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ALMOST ARTINIAN MODULES

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Introduction.

Let R be a commutative ring with identity. Call an R-module *almost Artinian* if every proper homomorphic image is Artinian. The purpose of this paper is to study the structure of an almost Artinian R-module A that is not Artinian. The group of all rational numbers of the form a/p^n, where p is a fixed non-zero prime, is an almost but not Artinian $\mathbf{Z}$-module. In fact, Corollary 2.8 shows that every almost but not Artinian abelian group is a finite sum of groups of this type and a cyclic subgroup of the rationals. The special case when $A=R$ is known to be equivalent to the property that R is a one dimensional Noetherian domain [3, Corollary 1]. And the case when A is the total ring of fractions of R is equivalent to the property that R is a semi-local one dimensional Noetherian domain. This is an easy consequence of a result of Matlis [5, Theorem 1].

It is shown in general that for any almost but not Artinian R-module A, the annihilator of A is a prime ideal of R, A is isomorphic to a submodule of the quotient field of $R/\mathrm{Ann}_R(A)$, and that $R/\mathrm{Ann}_R(A)$ is a one dimensional Noetherian domain. The main result of this paper is Theorem 2.6: If R is a Dedekind ring (an integrally closed Noetherian domain of dimension one) with quotient field Q, then there is an ideal I in R, a non-zero element $d \in I$, and a finite set $P_1,\ldots,P_n$ of maximal ideals of R such that A is isomorphic to $G(P_1)+\ldots+G(P_n)+I(1/d)$, where $G(P)=\{x \in Q \mid P^m x \subseteq R \text{ for some } m \geq 0\}$.

1. General Properties.

First we remark that the property of being almost Artinian is equivalent to the apparently more general property that every proper homomorphic image is finitely embedded, where a module is finitely embedded if it is an essential extension of it's finitely generated socle. Vamos studied this property as dual to the notion of finitely generated. For example, a module is Artinian if and only if every homomorphic image is finitely embedded [7, Proposition 2]. The above remark follows from this statement. It is also easy to see that if every proper submodule of A is almost Artinian, then A is almost Artinian.

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The following theorem records the general facts about an almost but not Artinian module.

THEOREM 1.1 *Let R be a commutative ring and A an almost but not Artinian R-module. Then*:

(a) *Every non-zero submodule of A is almost but not Artinian.*
(b) *For each $r \in R$ either* $\mathrm{Ann}_A(r)=0$ *or* $\mathrm{Ann}_A(r)=A$.
(c) $\mathrm{Ann}_R(A)$ *is a prime ideal of R*:
(d) *A is a torsion-free almost but not Artinian* $R/\mathrm{Ann}_R(A)$*-module.*
(e) *A is an indecomposable R-module.*
(f) *A is isomorphic to a submodule of the quotient field of* $R/\mathrm{Ann}_R(A)$.
(g) $\mathrm{End}_R(A)$ *has no non-zero zero divisors.*

PROOF. (a) If B is a non-zero submodule of A, then A/B is Artinian and it follows that B is not Artinian. If C is a non-zero submodule of B, then A/C is Artinian and it follows that B/C is Artinian. Therefore B is an almost but not Artinian R-module.

(b) For each $r \in R$ there is an isomorphism $rA \cong A/\mathrm{Ann}_A(r)$. If $rA=0$, then $\mathrm{Ann}_A(r)=A$. If $rA \neq 0$, then rA is not Artinian by (a). Therefore $A/\mathrm{Ann}_A(r)$ is an improper image of A and it follows that $\mathrm{Ann}_A(r)=0$.

(c) Let $rs \in \mathrm{Ann}_R(A)$ and suppose that $s \notin \mathrm{Ann}_R(A)$. Then $rsA=0$ and $sA \neq 0$, and it follows that $\mathrm{Ann}_A(s)=0$ by (b). Since $srA=rsA=0$ it follows that $rA \subseteq \mathrm{Ann}_A(s)=0$. Therefore $r \in \mathrm{Ann}_R(A)$ and $\mathrm{Ann}_R(A)$ is a prime ideal of R.

(d) If B is a $R/\mathrm{Ann}_R(A)$-homomorphic image of the $R/\mathrm{Ann}_R(A)$-module A, then B is a R-homomorphic image of the R-module A, and the R-submodules of B are the same as the $R/\mathrm{Ann}_R(A)$-submodules of B. Therefore A is an almost but not Artinian $R/\mathrm{Ann}_R(A)$-module. Since $\mathrm{Ann}_R(A)$ is a prime ideal of R we may assume that R is an integral domain and $\mathrm{Ann}_R(A)=0$. So let $r \in R$ and $a \in A$ be such that $ra=0$ and $a \neq 0$. Then $\mathrm{Ann}_A(r) \neq 0$ and it follows that $\mathrm{Ann}_A(r)=A$. Therefore $r \in \mathrm{Ann}_R(A)=0$ and it follows that A is torsion-free.

(e) Suppose that $A=B+C$ where $B \cap C=0$ and $B \neq 0$. Then $C \cong A/B$ which is Artinian. Therefore $C=0$ and A is indecomposable.

(f) Since each non-zero submodule of A is indecomposable, it follows that no two non-zero submodules of A can have a zero intersection. Therefore A is a rank one torsion-free $R/\mathrm{Ann}_R(A)$-module and it follows that A is isomorphic to a submodule of the quotient field of $R/\mathrm{Ann}_R(A)$.

(g) Let $f \in \mathrm{End}_R(A)$ and suppose that $f \neq 0$. Then $f(A)$ is not Artinian. Therefore $f(A)$ must be an improper image of A. Therefore f is a monomorphism and $\mathrm{End}_R(A)$ can have no non-zero zero divisors.

The next theorem, due to Cohen [3, Corollary 1], characterizes those rings that are almost Artinian. We include a proof for the sake of completeness.

THEOREM 1.2 (Cohen). *A commutative ring R is almost Artinian if and only if either*

(a) *R is an Artinian ring or*

(b) *R is a Noetherian domain of dimension one.*

PROOF. Suppose that R is almost but not Artinian. If $r \in R$, $r \neq 0$ then $rR \neq 0$ and it follows that $\text{Ann}_R(r) = 0$. Therefore R is a domain. Let $\{I_i\}_1^\infty$ be an ascending chain of ideals of R and assume that $I_1 \neq 0$. Then $\{I_i/I_1\}_i^\infty$ is an ascending chain in the Artinian ring R/I_1, and it follows that both chains are finite. Therefore R is Noetherian. Since R is not Artinian it must have positive dimension. Let $0 = P_0 \subset P_1 \subseteq P_2 \ldots$ be a chain of prime ideals of R where $P_1 \neq 0$. Then $P_2/P_1 \ldots$ is a chain of prime ideals in R/P_1. But R/P_1 is Artinian and is therefore a field. Hence P_1 is a maximal ideal and R has dimension one. The converse is clear.

Matlis showed that if R is a noetherian domain then R is semi-local of dimension one if and only if it's field of quotients is almost Artinian [5, Theorem 1]. The following theorem is an easy consequence of this result.

THEOREM 1.3. *Let R be a commutative ring and Q it's total ring of fractions. Then Q is an almost Artinian R-module if and only if either*

(a) *R is an Artinian ring or*

(b) *R is a semi-local Noetherian domain of dimension one.*

PROOF. If Q is Artinian then R is also Artinian. So assume that Q is almost but not Artinian. Then R is almost but not Artinian and it follows from Theorem 1.2 that R is a one dimensional Noetherian domain. Thus R is semi-local by the result of Matlis. Conversely, if R is Artinian then every non-zero divisor is a unit and it follows that $R = Q$. If R is a semi-local Noetherian domain of dimension one, then Q is almost Artinian by the result of Matlis.

The dual to the above situation for almost Noetherian modules has been studied by Armendariz [2, Theorem 2.1]. The next theorem describes those rings that possess an almost but not Artinian module.

THEOREM 1.4. *Let R be a commutative ring with an almost but not Artinian module A. Then either*

(a) *R is a Noetherian domain of dimension one (equivalently every almost but not Artinian R-module is faithful) or*

(b) *R has a non-zero prime ideal P such that R/P is a Noetherian domain of dimension one (equivalently every almost but not Artinian R-module is not faithful).*

Proof. If A is faithful, then $\mathrm{Ann}_R(A)=0$ is a prime ideal of R. Therefore A is torsion-free over the domain R. If $x \in A$ and $x \neq 0$, then $Rx \cong R$ and it follows that R is Noetherian of dimension one. If A is not faithful, then $P=\mathrm{Ann}_R(A)$ is a non-zero prime ideal of R and A is a faithful almost but not Artinian R/P-module. Therefore R/P is a Noetherian domain of dimension one. The equivalences follow from the fact that R cannot be a ring of type (a) and a ring of type (b).

We remark that any Noetherian ring of dimension $\geqq 1$ has an almost but not Artinian module. On the other hand a perfect ring has none. If we define the term torsion in terms of regular elements, then the words faithful and not faithful in Theorem 1.4 can be replaced by the words torsion-free and torsion of bounded order. We also note that it is easy to show that if R is a Noetherian domain of dimensi ɔn one and A is a non-zero submodule of the quotient field of R, then A is almost but not Artinian if and only if $A/A \cap R$ is Artinian.

2. The Dedekind case.

We assume in this section that R is a Dedekind ring (an integrally closed Noetherian domain of dimension one) and Q is it's field of quotients. For a maximal ideal P of R let $R(P^\infty)$ be the P-primary submodule of Q/R. So

$$R(P^\infty) = \{x+R \in Q/R \mid P^m x \subseteq R \text{ for some } m \geqq 0\} .$$

Let $G(P)=\{x \in Q \mid P^m x \subseteq R \text{ for some } m\geqq 0\}$. Note that $G(P)\cap G(P')=R$ if P and P' are distinct maximal ideals of R. Recall that an Artinian R-module is isomorphic to a finite direct sum of modules $R(P^\infty)$ and a module of finite length [4, Theorems 7, 8 and 9].

The following five Lemmas will constitute the proof of Theorem 2.6. We assume that A is an almost Artinian R-module and $R \subseteq A \subseteq Q$.

Lemma 2.1. *If P is a maximal ideal of R and $f\colon A \to R(P^\infty)$ is an epimorphism with $R \subseteq \mathrm{Ker}(f)$, then $G(P) \subseteq A$.*

Proof. Since $R(P^\infty)$ is divisible and hence injective, there exists an extension $g\colon Q \to R(P^\infty)$ of f. Let $x/y \in G(P)$. Then there is an element $a/b \in A$ such that $f(a/b)=x/y+R$. Let $g(1/b)=u/v+R$. Then $g(1)=bu/v+R=0$. Therefore there exists an element $r \in R$ such that $g(1/b)=r/b+R$, and it follows that $f(a/b)=g(a/b)=ar/b+R$. Thus $x/y=ra/b+s$ for some $s \in R$, so it follows that $x/y \in A$ and $G(P) \subseteq A$.

Lemma 2.2. *Let $P_1,\ldots,P_n$ be the distinct maximal ideals such that $R(P_i^\infty)$ is a* *direct summand of* A/R. *Then* $\sum_{i=1}^{n} G(P_i) \subseteq A$, *and if* P *is a maximal ideal such that* $G(P) \subseteq A$ *then* $P = P_i$ *for some* i.

PROOF. For each i let f_i be the composition of the natural mapping $A \to A/R \to 0$ followed by the projection map $A/R \to R(P_i^\infty) \to 0$. Then f_i is an epimorphism and $R \subseteq \mathrm{Ker}(f_i)$. Therefore by Lemma 2.1 it follows that $G(P_i) \subseteq A$ for each i. Now suppose that $G(P) \subseteq A$. Then $R(P^\infty) \subseteq A/R$ and it follows that $R(P^\infty)$ is a direct summand of A/R. Therefore $P = P_i$ for some i.

Let $P_1, \ldots, P_n$ be the distinct maximal ideals such that $G(P_i) \subseteq A$ for each i. Define the submodule T of A as follows: $T = \{a/b \in A \mid b \notin P_i \text{ for each } i\}$. Notice that T is indeed a module and if $n = 0$ then $T = A$.

LEMMA 2.3. *If* $n > 0$ *then* $\sum_{i=1}^{n} G(P_i) \cap T = R$.

PROOF. It is easy to see that $R \subseteq \sum_{i=1}^{n} G(P_i) \cap T$, so we show the other inclusion. Let $x \in \sum_{i=1}^{n} G(P_i) \cap T$. Then $x = a/b$ where $b \notin P_i$ for each i, and $x = \sum_{i=1}^{n} (c_i/d_i)$ where for each i there exists $m_i \geq 0$ such that $P_i^{m_i}(c_i/d_i) \subseteq R$. Let $J = \prod_{i=1}^{n} P_i^{m_i}$ and let $J_j = \prod_{i \neq j} P_i^{m_i}$. Then we have

$$Jx = J\left(\sum_{i=1}^{n} c_i/d_i\right) \subseteq \sum_{i=1}^{n} J(c_i/d_i) = \sum_{i=1}^{n} J_i P_i^{m_i}(c_i/d_i) \subseteq \sum_{i=1}^{n} J_i = R\,.$$

Since $b \notin J$ we can write $1 = rb + s$ for elements $r \in R$ and $s \in J$, and it follows that $a/b = ra + sa/b$. But $sa/b \in Jx \subseteq R$ and we have $x = a/b \in R$.

LEMMA 2.4. $A = \sum_{i=1}^{n} G(P_i) + T$.

PROOF. Let $a/b \in A$. Then there exist integers $m_i \geq 0$ and an ideal I such that $Rb = \prod_{i=1}^{n} P_i^{m_i} I$ where $I \not\subseteq P_i$ for each i. Pick $c \in I - \bigcup_i P_i$ and let $J = \prod_{i=1}^{n} P_i^{m_i}$. Then we can write $1 = rc + s$ for elements $r \in R$ and $s \in J$. Further, we have $sc \in JI = Rb$ and it follows that $sc = tb$ for some $t \in R$. Therefore we can write $a/b = rca/b + sa/b$. But we also have $sa/b = sat/bt = sat/sc = at/c$ and it follows that $sa/b \in T$. Now let $J_j = \prod_{i \neq j} P_i^{m_i}$. Then we can write $1 = \sum_{i=1}^{n} u_i$ where $u_i \in J_i$ for each i. Therefore $c = \sum_{i=1}^{n} u_i c$ and $P_i^{m_i} u_i c \subseteq Rb$ for each i. This says that $u_i c/b \in G(P_i)$. Thus we have

$$rac/b = ra \sum_{i=1}^{n} (u_i c/b) = \sum_{i=1}^{n} rau_i c/b \in \sum_{i=1}^{n} G(P_i)\,.$$

Therefore $a/b = rac/b + sa/b \in \sum_{i=1}^{n} G(P_i) + T$.

LEMMA 2.5. *Every proper homomorphic image of T has finite length. Thus there is an ideal I in R and a non-zero element $d \in I$ such that $T=I(1/d)$.*

PROOF. Suppose that T/R does not have finite length. Then T/R has a non-zero direct summand of the form $R(P^\infty)$ for some maximal ideal P of R. Therefore $G(P) \subseteq T$ and it follows that $P=P_i$ for some i. Let $a/b \in G(P)$. Then there exists an integer $m \geqq 0$ such that $P^m(a/b) \subseteq R$, and $b \notin P$. Therefore there are elements $r \in R$ and $s \in P^m$ such that $1=rb+s$, and it follows that $a/b=ra+sa/b$. This implies that $a/b \in R$, and it follows that $R(P^\infty)=0$, a contradiction. Therefore T/R has finite length. Now let B be a non-zero submodule of T and pick $x \in B \cap R$ where $x \neq 0$. Then there is an exact sequence $0 \to R/Rx \to T/Rx \to T/R \to 0$ with the two outside terms having finite length. Therefore T/Rx has finite length and it follows that T/B has finite length. Thus T is finitely generated and, since $R \subseteq T$, there is an ideal I of R and a non-zero element $d \in I$ such that $T=I(1/d)$.

The next theorem is the main result. It follows from the preceding lemmas together with the fact that any non-zero submodule of Q is isomorphic to one that contains R.

THEOREM 2.6. *Let R be a Dedekind ring and A an almost but not Artinian R-module. Then there is an ideal I in R, a non-zero element $d \in I$, and a finite set $P_1,\ldots,P_n$ of maximal ideals of R such that*

$$A \cong \sum_{i=1}^{n} G(P_i)+I(1/d)\,.$$

If R is a principal ideal domain and $p \in R$ is an non-zero prime we define

$$G(p) = \{a/p^m \mid a \in R \text{ and } m \geqq 0\}\,.$$

Then $G(p)=G(Rp)$ and we record the following two corollaries:

COROLLARY 2.7 *Let R be a principal ideal domain and A an almost but not Artinian R-module. Then there is a non-zero element $d \in R$ and a finite set $p_1,\ldots,p_n$ of non-zero primes in R such that*

$$A \cong \sum_{i=1}^{n} G(p_i)+R(1/d)\,.$$

COROLLARY 2.8. *An almost but not Artinian abelian group is isomorphic to a group of the form*

$$\sum_{i=1}^{n} G(p_i) + \mathbf{Z}(1/d) .$$

We remark that the structure of an almost Artinian module over a Dedekind ring depends completely on the structure of the Artinian modules. In the general case of an arbitrary one dimensional Noetherian domain the structure of it's Artinian modules is more complicated. Matlis has described the structure of an Artinian module over a Noetherian domain of dimension one, in fact, over a Cohen–Macaulay ring of dimension one [6]. We have not yet found a corresponding structure for the almost Artinian modules in this case.

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