorem 3.2] of $\Gamma(\alpha)$ as the kernel of the restriction of $\hat{\alpha}$ to the center $Z(G \times_\alpha M)$ of the W*-crossed product $G \times_\alpha M$ we obtain the following generalization of [4, III, Corollary 3.4].

THEOREM 4.1. *Let (M, G, α) be a W*-dynamical system, $(G \times_\alpha M, \hat{G}, \hat{\alpha})$ its dual system, H a closed subgroup of G with annihilator $H^\perp$ in $\hat{G}$. Then the following are equivalent*

(i) *$G \times_\alpha M$ is a factor*

(ii) (a) *α^H acts centrally ergodic on $H \times_\alpha M$*
and
(b) $\Gamma(\alpha) \supseteq H^\perp$.

To see this, let us first prove the following

LEMMA 4.2. *Let (M, G, α), $(G \times_\alpha M, \hat{G}, \hat{\alpha})$, H and $H^\perp$ be as above. The $H^\perp$-invariant central projections in $G \times_\alpha M$ identify with G-invariant central projections in $H \times_\alpha M$.*

PROOF. Let p be a G-invariant central projection in $H \times_\alpha M$. Since M is a subalgebra of $H \times_\alpha M$, p commutes with M, and the G-invariance ensures that p commutes with $\{\lambda_G\}$, thus p is central in $G \times_\alpha M$. Since $H \times_\alpha M$ by [12, Theorem 7.1] identifies with

$$\{y \in G \times_\alpha M \mid \hat{\alpha}(y) = y \ \forall \gamma \in H^\perp\},$$

p is $H^\perp$-invariant.

Conversely, assume p to be central and $H^\perp$-invariant in $G \times_\alpha M$, then by the above $p \in H \times_\alpha M$ and is G-invariant and central since it is central in $G \times_\alpha M$.

PROOF OF THEOREM 4.1. Assume (i), then (ii) (a) holds by 4.2, and by [4, III, Theorem 3.2] $\Gamma(\alpha) = \hat{G}$, a fortiori $\Gamma(\alpha) \supseteq H^\perp$. Conversely, assuming (ii) (a) we know by 4.2 that $G \times_\alpha M$ has no $H^\perp$-invariant non-trivial central projections and now the characterization

$$\Gamma(\alpha) = \ker \hat{\alpha} \,|\, Z(G \times_\alpha M)$$

tells us that (ii) (b) implies that all central projections in $G \times_\alpha M$ must be $H^\perp$-invariant. Thus $G \times_\alpha M$ is a factor.

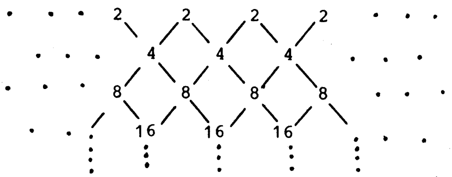
5. An example: the gauge group on the Fermion algebra.

Let (F, T, α) be the C*-dynamical system where F is the Fermion algebra, T the circle group and α the action of T as gauge automorphisms. One way of viewing this is to see F as the infinite tensor product of 2×2-matrices, and the gauge automorphism α_t as the product type automorphism which on each 2×2-matrix algebra is implemented by the unitary u_t,

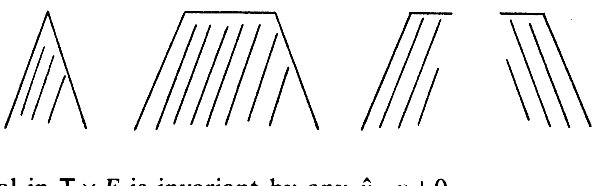
$$u_t = \begin{pmatrix} e^{it} & 0 \\ 0 & 1 \end{pmatrix}$$

(see [2, Section 4] or [9, 8.12.11]).

In [2, Section 4] the diagram of the crossed product $\mathrm{T} \times_\alpha F$ was shown to be



and the dual action represents a "shift" in the diagram. The non-trivial ideals in $\mathrm{T} \times_\alpha F$ correspond to subsets of the diagram of the form ([2, Section 4])



thus no ideal in $\mathrm{T} \times_\alpha F$ is invariant by any $\hat{\alpha}_\gamma$, $\gamma \neq 0$.

This shows that only one class of ideals in $\mathrm{T} \times_\alpha F$ has non-trivial elements, namely the 0-class. Whenever N is a proper subgroup of T, the class of ideals in $\mathrm{T} \times_\alpha F$ invariant under $N^\perp$ contains only 0 and $\mathrm{T} \times_\alpha F$. This corresponds according to section 2, to $N \times_\alpha F$ being T-simple. But in fact we have even more here: $N \times_\alpha F$ is always simple. Indeed, N is a finite (thus discrete) group and so by [8, Theorem 6.5] $N \times_\alpha F$ must be simple if F is N-simple and $\Gamma(\alpha \,|\, N) = \hat{N}$. But F is actually simple, and every gauge automorphism is outer, thus by [7, Theorem 4.2] $\Gamma(\alpha \,|\, N)^\perp = \{0\}$, so $\Gamma(\alpha \,|\, N) = \hat{N}$.

6. Actions on the compact operators.

Let $A = C(H)$, the algebra of all compact operators on a Hilbert space H, and let G be any locally compact abelian group acting on $C(H)$. Combining [5, Theorem 18 and Proposition 34] with [7, 4.2 Theorem] we get the following

PROPOSITION 6.1. *Let $(C(H), G, \alpha)$ be a C*-dynamical system, $(G \underset{\alpha}{\times} C(H), \hat{G}, \hat{\alpha})$ its dual system. The ideals of $G \underset{\alpha}{\times} C(H)$ all have the same stabilizer under the dual action, and this coincides with the Connes spectrum $\Gamma(\alpha)$.*

PROOF. By [5, Theorem 18] the crossed product $G \underset{\alpha}{\times} C(H)$ is isomorphic to the tensor product $C(H) \otimes (G' \overset{\lambda}{\times} \mathbf{C})$ where, denoting by U the unitary group in $B(H)$

$$G' = \{(s,u) \in G \times U \mid \alpha_s = u \cdot u^*\},$$

and the twisting map is defined by mapping the subgroup N' of G' consisting of elements $(0, e^{it} 1)$ into the circle group by the canonical procedure. Obviously, this is an isomorphism of the group N' onto $\mathbf{T}$, so the twisted C*-dynamical system in question is *a reduced abelian system* in the sense of [5]. Hence by [5, Proposition 32 (ii)] the stabilizer of any primitive ideal P in $G' \overset{\lambda}{\times} \mathbf{C}$ is the annihilator in $\hat{G}$ of the quotient Z/N' where Z denotes the center of G'. By the proof of [7, 4.2 Theorem] this annihilator always contains $\Gamma(\alpha)$, and by [8, 3.2 Lemma] the stabilizer is always contained in $\Gamma(\alpha)$, hence equality holds between the stabilizer and $\Gamma(\alpha)$.

The above exhibits the marked contrast between the Fermion algebra case treated in section 5 and the case for any C*-dynamical system based on the compact operators. In the latter case, all ideals of the crossed product belong to the Connes spectrum-class.

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