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DEFORMATIONS OF GRADED ALGEBRAS

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Introduction.

In this paper we study formal deformations of graded algebras and corresponding problems in projective geometry. Given a graded algebra A, we may forget the graded structure and deform A as an algebra. Clearly we also have a deformation theory respecting the given graded structure of A. This is closely related to the corresponding deformation theory of $X = \mathrm{Proj}\,(A)$. One objective of this paper is to compare these three theories of deformation.

A basic tool is the cohomology groups of André and Quillen. Let $S \to A$ be a graded ring homomorphism and let M be a graded A-module. We shall see that the groups $H^i(S,A,M)$ are graded A-modules whenever S is noetherian and $S \to A$ is finitely generated. In fact, if we let ${}_vH^i(S,A,M)$ correspond to S-derivations of degree v, we shall prove that there are canonical isomorphisms

$$\coprod_{v=-\infty}^{\infty} {}_vH^i(S,A,M) \cong H^i(S,A,M)$$

for every $i \geqq 0$.

Let $\pi\colon R \to S$ be a graded surjection satisfying $(\ker \pi)^2 = 0$. Then the problem of lifting the S-algebra A to R as a graded algebra (respectively as an algebra) is given as an obstruction sitting in ${}_0H^2(S,A,A\otimes_S \ker \pi)$ (respectively in $H^2(S,A,A\otimes_S \ker \pi)$). Moreover the graded deformations (respectively deformations) of A to R modulo equivalence are in $1-1$ correspondence with ${}_0H^1(S,A,A\otimes_S \ker \pi)$ (respectively $H^1(S,A,A\otimes_S \ker \pi)$). These generalities are discussed in chapter 1.

Since there is an injection

$${}_0H^2(S,A,A\otimes_S \ker \pi) \to H^2(S,A,A,\otimes_S \ker \pi)$$

sending the obstructions onto each other, we deduce that A is liftable to R iff A is liftable to R as a *graded* algebra. We would like to be able to omit the condition $(\ker \pi)^2 = 0$. Suppose R is a complete local ring with residue field S and let $\pi\colon R \to S$ be the canonical surjection and assume

$${}_vH^1(S,A,A) = 0$$

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for $v>0$ or for $v<0$, then the statement above follows from (2.7) of chapter 2.

In chapter 3 we compare the groups ${}_vH^i(S,A,M)$ with the corresponding groups $A^i(S,X,\tilde{M}(v))$ in projective geometry, $X=\operatorname{Proj}(A)$. The groups $A^i(S,X,-)$ were introduced by Illusie in [3] and by Laudal [6]. Recall that if X is S-smooth, then

$$A^i(S,X,\tilde{M}) = H^i(X,\theta_X \underset{S}{\otimes} \tilde{M})$$

where θ_X is the sheaf of S-derivations on X. If the depth of M with respect to the ideal $m=\coprod_{v=1}^{\infty} A_v$ is sufficiently large, we prove that

$${}_vH^i(S,A,M) \cong A^i(S,X,\tilde{M}(v)).$$

In particular, if $\operatorname{depth}_m A \geqq 4$, the graded deformations of A and the deformations of X as scheme are in $1-1$ correspondence. In the relative case, let $\varphi\colon B \to A$ be a surjective morphism of graded S-algebras such that $B_0=A_0=S$ and let

$$f\colon X = \operatorname{Proj}(A) \to Y = \operatorname{Proj}(B)$$

be the induced embedding. We would like to compare the groups ${}_vH^i(B,A,M)$ and $A^i(S,f,\tilde{M}(v))$. If f is locally a complete intersection, recall that

$$A^i(S,f,O_X(v)) \cong H^{i-1}(X,N_f(v))$$

where N_f is the normal bundle of X in Y. Again putting depth conditions on M with respect to the ideal m, we conclude that ${}_vH^i(B,A,M)$ and $A^i(S,f,\tilde{M}(v))$ coincide. It follows that, if $\operatorname{depth}_m A \geqq 2$,

$$\operatorname{Def}^0(\varphi,-) \cong \operatorname{Hilb}_f(-)$$

where $\operatorname{Def}^0(\varphi,-)$ is the graded deformation functor of φ and where $\operatorname{Hilb}_f(-)$ is the local Hilbert functor at X in Y.

Let $f\colon X \to \boldsymbol{P}_S^n$ be a closed subscheme. In chapter 4 we give conditions on f under which the graded group $H^1(S,B,B)$ has positive or negative grading, where B is the minimal cone of some twisted embedding of X in $\boldsymbol{P}_S^N$. Using this, together with (2.7) we find that the smooth non-liftable projective variety of Serre [12] gives rise to a graded k-algebra of characteristic p which is non-liftable *as an algebra* to characteristic zero. This is done in chapter 5. The problem of constructing such a k-algebra is, in fact, the main point of this paper. The proof given here is due to O. A. Laudal and the author. I would also like to thank G. Ellingsrud for discussions and in particular for pointing out to us the use of a theorem of M. Schlessinger related to (4.8) of this paper.

1. Cohomology groups of graded algebras.

Rings will be commutative with unit. Let $\underline{S\text{-alg}}$ be the category of S-algebras and denote by $\underline{SF}$ the full subcategory of free S-algebras, i.e. the polynomial algebras in any set of indeterminates defined over S. Given an S-algebra A and an A-module M, consider the category $\underline{SF}/A = \underline{C}$ of morphisms of S-algebras $\varphi\colon F \to A$ where F is an object of $\underline{SF}$. Define

$$H^i(S, A, M) = \varprojlim{}^{(i)}_{\underline{C}^0} \operatorname{Der}_S(-, M)$$

where $\operatorname{Der}_S(-, M)\colon \underline{C}^0 \to \underline{\mathrm{Ab}}$ is the functor given by

$$\operatorname{Der}_S(-, M)(F \xrightarrow{\varphi} A) = \operatorname{Der}_S(F, M),$$

M being an F-module via φ. See [1] and [6].

If $S \to A$ is a homogeneous morphism of graded rings and if M is a graded A-module, we may of course consider the category of graded S-algebras $\underline{Sg\text{-alg}}$ and the corresponding subgategory $\underline{SgF}$ of free graded S-algebras. Let $\underline{C}^g$ be the category $\underline{SgF}/A$ of homogeneous morphisms of graded S-algebras $\varphi\colon F \to A$, and let ${}_v\operatorname{Der}_S(-, M)\colon \underline{C}^{g^0} \to \underline{\mathrm{Ab}}$ be the functor defined by

$$\begin{aligned} {}_v\operatorname{Der}_S(-, M)(F \xrightarrow{\varphi} A) &= {}_v\operatorname{Der}_S(F, M) \\ &= \{D \in \operatorname{Der}_S(F, M) \mid D \text{ is homogeneous of degree } v\}. \end{aligned}$$

DEFINITION 1.1.

$${}_vH^i(S, A, M) = \varprojlim{}^{(i)}_{\underline{C}^{g0}} {}_v\operatorname{Der}_S(-, M).$$

DEFINITION 1.2. Given any S-algebra A and any surjective $\pi\colon R \to S$ with nilpotent kernel, we say the R-algebra A' is a lifting, or deformation, of A to R if there exists a morphism $A' \to A$ such that the diagram

$$\begin{array}{ccc} R & \to & A' \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ S & \to & A \end{array}$$

is commutative and

1) $A = A' \underset{R}{\otimes} S$

2) $\operatorname{Tor}_1^R(A', S) = 0$.

Two liftings A' and A'' are considered equivalent if there is an R-algebra isomorphism $A' \cong A''$ reducing to the identity on A. If $\varphi\colon A \to B$ is a morphism of S-algebras and A' and B' are liftings of A and B respectively to R, we say that a morphism

$$\varphi'\colon A' \to B'$$

is a lifting, or deformation, of φ to R with respect to A' and B' if $\varphi' \otimes_R \mathrm{id}_S = \varphi$. We define graded liftings of graded algebras and homogeneous liftings of homogeneous morphisms in exactly the same way.

Assume that $R \xrightarrow{\pi} S$ satisfies $(\ker \pi)^2 = 0$; then the following results are proved in [6], see [6, (2.2.5) and (2.3.3)].

THEOREM 1.3. *There is an obstruction*

$$o(A) \in H^2(S, A, A \underset{S}{\otimes} \ker \pi)$$

which is zero if and only if A can be lifted to R. If $o(A) = 0$, then the set of equivalence-classes of liftings is a principal homogeneous space over $H^1(S, A, A \otimes_S \ker \pi)$.

THEOREM 1.4. *There is an obstruction*

$$o(\varphi;\, A', B') \in H^1(S, A, B \underset{S}{\otimes} \ker \pi)$$

which is zero if and only if φ can be lifted to R with respect to A' and B'. If $o(\varphi;\, A', B') = 0$ then the set of liftings is a principal homogeneous space over

$$H^0(S, A, B \underset{S}{\otimes} \ker \pi) = \mathrm{Der}_S(A, B \underset{S}{\otimes} \ker \pi) .$$

The results (1.3) and (1.4) carry over to the graded situation as follows:

THEOREM 1.5. *There is an obstruction*

$$o_0(A) \in {}_0H^2(S, A, A \underset{S}{\otimes} \ker \pi)$$

which is zero if and only if A can be lifted to a graded R-algebra. If $o_0(A) = 0$, then the set of equivalence-classes of graded liftings is a principal homogeneous space over ${}_0H^1(S, A, A \otimes \ker \pi)$.

THEOREM 1.6. *Given graded liftings A' and B' of A and B respectively, there is an obstruction*

$$o_0(\varphi;\, A', B') \in {}_0H^1(S, A, B \underset{S}{\otimes} \ker \pi)$$

which is zero if and only if φ can be lifted as a homogeneous morphism to R with respect to A' and B'. Moreover, if $o_0(\varphi;\, A', B') = 0$, then the set of homogeneous liftings is a principal homogeneous space over

$${}_0H^0(S, A, B \underset{S}{\otimes} \ker \pi) = {}_0\mathrm{Der}_S(A, B \underset{S}{\otimes} \ker \pi) .$$

If we want to compare the graded and the non-graded theories of deformation, we need to know the relations between the groups ${}_0H^i(S,A,M)$ and $H^i(S,A,M)$. These are given by the following theorem. A proof of this can also be found in [3].

THEOREM 1.7. *Let $S \to A$ be a homogeneous ring homomorphism and let M be a graded A-module. If S is noetherian and $S \to A$ is finitely generated, then there is a canonical isomorphism*

$$\coprod_{\nu=-\infty}^{\infty} {}_\nu H^i(S,A,M) \to H^i(S,A,M)$$

for every $i \geqq 0$.

REMARK 1.8. In general, there is an injection

$$\coprod_{\nu=-\infty}^{\infty} {}_\nu H^i(S,A,M) \to H^i(S,A,M)$$

for every $i \geqq 0$.

PROOF. Let $\underline{C}^g_{fg} \subseteq \underline{C}^g = \underline{SgF}/A$ be the full subcategory defined by the objects $\varphi\colon F \to A$ with F a finitely generated S-algebra. The inclusion $\underline{C}^g_{fg} \subseteq \underline{C}^g$ and the forgetful functor $\underline{C}^g \to \underline{C}$ induce morphisms

1) $\varprojlim^{(i)}_{\underline{C}^{g0}} \mathrm{Der}_S(-,M) \to \varprojlim^{(i)}_{\underline{C}^{g0}_{fg}} \mathrm{Der}_S(-,M)$

2) $\varprojlim^{(i)}_{\underline{C}^0} \mathrm{Der}_S(-,M) \to \varprojlim^{(i)}_{\underline{C}^{g0}} \mathrm{Der}_S(-,M)$.

I claim that these morphisms are isomorphisms for $i \geqq 0$. This will prove (1.7) since, as one easily sees, there is a canonical isomorphism of functor

$$\coprod_{\nu=-\infty}^{\infty} {}_\nu \mathrm{Der}_S(-,M) \to \mathrm{Der}_S(-,M)$$

on $\underline{C}^{g0}_{fg}$

For $i=0$, 1) and 2) are easily proved. For $i>0$ we need some preliminaries.

Let $\varphi\colon F \to A$ be a homogeneous morphism of S-algebras, $F \in \mathrm{ob}\,\underline{SgF}$, and suppose φ is surjective. Put

$$F_i = F \times_A F \times_A \ldots \times_A F \quad (i+1)\text{-times}.$$

Consider the complex

$$\varprojlim^{(q)}_{\underline{SF}/F^0} \mathrm{Der}_S(-,M) \to \varprojlim^{(q)}_{\underline{SF}/F_1^0} \mathrm{Der}_S(-,M) \to \ldots$$

$$\to \varprojlim^{(q)}_{\underline{SF}/F_i^0} \mathrm{Der}_S(-,M) \to$$

where the differentials are the alterating sums of morphisms

$$\varprojlim^{(q)}_{\underline{SF}/F^{o}_{i-1}} \operatorname{Der}_S(-,M) \to \varprojlim^{(q)}_{\underline{SF}/F^{o}_{i}} \operatorname{Der}_S(-,M)$$

induced by the projections $F_i \to F_{i-1}$. In this situation there is a Leray spectral sequence given by the term

$$'E_2^{p,q} = H^p(\varprojlim^{(q)}_{\underline{SF}/F^{o}_{\cdot}} \operatorname{Der}_S(-,M))$$

converging to

$$\varprojlim^{(\cdot)}_{\underline{C}^{o}} \operatorname{Der}_S(-,M) = H^{(\cdot)}(S,A,M)\ .$$

For a proof see [6, (2.1.3)]. Similarly, there is a Leray spectral sequence with

$$'E_2^{p,q} = H^p(\varprojlim^{(q)}_{\underline{SgF}/F^{o}_{\cdot}} \operatorname{Der}_S(-,M))$$

converging to

$$\varprojlim^{(\cdot)}_{\underline{C}'^{o}} \operatorname{Der}_S(-,M)\ .$$

We shall prove that for all objects A of $\underline{Sg\text{-alg}}$ 2) is an isomorphism. Given A and F as above there is a morphism of Leray spectral sequences given by

3) $E_2^{p,q} \to {'E_2^{p,q}}$

converging to 2). Suppose that 2) is an isomorphism for $i<n$ and for every object A in $\underline{Sg\text{-alg}}$. Then 3) is an isomorphism for $q<n$ and for every p. Since

$$E_2^{0,q} \subseteqq \varprojlim^{(q)}_{\underline{SF}/F^{o}} \operatorname{Der}_S(-,M)\ ,$$

and since $\underline{SF}/F$ has a final object, so that $\varprojlim^{(q)}_{\underline{SF}/F^{o}} = 0$ for $q \geqq 1$, we find

$$E_2^{0,q} = 0$$

for $q \geqq 1$. In the same way

$$'E_2^{0,q} \subseteqq \varprojlim^{(q)}_{\underline{SgF}/F^{o}} \operatorname{Der}_S(-,M) = 0$$

for $q \geqq 1$. Since the differentials of the spectral sequences E_r and $'E_r$ are of bidegrees $(r, 1-r)$, and since for p and q given, $E_\infty^{p,q} = E_r^{p,q}$ and $'E_\infty^{p,q} = {'E_r^{p,q}}$ for some r, the morphisms 3) induce isomorphisms

$$E_\infty^{p,q} \to {'E_\infty^{p,q}}$$

for every p and q with $p+q \leqq n$. Hence 2) is an isomorphism for $i=n$. By induction 2) is an isomorphism for all A and $i \geqq 0$.

In the same way we prove that 1) is an isomorphism for $i \geqq 0$.

Let $R \xrightarrow{\pi} S$ be a graded surjection such that

$$(\ker \pi)^2 = 0 .$$

It is easy to see that the injection

$$_0H^2(S, A, A \otimes_S \ker \pi) \to H^2(S, A, A \otimes_S \ker \pi)$$

maps the obstruction $o_0(A)$ onto $o(A)$. This follows from the construction of the obstructions (see [6, (2.2)]). Thus we have proved

COROLLARY 1.9. *Let* $R \xrightarrow{\pi} S$ *be a graded surjection such that* $(\ker \pi)^2 = 0$. *If A is a graded S-algebra, then A can be lifted to R iff A can be lifted to R as a graded algebra.*

REMARK. 1.10. Let F_A be the set of equivalence-classes of liftings of A to R and F_A^0 the corresponding set of graded liftings. If A' is a graded lifting of A to R, then there are isomorphisms and obvious vertical injections fitting into a commutative diagram

$$\begin{array}{ccc} F_A & \cong & H^1(S, A, A \otimes \ker) \\ \uparrow & & \uparrow \\ F_A^0 & \cong & {}_0H^1(S, A, A \otimes \ker \pi) \end{array}$$

the horizontal isomorphisms are defined via A'. Hence there is a retraction $p_A \colon F_A \to F_A^0$. Now (1.9) can be generalized as follows. Let $\varphi \colon A \to B$ be a homogeneous morphism. Assume there are liftings A'' and B'', not necessarily graded, of A and B such that φ is liftable to R with respect to A'' and B''. Then φ admits a graded lifting to R with respect to $p_A(A'')$ and $p_B(B'')$. We omit the proof.

Similar results are valid for graded S-modules and for homogeneous module morphisms.

2. Deformation functors and formal moduli.

We shall in the rest of this paper confine ourselves to the study of deforming finitely generated algebras.

Consider the following problem. Let $\pi \colon R' \to R$ be a surjective ring homomorphism and let A be any graded R-algebra. Suppose A is liftable to R' as an ungraded algebra. When will there exist a graded lifting? By (1.9) there exists a graded lifting if $(\ker \pi)^2 = 0$. The general situation is, however, more complicated and leads to the study of hulls of the ungraded and graded deformation functors of A.

Let V be a complete local ring with maximal ideal m_V and residue field

$k=V/m_V$. Let $\underline{l}$ be the category whose objects are artinian local V-algebras with residue fields k and whose morphisms are local V-homomorphisms. In the following all liftings are assumed to be flat over R when R is in $\underline{l}$ (or in pro-$\underline{l}$). Let S be a finitely generated graded k-algebra and assume that we can find graded liftings S_R of S to R for any $R \in \text{ob}\,\underline{l}$ such that for any morphism $\pi: R' \to R$ of $\underline{l}$ there is a morphism $S_{R'} \to S_R$ with $S_{R'} \otimes_{R'} R = S_R$. For each R, fix one S_R with this property and let $\varphi: S \to A$ be a finitely generated graded S-algebra. Relative to the choice of liftings S_R we define

$$\text{Def}^0(A/S, R) = \left\{ \begin{matrix} S_R & \to & A' \\ \downarrow & & \downarrow \\ S & \to & A \end{matrix} \;\middle|\; A' \text{ is a graded lifting of } A \text{ to } S_R \right\} \Big/ \sim$$

It is easy to see that $\text{Def}^0(A/S, -)$ is a covariant functor on $\underline{l}$ with values in $\underline{\text{Sets}}$. This is the *graded deformation functor* of A/S. Correspondingly, we denote by $\text{Def}(A/S, -)$ the non-graded deformation functor of A/S.

Recall that a morphism of covariant functors $F \to G$ on $\underline{l}$ is smooth iff the map

$$F(R') \to F(R) \underset{G(R)}{\times} G(R')$$

is surjective whenever $R' \to R$ is surjective. The tangent space t_F of F is defined to be

$$t_F = F(k[\varepsilon])$$

where $k[\varepsilon] \in \text{ob}\,\underline{l}$ is the ring of dual numbers.

When we work in the pro-category of $\underline{l}$, we denote by $\text{Hom}^c_{\text{pro-}\underline{l}}(-,-)$ the continuous hom-functor in this category.

Definition 2.1. A pro-$\underline{l}$ object $R_V(A/S)$ is called a hull for $\text{Def}(A/S, -)$ if there is a smooth morphism of functors

$$\text{Hom}^c_{\text{pro-}\underline{l}}(R_V(A/S), -) \to \text{Def}(A/S, -)$$

on $\underline{l}$ which induces an isomorphism on the tangent spaces.

Similarly we define the hull $R^0_V(A/S)$ of $\text{Def}^0(A/S, -)$. Note that by definition there is a lifting A_v of A to S_H where $H=R_V(A/S)$, which will be called a versal lifting, with the property that each time there is a lifting A' of A to S_R where $R \in \text{ob}\,\underline{l}$, there exists a morphism $H \to R$ in pro-$\underline{l}$ such that $A_v \otimes_H R \cong A'$. For further details, see [9] and [6].

Look at the canonical morhism of functors

$$\text{Def}^0(A/S, -) \to \text{Def}(A/S, -)$$

and at a corresponding V-morphism

$$R_V(A/S) \to R_V^0(A/S) .$$

If this morphism splits we have solved the problem mentioned at the beginning of this paragraph with R' in pro-$\underline{l}$ and $R=k$.

In [6] we find a very general theorem describing these hulls. Following [6] we notice that since A is a finitely generated S-algebra, the group $H^i(S,A,A)$ for a given i is finite as an A-module. We pick a countable k-basis $\{v_j\}_{j\in J}$ of homogeneous elements of $H^i = H^i(S,A,A)$ and define a topology on H^i in which a basis for the neighbourhoods of zero are those subspaces containing all but a finite number of these v_j. Let

$$H^{i*} = \operatorname{Hom}_k^c(H^i, k)$$

for $i=1,2$ be endowed with the natural topology defined by the dual basis, and let T^i for $i=1,2$ be the completion of $\operatorname{Free}_V(H^{i*}) = V[v_j^*]_{j\in J}$ in the topology in which a basis for the neighbourhoods of zero are those ideals containing some power of the maximal ideal and intersecting H^{i*} in an open subspace. If H^i is a finite k-vector space then T^i is a convergent power series algebra on V.

THEOREM 2.2. *There is a morphism of complete local rings*

$$o = o(A)\colon T^2 \to T^1$$

such that

$$R_V(A/S) \cong T^1 \hat{\otimes}_{T^2} V$$

We shall call the morphism $o\colon T^2 \to T^1$ an obstruction morphism. For a proof of (2.2), see [6, (4.2.4)]. However, the main ideas of the proof are contained in the proof of (2.3).

Similar results are true for $R_V^0(A/S)$. If

$${}_0H^{i*} = \operatorname{Hom}_k^c({}_0H^i(S,A,A), k)$$

and ${}^0T^i$ for $i=1,2$ are the completion of $\operatorname{Free}_V({}_0H^{i*})$ in the corresponding topology, then there is a morphism of complete local rings

$$o_0 = o_0(A)\colon {}^0T^2 \to {}^0T^1$$

such that

$$R_V^0(A/S) \cong {}^0T^1 \hat{\otimes}_{{}^0T^2} V .$$

In the proof of the splitting of $R_V(A/S) \to R_V^0(A/S)$, the following theorem is essential. Consider T^i for $i=1,2$ as graded V-algebras by defining the degrees of the indeterminates v_j^* to be equal to their degrees considered as elements of H^{i*}. If we grade ${}^0T^i$ in the same way, then all elements of ${}^0T^i$ will be of degree zero.

THEOREM 2.3. *Let A be a finitely generated graded S-algebra. Then there is a homogeneous obstruction morphism. Moreover, the versal lifting A_v is a graded V-algebra and the structure morphism $S_H \to A_v$ is homogeneous where $H = R_V(A/S)$.*

PROOF. To simplify ideas, assume $V=k$ and $H^1(S,A,A)$ finite as a k-vector space. Let $\underline{l}_n \subseteqq \underline{l}$ be the full subcategory consisting of objects R satisfying $m_R^n = 0$ where $m_R \subseteqq R$ is the maximal ideal. In this proof we shall denote by B_n the algebra B/m_B^n when $B \in \mathrm{ob}\,\underline{l}$. If we say that A_n is a versal lifting of A to S_{H_n}, then

$$A_n \cong A_v \underset{H}{\otimes} H_n$$

where A_v is a versal lifting of A to S_H.

The obstruction morphism found in [6] is constructed step by step, i.e. he constructs morphisms $o_n\colon T_n^2 \to T_n^1$ for each $n \geqq 2$ extending o_{n-1} when $n \geqq 3$ and such that

$$H_n \cong T_n^1 \underset{T_n^2}{\otimes} k \qquad \text{for } n \geqq 2\,.$$

We shall follow this construction, and each time investigate when o_n, and the corresponding versal lifting to S_{H_n}, are graded.

If $R \in \mathrm{ob}\,\underline{l}_2$, then by (1.3) we get

$$\mathrm{Def}\,(A/S,R) = H^1(S,A,A)\otimes m_R = \mathrm{Hom}_k\,(H^{1*},m_R) = \mathrm{Hom}_{\underline{l}_2}\,(T_2^1,R)\,.$$

Hence, letting $H_2 = T_2^1$ and letting o_2 be trivial, i.e. be the composition of $T_2^2 \to k \to T_2^1$, we find that H_2 represents the functor $\mathrm{Def}\,(A/S,-)$ on $\underline{l}_2$. Let A_2 be the universal lifting of A to S_{H_2}. This is the first step of the construction in [6]. Note, however, that A_2 is a graded S_{H_2}-algebra since by (1.5)

$$\mathrm{Def}^0\,(A/S,R) = {}_0H^1(S,A,A\otimes m_R) = {}_0\mathrm{Hom}_{\underline{l}_2}\,(T_2^1,R)\,.$$

By induction we can assume that the morphism

$$o_{n-1}\colon T_{n-1}^2 \to T_{n-1}^1$$

is homogeneous and that if $H_{n-1} = T_{n-1}^1 \otimes_{T_{n-1}^2} k$, the versal lifting A_{n-1} of A to $S_{H_{n-1}}$ is a graded $S_{H_{n-1}}$-algebra. To construct $o_n\colon T_n^2 \to T_n^1$ it is enough to construct its k-linear restriction $o_n\colon H^{2*} \to T_n^1$. Since the morphism $T_{n-1}^1 \to H_{n-1}$ is surjective, it allows us to write

$$H_{n-1} = T_{n-1}^1/\mathfrak{a}_{n-1}$$

for a certain ideal $\mathfrak{a}_{n-1}$ in T^1_{n-1}. Let $\pi_n\colon T^1_n \to T^1_{n-1}$ be the canonical surjection and define H'_n to be

$$H'_n = T^1_n/\mathfrak{a}'_n\,,$$

where $\mathfrak{a}'_n = \pi_n^{-1}(\mathfrak{a}_{n-1})m_{T^1_n}$. Observe that π_n induces a map

$$\pi'\colon H'_n \to H_{n-1}$$

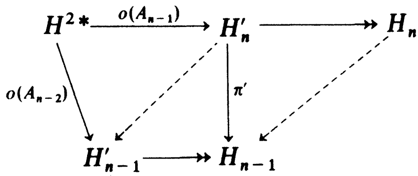
and that $\ker \pi'$ is a k-module via $H'_n \to k$. Now the problem of lifting the $S_{H_{n-1}}$-algebra A_{n-1} to $S_{H'_n}$ is by (1.3) given as an obstruction

$$(*) \qquad o(A_{n-1}) \in H^2(S_{H_{n-1}}, A_{n-1}, A_{n-1}\otimes \ker \pi') \cong H^2(S,A,A)\otimes \ker \pi'$$
$$\cong \operatorname{Hom}_k(H^{2*}, \ker \pi')\,.$$

Let H_n be the cokernel of the composition

$$H^{2*} \xrightarrow{o(A_{n-1})} \ker \pi' \subseteq H'_n\,.$$

Thus killing the obstruction of lifting, we conclude that A_{n-1} is liftable to S_{H_n}. Let A_n be any lifting of A_{n-1} to S_{H_n}. Moreover, by (2.6) there are dotted arrows fitting into a commutative diagram

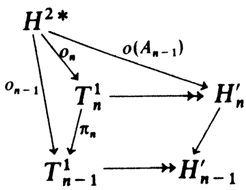


This diagram is what is needed to conclude that the equality of $\pi_n(\mathfrak{a}'_n) = \mathfrak{a}'_{n-1}$ implies the equality $\pi_n(\mathfrak{a}_n) = \mathfrak{a}_{n-1}$. Since it is easy by the definition of $\mathfrak{a}'_n$ to see that the equality of $\pi_{n-1}(\mathfrak{a}_{n-1}) = \mathfrak{a}_{n-2}$ implies the equality of $\pi_n(\mathfrak{a}'_n) = \mathfrak{a}'_{n-1}$, it follows by induction that

$$\pi_n(\mathfrak{a}'_n) = \mathfrak{a}'_{n-1}\,.$$

Therefore we can find a k-linear map o_n and a commutative diagram

(**)



such that

$$H_n \cong T_n^1 \otimes_{T_n^2} k \, .$$

This is the general step of Laudals construction. Finally we set

$$R_V(A/S) = \varprojlim H_n' \quad \text{and} \quad o = \varprojlim o_n \, .$$

Note that the obstruction $o(A_{n-1})$ in (*) is homogeneous. This follows from (1.5) and the proof of (1.9) since A_{n-1} is a graded $S_{H_{n-1}}$-algebra. Moreover, we can find a homogeneous k-linear map o_n in (**) and since it follows that S_{H_n} is graded and since any lifting A_n of A_{n-1} to S_{H_n} serves as a versal lifting when $n \geqq 3$, then by (1.9) we may choose a graded lifting to S_{H_n}.

If $V \neq k$ and $H^1(S, A, A)$ is finite as a k-vector space, just as in the general step, let V_2' be the largest quotient of $V_2 = V/m_V^2$ such that A is liftable to $S_{V_2'}$. Note that if $R \in \operatorname{ob} \underline{l}_2$, then A is liftable to S_R iff $V_2 \to R$ factorizes via $V_2 \to V_2'$, see (2.6). And one easily defines a morphism $o_2 \colon T_2^2 \to T_2^1$ such that

$$T_2^1 \otimes_{T_2^2} V_2 \cong T_2^1 \otimes_{V_2} V_2'$$

where the morphism $T_2^2 \to V_2$ is V_2-linear and sends the images of the indeterminates in T_2^2 to zero in V_2. Put

$$H_2 = T_2^1 \otimes_{T_2^2} V_2 \, .$$

Moreover, pick a lifting of A to $S_{V_2'}$. If $V_2 \to R$ factors via $V_2 \to V_2'$, we may use this lifting to define an isomorphism

$$\operatorname{Def}(A/S, R) \cong H^1(S, A, A) \otimes m_R = \operatorname{Hom}_{\underline{l}_2}(T_2^1, R) \, .$$

Using this we get an isomorphism

$$\operatorname{Def}(A/S, R) \cong \operatorname{Hom}_{\underline{l}_2}(H_2, R)$$

for *any* $R \in \operatorname{ob} \underline{l}_2$. Both sets might however be empty. This is the corresponding first step of the construction when $V \neq k$. By (1.9) there is a graded lifting of A to $S_{V_2'}$, and we may use this lifting to define isomorphisms

$$\operatorname{Def}(A/S, R) \cong \operatorname{Hom}_{\underline{l}_2}(H_2, R)$$

$$\operatorname{Def}^0(A/S, R) \cong {}_0\operatorname{Hom}_{\underline{l}_2}(H_2, R) \, .$$

Clearly the obstruction morphism $o_2 \colon T_2^2 \to T_2^1$ is homogeneous and the universal lifting A_2 of A to S_{H_2} is contained in the subset $\operatorname{Def}^0(A/S, H_2)$ of $\operatorname{Def}(A/S, H_2)$. The arguments for the rest of the proof in this case are as for the case $V = k$. Finally if $H^1(S, A, A)$ has infinite k-dimension, using the same arguments as above, there is no trouble in proving (2.3), see [6, (4.2.4)].

REMARK 2.4. Theorem (2.3) is essentially proved by Pinkham in [8, (2.3)] under some restrictions such as $V=k$ and by different methods.

The following proposition may throw some light on the morphism $R_V(A/S) \to R_V^0(A/S)$ and is needed in our proof of the splitting of $R_V(A/S) \to R_V^0(A/S)$.

PROPOSITION 2.5. *Let $p^i\colon T^i \to {}^0T^i$ for $i=1,2$ be the surjections induced from the natural injections ${}_0H^i(S,A,A) \to H^i(S,A,A)$. Then there are homogeneous obstruction morphisms*

$$o\colon T^2 \to T^1$$

$$o_0\colon {}^0T^2 \to {}^0T^1$$

such that $p^1 o = o_0 p^2$ and such that the induced morphism $R_V(A/S) \to R_V^0(A/S)$ makes the following diagram commutative

$$\begin{array}{ccc} \mathrm{Def}^0\,(A/S,-) & \to & \mathrm{Def}\,(A/S,-) \\ \uparrow & & \uparrow \\ \mathrm{Hom}\,(R_V^0(A/S),-) & \to & \mathrm{Hom}\,(R_V(A/S),-) \end{array}$$

where the morphism of the deformation functors is the natural one and where the vertical morphisms are as in (2.1).

In the proof of (2.5), we need the following easy lemma which we will not prove. See [6, (4.2.3)].

LEMMA 2.6. *Consider the commutative diagram*

$$\begin{array}{ccc} R_1' & \to & R_2' \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ R_1 & \to & R_2 \end{array}$$

where objects and morphisms are in $\underline{l}$. Assume π_1 and π_2 surjective and $(\ker\pi_1)^2 = (\ker\pi_2)^2 = 0$. If A_1 is a lifting of A to S_{R_1} and $A_2 = A_1 \otimes_{R_1} R_2$, then

$$H^2(S_{R_1}, A_1, A_1 \underset{R_1}{\otimes} \ker\pi_1) \to H^2(S_{R_2}, A_2, A_2 \underset{R_2}{\otimes} \ker\pi_2)$$

maps the obstruction $o(A_1)$ onto $o(A_2)$.

PROOF OF (2.5). Assume $H^1(S,A,A)$ has finite k-dimension and use the notations from the proof of (2.3). In the same way, let

$$R_V^0(A/S) = \varprojlim H_n^0 \quad \text{and} \quad o_0 = \varprojlim o_n^0\,.$$

(2.5) follows if we prove that there are homogeneous morphisms o_n and $o_n^{\text{υ}}$ such that the diagram

1)
$$\begin{array}{ccc} T_n^2 & \to & {}^0T_n^2 \\ \downarrow o_n & & \downarrow o_n^0 \\ T_n^1 & \to & {}^0T_n^1 \end{array}$$

is commutative for $n \geqq 2$ and that

2) if A_n is a graded versal lifting of A to S_{H_n} for the non-graded deformation functor, then $A_n \otimes_{H_n} H_n^0$ is a versal lifting to $S_{H_n^0}$ for the graded deformation functor.

For $n=2$, 1) and 2) are easy. Assume 1) and 2) for $n \leqq t-1$. It follows from (2.6) and from the construction of o_n and o_n^0 that 1) holds for $n=t$. Since A_t is a graded lifting of A_{t-1} to S_{H_t} and since $H_t \to H_t^0$ is homogeneous, it follows that $A_t \otimes_{H_t} H_t^0$ is a graded lifting of $A_{t-1} \otimes_{H_{t-1}} H_{t-1}^0$. So 2) holds for $n=t$.
The case where $H^1(S,A,A)$ has infinite k-dimension is similarly treated.

The main result we have been aiming at, is now a consequence of (2.3) and (2.5).

THEOREM 2.7. *If* ${}_vH^i(S,A,A)=0$ *for* $v>0$ *or if* ${}_vH^i(S,A,A)=0$ *for* $v<0$ *then the morphism*

$$R_V(A/S) \to R_V^0(A/S)$$

of (2.5) *splits and its right-inverse is a local V-homomorphism.*

PROOF. Denoting by ${}_0(T^1)$ the elements of T^1 of degree zero, there are by (2.3) and (2.5) commutative diagrams

$$\begin{array}{ccccc} {}_0H^{2*} & \subseteq & H^{2*} & \longrightarrow & {}_0H^{2*} \\ \downarrow & & \downarrow o & & \downarrow o_0 \\ {}_0(T^1) & \overset{i^1}{\subseteq} & T^1 & \xrightarrow{p^1} & {}^0T^1 \end{array}$$

However, since $H^1(S,A,A)$ is positively or negatively graded, the composition p^1i^1 is the identity. Using (2.2) and the corresponding characterization for the graded case, we are done.

Note also the following consequences of (2.2) and (2.5).

COROLLARY 2.8. i) *If* $V \to R_V(A/S)$ *is formally smooth, so is* $V \to R_V^0(A/S)$
ii) *If* $V \to R_V^0(A/S)$ *is formally smooth, then*

$$R_V(A/S) \to R_V^0(A/S)$$

splits.

PROOF. I claim that $V \to R_V(A/S)$ is formally smooth iff the obstruction morphism $o: T^2 \to T^1$ is trivial. Clearly if $o: T^2 \to T^1$ is trivial, then

$$R_V(A/S) = T^1 .$$

Conversely if $R_V(A/S)$ is formally smooth, then there is a morphism $R_V(A/S) \to T^1$, thus proving that $S \to A$ is liftable to S_{T^1}. This, together with the construction of $R_V(A/S)$, see the proof of (2.3), we conclude that $T^1 \to R_V(A/S)$ is the identity. It follows that o is trivial. Clearly the same arguments work for the graded case too.

By (2.5) we know that $p^1 o = o_0 p^2$. Assuming that o is trivial, we conclude that o_0 is trivial too since p^2 is surjective. Clearly (ii) is easy since by assumption $R_V^0(A/S) = {}^0T^1$.

REMARK 2.9. i) If $H^1(S, A, A)$ is negatively (respectively positively) graded and if $A[T]$ is a polynomial algebra in one variable over A where $\deg T = 1$ (respectively $\deg T = -1$), then one may prove that there is an isomorphism

$$R_V^0(A[T]/S) \cong R_V(A/S) .$$

From this isomorphism, noting that there exists a natural morphism

$$R_V^0(A[T]/S) \to R_V^0(A/S) ,$$

(2.7) will follow, see [5]. The idea of introducing the algebra $A[T]$ and comparing graded deformations of this algebra with non-graded deformations of A is due to Pinkham, see [8, (4.2) and (5.1)], and also (3.8) of this paper where I formulate a slight generalization of one of his theorems. So (2.7) follows from his results when $H^1(S, A, A)$ is negatively graded and $X = \operatorname{Proj}(A)$ is S-smooth where $S = k = V$ is an algebraically closed field.

ii) Without any assumptions on the grading of $H^1(S, A, A)$ we may prove

$$R_V^0(A[T]_T/S) \cong R_V(A)$$

where $\deg T = 1$.

REMARK 2.10. Let S_0 be the elements of degree zero in S, and assume that $S_0 \to S$ is smooth. It suffices to assume that $H^1(S_0, A, A)$ is negatively or positively graded to deduce a splitting of

$$R_V(A/S) \to R_V^0(A/S) .$$

3. Relation to projective geometry.

We know the theory of graded algebras is closely related to projective geometry. In what follows we shall compare the groups ${}_vH^i(S, A, M)$ with $A^i(S, X, \tilde{M}(v))$ when $X = \operatorname{Proj}(A)$. Moreover if

$$\varphi\colon B \to A$$

is a surjective graded morphism of S-algebras, and

$$f\colon \mathrm{Proj}\,(A) \to \mathrm{Proj}\,(B)$$

is the induced embedding, we shall relate the groups ${}_vH^i(B,A,M)$ to $A^i(S,f,\tilde{M}(v))$. Schlessinger establishes isomorphisms of this kind, using his Comparison Theorem, see [10] and [11]. In (3.3) we recall a generalization of the Comparison Theorem due to Laudal, and using this, we may prove the relationship between the groups above, as announced, so extending some theorems from [10] and [11].

In general, let X be any S-scheme, $\underline{M}$ any quasicoherent O_X-Module and let $f\colon X \to Y$ be a morphism of S-schemes. Using [6] we shall summarize some properties needed in the sequel.

3.1) [6, (3.2.7)] states that there are groups $A^i(S,X,\underline{M})$, called global cohomology groups of algebras, which are the abutment of a spectral sequence given by the term

$$E_2^{p,q} = H^p(X,\underline{A}^q(S,\underline{M}))\,.$$

If $U = \mathrm{Spec}\,(A)$ is an open affine subscheme of X, the O_X-Module $\underline{A}^q(S,\underline{M})$, or just $\underline{A}^q(\underline{M})$, is given by

$$\underline{A}^q(\underline{M})(U) = H^q(S,A,\underline{M}(U))\,.$$

If X is affine, say $X = \mathrm{Spec}\,(A)$, and $\underline{M} = \tilde{M}$ for some A-module M, we deduce

$$A^i(S,X,\underline{M}) = H^i(S,A,M)\,.$$

Moreover if X is S-smooth, we find

$$A^i(S,X,\underline{M}) = H^i(X,\theta_X \otimes_S \underline{M})$$

where $\theta_X = \underline{A}^0(O_X)$ is the sheaf of S-derivations on X.

3.2) By [6, (3.2.9)] and [6, (3.2.5)] the relative cohomology groups of algebras $A^i(S,f,\underline{M})$ are the abutment of the spectral sequence given by

$$E_2^{p,q} = H^p(Y,\underline{A}^q(f,\underline{M}))$$

if $\underline{A}^q(f,M)$ is quasi-coherent. By definition

$$A^q(f,\underline{M})(V) = A^q(B,f^{-1}(V),\underline{M})$$

where $V = \mathrm{Spec}\,(B)$ is any open affine subscheme of Y. Therefore if f is affine, say $f^{-1}(V) = \mathrm{Spec}\,(A)$,

$$A^q(B,f^{-1}(V),\underline{M}) = H^q(B,A,H^0(f^{-1}(V),\underline{M}))\,.$$

3.3) Let $Z \subseteqq X$ be locally closed. By [6, (3.2.10)] and [6, (3.2.11)] there is an exact sequence

$$\to A_Z^n(S,X,\underline{M}) \to A^n(S,X,\underline{M}) \to A^n(S,X-Z,\underline{M}) \to A_Z^{n+1}(S,X,\underline{M}) \to$$

where the groups $A_Z^n(S,X,\underline{M})$ are the abutment of a spectral sequence given by the term

$$E_2^{p,q} = A^p(S,X,\underline{H}_Z^q(\underline{M})) .$$

If $X = \mathrm{Spec}\,(A)$ and $Z = V(I)$ for a suitable ideal $I \subseteqq A$, we write

$$H_I^n(S,A,M) = A_Z^n(S,X,\tilde{M}) .$$

3.4) Let $f\colon X \to Y$ be an affine morphism of S-schemes. By [6, (3.3.4)] there is a long exact sequence

$$\to A^n(S,f,\underline{M}) \to A^n(S,X,\underline{M}) \to A^n(S,Y,f_*\underline{M}) \to A^n(S,f,\underline{M}) \to$$

Let us now turn to the problem of relating the groups ${}_vH^i(B,A,M)$ to $A^i(S,f,M(v))$. Suppose S is noetherian and let A and B be finitely generated, positively graded S-algebras generated by elements of degree 1. Assume $A_0 = B_0 = S$. Let

$$\varphi\colon B \to A$$

be a surjective graded S-algebra morphism and let

$$f\colon X = \mathrm{Proj}\,(A) \to \mathrm{Proj}\,(B) = Y$$

be the corresponding embedding. Put

$$m = \coprod_{v=1}^{\infty} A_v \quad \text{and} \quad X' = \mathrm{Spec}\,(A) - V(m) .$$

Let $\Pi\colon X' \to X$ be the obvious morphism. Π is an affine smooth surjection. Suppose $b \in B$ is homogeneous and $a = \varphi(b)$. If M is a graded A-module of finite type, we shall denote by M_a the localization of M in $\{1,a,a^2,\ldots\}$. Let $M_{(a)}$ be the homogeneous part of M_a of degree zero.

Since $B_{(b)} \to B_b$ is flat and since $A_{(a)} \otimes_{B_{(b)}} B_b \cong A_a$, it follows that

$$H^q(B_{(b)},A_{(a)},M_a) \cong H^q(B_b,A_{(a)} \underset{B_{(b)}}{\otimes} B_b,M_a) \cong H^q(B,A,M)_a .$$

Using this and (3.2) we find isomorphisms of O_Y-Modules

$$\underline{A}^q(f,\tilde{M}(v))(\mathrm{Spec}\,B_{(b)}) = H^q(B_{(b)},A_{(a)},M(v)_{(a)}) \cong H^q(B,A,M(v))_{(a)} .$$

Similarly by (3.1) we find that

$$A^q(B,\tilde{M})(\mathrm{Spec}\,A_a) = H^q(B,A_a,M_a) = H^q(B,A,M)_a$$

as $O_{X'}$-Modules. This proves

$$f_* \Pi_* A^q(B, \tilde{M}) \cong \coprod_v A^q(f, \tilde{M}(v)) .$$

LEMMA 3.5. *With notations as above there is an isomorphism*

$$A^i(S, f, \tilde{M}(v)) \cong {}_vA^i(B, X', \tilde{M})$$

where ${}_vA^i(B, X', \tilde{M})$ *is the homogeneous piece of* $A^i(B, X', \tilde{M})$ *of degree* v.

PROOF. Going back to the definitions of $A^i(S, f, \tilde{M}(v))$ and $A^i(B, X', \tilde{M})$ in [6], we deduce a morphism

$$A^i(S, f, \tilde{M}(v)) \to {}_vA^i(B, X', \tilde{M}) .$$

The corresponding morphism of spectral sequences $H^p(Y, \underline{A}^q(f, \tilde{M}(v))) \to {}_vH^p(X', \underline{A}^q(B, \tilde{M}))$ is an isomorphism for every p and q since

$${}_vH^p(X', \underline{A}^q(B, M)) \cong {}_vH^p(Y, f_* \Pi_* \underline{A}^q(B, M)) .$$

THEOREM 3.6. *Let n be an integer. If φ: $B \to A$ is surjective and if* $\operatorname{depth}_m M \geqq n$ *then the morphisms*

$${}_vH^i(B, A, M) \to A^i(S, f, \tilde{M}(v))$$

are isomorphisms for $i < n$ and injections for $i = n$.

PROOF. By (3.3) there is a long exact sequence

$$\to H^i_m(B, A, M) \to H^i(B, A, M) \to A^i(B, X', M) \to H^{i+1}_m(B, A, M) \to$$

Since $\operatorname{depth} M \geqq n$, we conclude that

$$H^q_m(M) = 0 \quad \text{for } q \leqq n-1 .$$

Moreover, $H^0(B, A, -) = 0$ since φ is surjective. By the spectral sequence of (3.3) we deduce

$$H^i_m(B, A, M) = 0 \quad \text{for } i \leqq n .$$

REMARK 3.7. If $\operatorname{depth}_m A \geqq 2$, then the morphisms

$${}_vH^1(B, A, A) \xrightarrow{\sim} A^1(S, f, O_X(v))$$

$${}_vH^2(B, A, A) \longrightarrow A^2(S, f, O_X(v))$$

are isomorphisms and injections respectively. Using this for $v = 0$, it follows from (2.2) and its corresponding theorem in [6] that their deformation theories are the same. In particular, let S be a field, say $S = k$, and let $\underline{l}$ be the category of local artinian k-algebras. If $\mathrm{Def}^0(\varphi, -) = \mathrm{Def}^0(A/B, -)$ are defined on this $\underline{l}$ as in chapter 2, using trivial liftings of B, and if $\mathrm{Hilb}_f(-)$ is similarly defined in the projective case, then

$$\mathrm{Def}^0(\varphi, -) \xrightarrow{\simeq} \mathrm{Hilb}_f(-) .$$

In fact, by the long exact sequence in the proof of (3.6), this isomorphism follows if

$$\mathrm{depth}_m A \geqq 1 \quad \text{and} \quad {}_0H^2_m(B, A, A) = 0 .$$

Note that if $\mathrm{depth}_m A \geqq 1$, then

$${}_0H^2_m(B, A, A) = {}_0\mathrm{Hom}_B(I, H^1_m(A))$$

where $I = \ker \varphi$.

In [2, prop. 1] there is essentially a direct proof for the isomorphism

$$\mathrm{Def}^0(\varphi, -) \xrightarrow{\cong} \mathrm{Hilb}_f(-)$$

assuming $\mathrm{depth}_m A \geqq 2$ and $Y = \boldsymbol{P}^n_k$.

REMARK 3.8. The isomorphism of deformation functors (3.7) together with (2.9) makes it easy to prove the following theorem, due to Pinkham: Let X be a closed subscheme of $\boldsymbol{P}^n_k$ and let A be the minimal cone (which is characterized by $\mathrm{depth}_m A \geqq 1$). If $\bar{X} = \mathrm{Proj}(A[T])$ is the projective cone in $\boldsymbol{P}^{n+1}_k$ and if $H^1(k, A, A)$ is negatively graded, then there is a smooth morphism of functors

$$\mathrm{Hilb}_{\bar{X}}(-) \to \mathrm{Def}(A/k, -)$$

on $\underline{l}$.

So far we have concentrated on deformations of embeddings. The relationship between the groups ${}_vH^i(S, A, M)$ and $A^i(S, X, \tilde{M}(v))$ is given by our next theorem, see also [6, (5.2.7)].

THEOREM 3.9. *There are canonical morphisms*

$${}_vH^i(S, A, M) \to A^i(S, X, \tilde{M}(v))$$

for any $i \geqq 0$ and any v. If $n \geqq 1$ and if $\mathrm{depth}_m M \geqq n+2$, *then the morphisms above are bijective for $1 \leqq i < n$ and injective for $i = n$.*

PROOF. Consider the following two exact sequences

$$\to H^i_m(S, A, M) \to H^i(S, A, M) \to A^i(S, X', \tilde{M}) \to H^{i+1}_m(S, A, M) \to$$

$$\to A^i(S, \Pi, \tilde{M}) \to A^i(S, X', \tilde{M}) \to A^i(S, X, \Pi_* \tilde{M}) \to A^{i+1}(S, \Pi, \tilde{M})$$

with

$$\Pi\colon X' = \operatorname{Spec}(A) - V(m) \to X = \operatorname{Proj}(A)$$

as before. By (3.2) $A^{(\cdot)}(S, \Pi, \tilde{M})$ is the abutment of a spectral sequence given by $E_2^{p,q} = H^p(X, \underline{A}^q(\Pi, \tilde{M}))$ where by definition

$$\underline{A}^q(\Pi, \tilde{M})(\operatorname{Spec} A_{(a)}) = H^q(A_{(a)}, A_a, M_a) .$$

It follows that

$$\underline{A}^q(\Pi, \tilde{M}) = \begin{cases} 0 & \text{for } q \neq 0 \\ \Pi_* \tilde{M} & \text{for } q = 0 \end{cases}$$

Since depth $M \geqq n+2$

$$A^i(S, \Pi, \tilde{M}) = H^i(X, \Pi_* \tilde{M}) = \coprod H^i(X, M(\nu)) = 0$$

for $1 \leqq i \leqq n$. Furthermore if $i \leqq n+1$, the depth condition implies that $H^i_m(M) = 0$. By the spectral sequence of (3.3) we conclude that

$$H^i_m(S, A, M) = 0$$

for $i \leqq n+1$. The theorem now follows from the two exact sequence stated at the beginning of this proof.

Remark 3.10. If $\operatorname{depth}_m A \geqq 3$ and if $A^1(S, X, O_X(\nu)) = 0$ for every ν, then $H^1(S, A, A) = 0$. By (3.1) it follows that if X is locally rigid, i.e. if $\underline{A}^1(O_X) = 0$, then

$$A^1(S, X, O_X(\nu)) = H^1(X, \theta_X(\nu))$$

and the statement above reduces to a rigidity theorem of Schlessinger. See [4, (2.2.6)] and [13].

4. Positive or negative grading of H^1.

In this paragraph we shall see that if $X \subseteq \boldsymbol{P}^N_S$ is closed and satisfies some weak conditions, then after a suitable twisting the minimal cone B of the corresponding embedding will satisfy ${}_\nu H^1(S, B, B) = 0$ for $\nu \geqq 1$ or for $\nu \leqq 1$.

Suppose S is noetherian and let

$$A = \coprod_{\nu=0}^{\infty} A_\nu$$

be a graded $A_0 = S$-algebra of finite type, generated by A_1. Denote by m the augmentation ideal of A; i.e., $m = \coprod_{\nu=1}^{\infty} A_\nu$. Let M be any graded A-module and put $M_{(d)} = \coprod_{\nu=-\infty}^{\infty} M_{d\nu}$ for $d \geqq 1$. First we shall be interested in relating the groups $H^i(S,A,M)$ to the groups $H^i(S,A_{(d)},M_{(d)})$; later we shall deduce some corollaries concerning the grading of $H^1(S,B,B)$ where $B=A_{(d)}$. Again the work of Schlessinger in [11] is good references for closely related results. See also [7].

Note that there are morphisms

$$\varphi^i_{d,v}\colon {}_{dv}H^i(S,A,M) \to {}_vH^i(S,A_{(d)},M_{(d)})$$

for any $i\geqq 0$, $d\geqq 1$ and any v. In fact if $D\colon A \to M$ is an S-derivation of degree dv, the composition $A_{(d)} \hookrightarrow A \to M$ factorizes via $A_{(d)} \to M_{(d)} \hookrightarrow M$ where $A_{(d)} \to M_{(d)}$ is an S-derivation of degree v, thus defining $\varphi^0_{d,v}$. The same idea defines $\varphi^i_{d,v}$ as well for any $i\geqq 0$ since $A_{(d)} \hookrightarrow A$ induces a morphism

$${}_{dv}H^i(S,A,M) \to H^i(S,A_{(d)},M)$$

and its image is contained in ${}_vH^i(S,A_{(d)},M_{(d)})$ via the injection

$${}_vH^i(S,A_{(d)},M_{(d)}) \hookrightarrow H^i(S,A_{(d)},M)$$

defined by $M_{(d)} \hookrightarrow M$. This follows since one may prove, using the isomorphism 2) of the proof of (1.7), that the composition

$${}_{dv}H^i(S,A,M) \to H^i(S,A_{(d)},M) \to H^i(S,A_{(d)},M/M_{(d)})$$

is zero.

LEMMA 4.1. *Let $d\geqq 1$ be an integer which is invertible in S. If $X'=\operatorname{Spec}(A)-V(m)$ and $X'_{(d)}=\operatorname{Spec}(A_{(d)})-V(m_{(d)})$, then*

$$A^i(S,X',\tilde{M})_{(d)} \to A^i(S,X'_{(d)},\tilde{M}_{(d)})$$

are isomorphic for every i.

PROOF. The canonical morphism $A_{(d)} \hookrightarrow A$ induces a morphism of schemes $X' \to X'_{(d)}$, thus a homomorphism

$$A^i(S,X',\tilde{M}) \to A^i(S,X'_{(d)},\tilde{M})\,.$$

The $A_{(d)}$-linear map $M_{(d)} \hookrightarrow M$, which is split, gives rise to an injection

$$A^i(S,X'_{(d)},\tilde{M}_{(d)}) \hookrightarrow A^i(S,X'_{(d)},\tilde{M})$$

and it follows that there is a morphism

$$A^i(S,X',\tilde{M})_{(d)} \to A^i(S,X'_{(d)},\tilde{M}_{(d)})$$

for any i. Consider the commutative diagram

$$\begin{array}{ccc} X' & \to & X'_{(d)} \\ & {}_{\pi}\searrow \quad \swarrow_{\pi_{(d)}} & \\ & X & \end{array}$$

where $X = \text{Proj}(A) \cong \text{Proj}(A_{(d)})$. The morphisms of this diagram give long exact sequences (see (3.4)) and morphisms between them as follows:

$$\begin{array}{ccccccccc} \to & A^i(S,\Pi,\tilde{M})_{(d)} & \to & A^i(S,X',\tilde{M})_{(d)} & \to & A^i(S,X,\Pi_*\tilde{M})_{(d)} & \to & A^{i+1}(S,\Pi,\tilde{M})_{(d)} & \to \\ & \downarrow & & \downarrow & & \downarrow\cong & & \downarrow & \\ \to & A^i(S,\Pi_{(d)},\tilde{M}_{(d)}) & \to & A^i(S,X'_{(d)},\tilde{M}_{(d)}) & \to & A^i(S,X,\Pi_{(d)*}\tilde{M}_{(d)}) & \to & A^{i+1}(S,\Pi_{(d)},\tilde{M}_{(d)}) & \to \end{array}$$

It is therefore enough to prove that

$$A^i(S,\Pi,\tilde{M})_{(d)} \to A^i(S,\Pi_{(d)},\tilde{M}_{(d)})$$

are isomorphic for any i. In the proof of (3.9) we found that

$$A^i(S,\Pi,\tilde{M})_{(d)} \cong H^i(X,\Pi_*\tilde{M})_{(d)} = \coprod_{\nu=-\infty}^{\infty} H^i(X,\tilde{M}(d\nu)) .$$

In the same way, we prove that

$$A^i(S,\Pi_{(d)},\tilde{M}_{(d)}) \cong \coprod_{\nu=-\infty}^{\infty} H^i(X,\tilde{M}(d\nu)) .$$

And it is not difficult to check that the morphism $A^i(S,\Pi,\tilde{M})_{(d)} \to A^i(S,\Pi_{(d)},\tilde{M}_{(d)})$, via the isomorphisms above, is given as the multiplication by the number d. This proves (4.1).

Having the following commutative diagrams of long exact sequences in mind

$$(*)\quad \begin{array}{ccccccccc} \to & {}_{d\nu}H^i(S,A,M) & \to & {}_{d\nu}A^i(S,X',\tilde{M}) & \to & {}_{d\nu}H^{i+1}_m(S,A,M) & \to & {}_{d\nu}H^{i+1}(S,A,M) & \to \\ & \downarrow\varphi^i_{d,\nu} & & \downarrow\cong & & \downarrow{}_m\varphi^i_{d,\nu} & & \downarrow\varphi^{i+1}_{d,\nu} & \\ \to & {}_\nu H^i(S,A_{(d)},M_{(d)}) & \to & {}_\nu A^i(S,X'_{(d)},\tilde{M}_{(d)}) & \to & {}_\nu H^{i+1}_{m_{(d)}}(S,A_{(d)},M_{(d)}) & \to & {}_\nu H^{i+1}(S,A_{(d)},M_{(d)}) & \to \end{array}$$

we may prove

THEOREM 4.2. *Let $d \geqq 1$ be invertible in S and let $n \geqq 0$ be an integer. If M is a graded A-module of finite type satisfying* depth ${}_mM \geqq n+1$ *then*

$$\varphi^i_{d,\nu}\colon {}_{d\nu}H^i(S,A,M) \to {}_\nu H^i(S,A_{(d)},M_{(d)})$$

are isomorphisms for $i<n$ and injections for $i=n$.

PROOF. Since depth $M \geqq n+1$ and since $H^i_m(M)_{(d)} \cong H^i_{m_{(d)}}(M_{(d)})$ for any i, it follows that depth $M_{(d)} \geqq n+1$. Hence by the spectral sequence in (3.3), we deduce

$$H^i_m(S,A,M) = H^i_{m_{(d)}}(S,A_{(d)},M_{(d)}) = 0 \quad \text{for } i \leqq n .$$

and (4.2) follows easily by (*).

In particular (4.2) states that if $\operatorname{depth}_m A \geqq 3$, then

$$(**) \qquad \begin{aligned} {}_{dv}H^1(S, A, A) &\xrightarrow{\sim} {}_{v}H^1(S, A_{(d)}, A_{(d)}) \\ {}_{dv}H^2(S, A, A) &\hookrightarrow {}_{v}H^2(S, A_{(d)}, A_{(d)}) \end{aligned}$$

for every v. We shall be interested in proving (**) under less restrictive hypotheses.

Assume for a moment $\operatorname{depth}_m A \geqq 2$. If ${}_m\varphi^2_{d,v}$ is injective, then using (*) for $i=1$ (**) follows. Note that the map

$${}_m\varphi^2_{d,v}\colon \operatorname{Der}_S(A, H^2_m(A)) \to \operatorname{Der}_S^{\bullet}(A_{(d)}, H^2_m(A)_{(d)})$$

is precisely the map $\varphi^0_{d,v}$ for $M = H^2_m(A)$. So when is $\varphi^0_{d,v}$ injective?

If M is any graded A-module, then define

$$\psi_{s,d}\colon M_{s-d} \to \operatorname{Hom}_S(A_d, M_s)$$

to be the S-linear map given by $[\psi_{s,d}(m)](g) = gm$ where $g \in A_d$.

LEMMA 4.3. *Let $d \geqq 1$ be invertible in S and let $v \in \mathbf{Z}$. Put $s = dv + d + 1$. If*

$$\psi_{s,d}\colon M_{s-d} \to \operatorname{Hom}_S(A_d, M_s)$$

is injective, so is the natural map

$$\varphi^0_{d,v}\colon {}_{dv}\operatorname{Der}_S(A, M) \to {}_{v}\operatorname{Der}_S(A_{(d)}, M_{(d)})\,.$$

PROOF. Let $D \in {}_{dv}\operatorname{Der}(A, M)$ satisfy $\varphi^0_{d,v}(D) = 0$, i.e. the composition $A_d \hookrightarrow A \xrightarrow[D]{} M$ is zero. I claim that the composition

$$A_{d+1} \hookrightarrow A \xrightarrow[D]{} M$$

is zero too. In fact, it is enough to prove that $D(f) = 0$ where $f = \prod_{i=1}^{d+1} f_i$ is a product of linear forms. By the product rule for derivations we have

$$D(f) = \sum_{i=1}^{d+1} g_i D(f_i)$$

where $g_i f_i = f$. Furthermore, since $\deg g_i = d$, it follows that

$$D(f) = g_i D(f_i) + f_i D(g_i) = g_i D(f_i)\,,$$

thus $dD(f) = 0$. To prove that $D = 0$, it suffices to prove that $D(x) = 0$ where x is any element in A_1. Since $D(x) \in M_{dv+1} = M_{s-d}$, it is enough to prove that $gD(x) = 0$ for any $g \in A_d$ by the injectivity of $\psi_{s,d}$. This follows from the first part of the proof since

$$gD(x) = D(gx) - xD(g)\,.$$

THEOREM 4.4. *Let $d \geqq 1$ be invertible in S and let $v \in \mathbf{Z}$. If* $\operatorname{depth}_m A \geqq 2$ *and if*

$$H^1(X, O_X(dv+1)) \to \operatorname{Hom}_S(A_d, H^1(X, O_X(dv+d+1)))$$

is injective, then (**) *holds.*

A consequence of (4.4) is that if

$$H^1(X, O_X(1)) \to \operatorname{Hom}_S(A_d, H^1(X, O_X(d+1)))$$

is injective and if $\operatorname{depth}_m A \geqq 2$, then

$$R_V^0(A_{(d)}/S) \xrightarrow{\sim} R_V^0(A/S)$$

are isomorphic.

REMARK 4.5. i) Let S be a field and assume that $X = \operatorname{Proj}(A)$ is of pure dimension 1. The injectivity of

$$H^1(X, O_X(s-d)) \to \operatorname{Hom}_S(H^0(X, O_X(d)), H^1(X, O_X(s)))$$

is by duality equivalent to the surjectivity of

$$H^0(X, O_X(d)) \underset{S}{\otimes} H^0(X, \omega_X(-s)) \to H^0(X, \omega_X(-s+d))$$

where ω_X is the dualizing sheaf. If $\omega_X(-s)$ is generated by its global sections, the surjectivity follows for $d \gg 0$. In particular if $s = dv+d+1$ and d is large, this morphism is surjective for $v \leqq -2$ and it is surjective for $v = -1$ if $\omega_X(-1)$ is generated by its global sections. The surjectivity for $v \geqq 1$ and $d \gg 0$ is trivially true.

EXAMPLE 4.6. Suppose S is a field and $d \geqq 1$ is invertible in S. Let $X \subseteq \boldsymbol{P}_S^N$ be a curve such that the dualizing sheaf ω_X satisfies $\omega_X \cong O_X(r)$ where $r \geqq 1$ and such that the minimal cone A satisfies $\operatorname{depth}_m A \geqq 2$. Let $s = dv+d+1$. By duality it is easy to see that

$$H^1(X, O_X(s-d)) \to \operatorname{Hom}_S(H^0(X, O_X(d)), H^1(X, O_X(s)))$$

is injective for $s \leqq r$. So (**) holds for $v \leqq -1$ and any invertible $d \geqq 1$. Moreover

$$R_V^0(A/S) \cong R_V^0(A_{(d)}/S)$$

when $d \leqq r-1$. Finally (**) holds for $v \geqq 1$ if $d > r-1$ since in this case we easily prove

$$H^1(X, O_X(dv+1)) = 0 .$$

A consequence of the isomorphism of the hulls above is that if X is reduced, and if $X \subseteq \boldsymbol{P}_S^N$ is locally a complete intersection, $R_V^0(A_{(d)}/S)$ for all invertible $d \geqq 1$ in S are smooth local S-algebras if $R_V^0(A/S)$ is.

Before deducing some corollaries on the grading of $H^1(S, A_{(d)}, A_{(d)})$, note the following result.

PROPOSITION 4.7. *Let $d \geqq 1$ be an integer which is invertible in S and let $\nu \in \mathbf{Z}$.*

i) *If* $\operatorname{depth}_m A \geqq 1$, *then* (**) *holds for* $\nu \geqq 1$ *and* $d \gg 0$.
ii) *If* $\operatorname{depth}_m A \geqq 2$, *and if* $H^1(X, O_X(n)) = 0$ *for* $n \ll 0$, *then* (**) *holds for* $\nu \leqq -1$ *and* $d \gg 0$.

PROOF. If $\operatorname{depth}_m A \geqq 2$, then i) and ii) are trivial consequences of (4.4). If $\operatorname{depth}_m A \geqq 1$, we have

$$_{d\nu}H^1_m(S, A, A) \;=\; _{d\nu}H^0(S, A, H^1_m(A)) \;=\; 0$$

$$_{\nu}H^1_{m_{(d)}}(S, A_{(d)}, A_{(d)}) \;=\; _{\nu}H^0(S, A_{(d)}, H^1_m(A)_{(d)}) \;=\; 0$$

for $\nu \geqq 1$ and large d since the group $_tH^1_m(A)$ vanishes for large t. So by (*) it suffices to prove that $_m\varphi^2_{d,\nu}$ is injective, hence it is enough to prove that

$$_{d\nu}H^2_m(S, A, A) \;=\; 0$$

for $\nu \geqq 1$ and large d. Clearly $_{d\nu}H^0(S, A, H^2_m(A)) = 0$ for such ν and d. Moreover if $\varphi\colon F \to A$ is a surjective graded S-algebra homomorphism where F is S-smooth, then

$$_{d\nu}\operatorname{Hom}_F(\ker \varphi, H^1_m(A)) \;=\; _{d\nu}H^1(F, A, H^1_m(A)) \to \, _{d\nu}H^1(S, A, H^1_m(A))$$

is surjective. Since $_{d\nu}\operatorname{Hom}_F(\ker \varphi, H^1_m(A)) = 0$ for ν and d as above, it follows that

$$_{d\nu}H^1(S, A, H^1_m(A)) \;=\; 0\,.$$

which proves (4.7).

Let $B = A_{(d)}$. If (**) is satisfied for $\nu \neq 0$, it is easy to see when $H^1(S, B, B)$ is negatively or positively graded. We only need to know when $_tH^1(S, A, A)$ vanishes for small, respectively large, t. Recall that if $\operatorname{depth}_m A \geqq 1$ and $X = \operatorname{Proj}(A)$ satisfies $\underline{A}^1(O_X) = 0$, then

$$_tH^1(S, A, A) \;=\; 0 \qquad \text{for large } t\,.$$

This follows since, as we have seen, there are exact sequences

$$_tH^0(S, A, H^1_m(A)) \to \, _tH^1(S, A, A) \to \, _tA^1(S, X', O_{X'}) \to \dots$$

and

$$_tA^1(S, \Pi, O_{X'}) \to \, _tA^1(S, X', O_{X'}) \to \, _tA^1(S, X, \Pi_* O_{X'}) \to \dots$$

where

$$ {}_tA^1(S,\Pi,O_{X'}) \cong H^1(X,O_X(t)) \quad \text{and} \quad {}_tA^1(S,X,\Pi_*O_{X'}) \cong H^1(X,\theta_X(t)) . $$

The vanishing of ${}_tH^1(S,A,A)$ for small t is always true since if $\varphi\colon F \to A$ is a surjective S-algebra morphism where F is S-smooth, then

$$ {}_t\mathrm{Hom}_F(\ker\varphi, A) \to {}_tH^1(S,A,A) $$

is surjective. Thus we have proved

THEOREM 4.8. *Let* $X = \mathrm{Proj}(A)$ *where* $\mathrm{depth}_m A \geqq 1$.

i) *If X is S-smooth, then there is a graded S-algebra B satisfying* $X \cong \mathrm{Proj}(B)$ *such that*

$$ {}_vH^1(S,B,B) = 0 \quad \textit{for } v \geqq 1 . $$

ii) *If* $\mathrm{depth}_m A \geqq 2$ *and if the morphism*

$$ H^1(X,O_X(t)) \to \mathrm{Hom}_S(H^0(X,O_X(-t+1)), H^1(X,O_X(1))) $$

is injective for small t, then there is a graded S-algebra B satisfying X $\cong \mathrm{Proj}(B)$ *and*

$$ {}_vH^1(S,B,B) = 0 \quad \textit{for } v \leqq -1 . $$

iii) *If X satisfies the conditions* i) *and* ii), *then there is an S-algebra B where* $X \cong \mathrm{Proj}(B)$ *such that*

$$ {}_0H^1(S,B,B) \cong H^1(S,B,B) . $$

REMARK 4.9. By (4.5), if $\dim X = 1$ and if $\omega_X(-1)$ is generated by its global sections, then (4.8, ii) holds. Mumford proves in [7] by different methods that if X is a smooth curve over the complex numbers and is not hyperelliptic, then (4.8, iii) holds. Schelssinger treats the higher dimensional cases, i.e. $\dim X \geqq 2$, over the complex numbers and proves (4.8, iii) in [11] using $B = A_{(d)}$ with $d \gg 0$ and (4.7, ii).

5. The existence of a k-algebra which is unliftable to characteristic zero.

In [12] Serre gives an example of a k-smooth projective variety X in characteristic p which cannot be lifted to characteristic zero. This means that for any complete local ring V of characteristic zero such that $V/m_V = k$, it is impossible to lift X to V. His variety is of the form $X = Y/G$ where Y is a complete intersection of dimension 3 and G is a finite group operating on Y without fixpoints. Furthermore p divides the order of G.

$X = \mathrm{Proj}\,(B)$. Hence (2.7) proves that B cannot be lifted to any complete local ring V of characteristic zero. In fact the example of Serre satisfies even (4.8, iii), thus proving the existence of a graded k-algebra C satisfying ${}_{\nu}H^1(k, C, C) = 0$ for $\nu \neq 0$, such that $X = \mathrm{Proj}\,(C)$. (2.7) reduces to the almost trivial result

$$R_V^0(C/k) \cong R_V(C/k)\,.$$

Clearly C is unliftable to any complete local ring V of characteristic zero.

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