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# LOCAL STRUCTURE OF THE ZERO-SETS OF DIFFERENTIABLE MAPPINGS AND APPLICATION TO BIFURCATION THEORY

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## 1. Introduction.

Let $f$ be a real $C^2$ function defined in a neighbourhood of the origin in $\mathbf{R}^n$, $n \geqq 2$. By the Taylor formula, $f(x) = f(0) + f_x(0)x + \frac{1}{2}f_{xx}(0)(x,x) + o(\|x\|^2)$, where $f_x(0)$ denotes grad $f(0)$ (interpreted as a linear form), $f_{xx}(0)$ is the bilinear form whose matrix consists of the second order partial derivatives of $f$ at the origin, and $f_{xx}(0)(y,y)$ means evaluation of $f_{xx}(0)$ at $(y,y)$. Suppose that $f(0)=0$ and $f_x(0)=0$. Suppose also that $f_{xx}(0)$ is nondegenerate (this means that the matrix of $f_{xx}(0)$ is nonsingular). It is known that under those assumptions there exists a local coordinate change $y(x)$ such that $f(x) \equiv \frac{1}{2}f_{xx}(0)(y(x),y(x))$. This is a special case of the Morse Lemma (see e.g. [6, p. 71] or [2, p. 145]). An easy consequence of this lemma is the following

(1.1) COROLLARY. *If $f_{xx}(0)$ is nondegenerate, then, in a neighbourhood of the origin, the zero-set of $f$ is homeomorphic to the zero-set of $f_{xx}(0)$.*

The development of local differential analysis has led to much more general results than Corollary (1.1). Let $p\colon \mathbf{R}^n \to \mathbf{R}^m$ be an $m$-tuple of polynomials of degree $k \geqq 1$ such that $p(0)=0$. In [3, Theorem 1], T. C. Kuo gives necessary and sufficient conditions that the zero-set of $p$ be locally homeomorphic at the origin to the zero-set of any $C^{k+1}$ mapping whose Taylor expansion of order $k$ is identical with $p$. A similar result for $C^k$ mappings is indicated in [3, Appendix 1].

In the present paper we give another generalization of Corollary (1.1) (see Theorem (1.2) below). Also, in order to illustrate possible applications of this theorem, we discuss some problems in bifurcation theory.

For a $C^1$ mapping $f\colon \mathbf{R}^n \to \mathbf{R}^m$ let $J_f(x)$ denote the Jacobi matrix of $f$ at $x$. Furthermore, let

$$Z_f = \{x \in \mathbf{R}^n : f(x)=0\} \quad \text{and} \quad B(a,r) = \{x \in \mathbf{R}^n : \|x-a\| \leqq r\} .$$

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The mapping $f$ is said to be homogeneous of degree $k$ if $f(tx)=t^k f(x)$ for all $x$ and all $t>0$.

Now we state the main result:

(1.2) THEOREM. *Let $f$ be a $C^1$ mapping of an open neighbourhood of the origin in $\mathbf{R}^n$ into $\mathbf{R}^m$, $n>m\geqq 1$. Suppose that*:

(i) $f=p+g=(p_1,\ldots,p_m)+(g_1,\ldots,g_m)$, *where all $p_i$ are homogeneous functions of (not necessarily integral) degree $k_i\geqq 1$ and $g_i$ are $o(\|x\|^{k_i})$ as $x\to 0$*;

(ii) *All first partial derivatives of $g_i$ are $o(\|x\|^{k_i-1})$ as $x\to 0$*;

(iii) *$J_p(x)$ has rank $m$ for all $x\in Z_p-\{0\}$.*

*Then $Z_p\cap B(0,r)$ and $Z_f\cap B(0,r)$ are homeomorphic for all sufficiently small positive $r$.*

Note that we do not assume any higher order differentiability of $f$. Also, the homogeneous functions $p_i$, even when of integral degree, need not be polynomials. However, if all $k_i$ are integers and $f_i\in C^{k_i}$, then the $p_i$ are polynomials (by the Taylor formula). In this case the condition (ii) of the theorem must necessarily be satisfied. Thus, for a sufficiently smooth mapping $f=(f_1,\ldots,f_m)$, Theorem (1.2) states that if $p_i$ is the first nonzero term in the Taylor expansion of $f_i$ $(i=1,\ldots,m)$ and if (iii) holds for $p=(p_1,\ldots,p_m)$, then the sets $Z_p\cap B(0,r)$ and $Z_f\cap B(0,r)$ are homeomorphic for small $r$.

If $k_i=k$ for all $i$ and $f_i\in C^k$, it follows from Theorem 1 of [3] that $Z_p$ and $Z_f$ are locally homeomorphic at the origin. In this special case Theorem (1.2) provides a slightly stronger result, that $Z_p$ and $Z_f$ are homeomorphic on any sufficiently small ball $B(0,r)$.

This work has been done without knowledge of Kuo's results and the proof of Theorem (1.2) is effected by different methods than those of [3].

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## 2. Preliminaries.

Let $M$ and $N$ be two differentiable manifolds without boundary and consider a $C^1$ mapping $f$ of an open subset of $M$ into $N$. Denote the tangent space of $M$ at $x$ by $TM_x$ and let $df_x\colon TM_x\to TN_{f(x)}$ be the derivative of $f$ at $x$. If $df_x$ is surjective, $x$ is called a regular point of $f$. If $df_x$ is surjective for all $x\in f^{-1}(y)$, or if $f^{-1}(y)=\varnothing$, then $y$ is called a regular value of $f$. Nonregular values are called singular. It is known (see e.g. [2, Chap. 1, Theorem 3.2]) that when $y$ is regular, $f^{-1}(y)$, if nonempty, is a differentiable manifold of dimension $\dim M - \dim N$.

Now let $M$ and $N$ be two oriented $m$-dimensional differentiable manifolds without boundary and let $\Omega$ be an open subset of $M$ with compact closure $\bar{\Omega}$. Suppose that $f$ is a continuous mapping of $\bar{\Omega}$ into $N$ and that $f$ *is* $C^1$ in $\Omega$. If $y_0 \in N - f(\partial\Omega)$ is a regular value of $f$, we define the topological degree of $f$ at $y_0$ by setting

$$\deg(f, \Omega, y_0) = \sum_{x \in f^{-1}(y)} \operatorname{sgn} df_x ,$$

where $\operatorname{sgn} df_x = 1$ if $df_x$ preserves and $-1$ if it reverses orientation. In the proof of Theorem (1.2) we shall need some basic properties of degree, which may be found e.g. in [6], Section 1.3.

Let $M$ and $N$ be two differentiable manifolds without boundary and let $N$ be a compact submanifold of $M$ of codimension $p$ (i.e. $\dim M - \dim N = p$). For each $x \in N$, denote the orthogonal complement of $TN_x$ in $TM_x$ by $TN_x^\perp$. If there exists a $C^1$ mapping $\eta$ which assigns to each $x \in N$ a basis $\eta(x) = (\eta_1(x), \ldots, \eta_p(x))$ for $TN_x^\perp$, then $\eta$ is called a framing of $N$ in $M$ and the pair $(N, \eta)$ is a framed submanifold of $M$ (see [5, p. 42]).

(2.1) PRODUCT NEIGHBOURHOOD THEOREM ([5, p. 46]). *Under the above assumptions there exists a neighbourhood of $N$ in $M$ which is diffeomorphic to $N \times \mathbf{R}^p$. Moreover, the diffeomorphism may be chosen so that*

(i) *each point $x \in N$ corresponds to $(x, 0) \in N \times \mathbf{R}^p$;*
(ii) *each basis $\eta(x)$ for $TN_x^\perp$ corresponds to the standard basis for $\mathbf{R}^p$.*

Let $f$ be a $C^1$ mapping of an open neighbourhood of the origin in $\mathbf{R}^n$ into $\mathbf{R}^m$, $n > m \geqq 1$, and let $H_f(x)$ be the linear subspace of $\mathbf{R}^n$ spanned by the rows of $J_f(x)$. In the proof of Theorem (1.2) it will be the convenient to use the following two lemmas:

(2.2) LEMMA. *Suppose that $J_f$ has rank $m$ at $x$. Then $df_x \mid H_f(x)\colon H_f(x) \to \mathbf{R}^m$ is an isomorphism.*

PROOF. $H_f(x)^\perp$ is the null space of $df_x$ and $\dim H_f(x) = m$.

(2.3) LEMMA. *Let $A\colon \mathbf{R}^n \to \mathbf{R}^m$ be a surjective linear mapping and let $L_1$ and $L_2$ be two $m$-dimensional linear subspaces of $\mathbf{R}^n$ satisfying the following conditions:*

(i) *$L_1^\perp$ is the null space of $A$;*
(ii) *$L_1^\perp \cap L_2 = \{0\}$.*

*Then $A \mid L_2\colon L_2 \to \mathbf{R}^m$ is an isomorphism.*

PROOF. It suffices to show that $A|L_2$ is a monomorphism. Let $a=a'+a'' \in L_2$, where $a' \in L_1$ and $a'' \in L_1^{\perp}$. Suppose that $Aa=0$. Then $Aa=Aa'+Aa''=Aa=0$. By (i), $A|L_1$ is an isomorphism and hence $a'=0$. So we find that $a \in L_1^{\perp}$, and by (ii), $a=0$.

## 3. Proof of Theorem (1.2).

First let us introduce some notation. Denote $\partial B(0,1)$, the unit sphere in $\mathbf{R}^n$, by $S$ and let $A=Z_p \cap S$. For a subset $F$ of $S$ and a real number $\varrho$ let

$$\varrho F = \{x \in \mathbf{R}^n : x=\varrho s \text{ for some } s \in F\}.$$

The proof of the theorem will proceed in five steps. Suppose that $A \neq \emptyset$. In step I we construct a small open neighbourhood $B$ of $A$ in $S$ such that $B=\bigcup_{a\in A} B_a$, all the sets $B_a$ being diffeomorphic to open bounded subsets of $\mathbf{R}^m$ and $B_a \cap B_{a'}=\emptyset$ for $a \neq a'$. We show that for a suitably chosen positive number $r_1$ and all $a \in A$ and $0<\varrho \leqq r_1$, $f|\varrho B_a$ has no singular values. In step II we consider $f_t=p+tg$ and show that for a small positive number $r \leqq r_1$, $f_t(\varrho x) \neq 0$ for any $x \notin B$, $0<\varrho \leqq r$ and $0 \leqq t \leqq 1$. In step III we use the result of step II to prove that $\deg(p,\varrho B_a,0)=\deg(f,\varrho B_a,0)=1$ or $-1$ for all $a \in A$ and $0<\varrho \leqq r$. By the basic existence theorem of topological degree theory there is a point $y \in \varrho B_a$ such that $f(y)=0$. This point is unique. To show it, we use the fact that $f|\varrho B_a$ has no singular values. In step IV we define a homeomorphism $\alpha$: $Z_p \cap B(0,r) \to Z_f \cap B(0,r)$ by letting $\alpha(\varrho a)$ be the unique point of $\varrho B_a \cap Z_f$ determined in step III. Finally, in step V, we consider the case $A=\emptyset$.

STEP I. Denote the $i$th row of $J_p(x)$ *by* $\eta_i(x)$. By the Euler identities for homogeneous functions, if $a=(a_1,\dots,a_n) \in A$, then

$$\sum_{j=1}^{n} a_j \frac{\partial p_i(a)}{\partial x_j} = k_i p_i(a) = 0 \qquad (1 \leqq i \leqq m).$$

It follows that $\eta_i(a) \in TS_a$ for all $i$ and hence $H_p(a) \subset TS_a$. Since by Lemma (2.2)

$$(1) \qquad dp_a|H_p(a)\colon H_p(a) \to \mathbf{R}^m \text{ is an isomorphism for all } a \in A,$$

0 is a regular value of $p|S$ and it follows that $A$ is an $(n-1-m)$-dimensional submanifold of $S$. Furthermore, for each $a \in A$, $\eta(a)=(\eta_1(a),\dots,\eta_m(a))$ is a basis for the orthogonal complement of $TA_a$ in $TS_a$ and therefore, if $\eta \in C^1$, $(A,\eta)$ is a framed submanifold of $S$ (we return to the case $\eta \notin C^1$ later). By the Product Neighbourhood Theorem (2.1), there is a neighbourhood $B'$ of $A$ in $S$ and a diffeomorphism $h$: $A \times \mathbf{R}^m \to B'$ such that $h(a,0)=a$. Let $\mathring{B}(0,\varepsilon)$ denote the interior of $B(0,\varepsilon)$ in $\mathbf{R}^m$ and let $B=h(A \times \mathring{B}(0,\varepsilon))$ for some positive number $\varepsilon$. If $\varepsilon$ is small enough, $J_p$ has rank $m$ for all $b \in \bar{B}$. Denote

$$B'_a = h(\{a\} \times \mathbf{R}^m), \quad B_a = h(\{a\} \times \mathring{B}(0,\varepsilon)) .$$

Using the statement (ii) of (2.1) we may assert that $H_p(a) = T(B_a)_a$.

We claim that for sufficiently small $\varepsilon$ there is a positive number $r_1$ such that no vector of $T(\varrho B_a)_{\varrho b}$ is orthogonal to $H_f(\varrho b)$ for any $a \in A$, $b \in B_a$ and $0 < \varrho \leqq r_1$. For otherwise we could find sequences $\{a_k\}$, $\{b_k\}$ and $\{\varrho_k\}$ such that $b_k \in B_{a_k}$, the distance $d(a_k, b_k) \to 0$, $\varrho_k \to 0$, and for each $k$ there is a unit vector

$$v_k \in T(\varrho_k B_{a_k})_{\varrho_k b_k} = T(B_{a_k})_{b_k} ,$$

which is orthogonal to $H_f(\varrho_k b_k)$. By compactness of $A$, assume that $a_k \to a \in A$. Then also $b_k \to a$. It follows that

$$\operatorname{Lim} T(B_{a_k})_{b_k} = T(B_a)_a$$

(Lim denotes the topological limit of a sequence of sets). The sequence $\{v_k\}$ has a convergent subsequence whose limit $v \in T(B_a)_a$. Recall that the rows of $J_p$ are homogeneous functions of degree $k_i - 1$ and the corresponding rows of $J_g$ are $o(\|x\|^{k_i - 1})$ as $x \to 0$. Since $H_f$ is spanned by the rows of $J_f = J_p + J_g$,

$$\operatorname{Lim} H_f(\varrho_k b_k) = H_p(a) .$$

It follows that $v$ is orthogonal to $H_p(a)$. This is a contradiction because $v$ is nonzero and $v \in T(B_a)_a = H_p(a)$. The claim is proven.

So, $H_f(\varrho b)^\perp \cap T(\varrho B_a)_{\varrho b} = \{0\}$ for all $a \in A$, $b \in B_a$ and $0 < \varrho \leqq r_1$. We may obviously choose $r_1$ so small that $J_f(\varrho b)$ have rank $m$. Then $df_{\varrho b}$ is surjective and it follows from Lemma (2.3) (with $A = df_{\varrho b}$, $L_1 = H_f(\varrho b)$ and $L_2 = T(\varrho B_a)_{\varrho b}$) that

(2) $\qquad df_{\varrho b} \mid T(\varrho B_a)_{\varrho b} \colon T(\varrho B_a)_{\varrho b} \to R^m$

is an isomorphism for all $a \in A$, $b \in B_a$ and $0 < \varrho \leqq r_1$ .

Hence $f \mid \varrho B_a$ has no singular values for any $a \in A$ and $0 < \varrho \leqq r_1$.

If $\eta \notin C^1$ (which corresponds to the case $p \notin C^2$), we may approximate $\eta$ by some $\eta' \in C^1$ such that for any $a \in A$, $\eta'(a)$ is a basis for a complement of $TA_a$ in $TS_a$. The pair $(A, \eta')$ need not be a framed submanifold according to our definition but it is easily seen from the proof in [5] that the conclusions of Theorem (2.1) remain true for $(A, \eta')$ (indeed, the proof does not use the orthogonality property of $\eta$ but the fact that $\eta$ is a basis for a complementary space). Therefore, though now $H_p(a) \neq T(B_a)_a$, we may repeat the above argument to obtain a unit vector $v \in T(B_a)_a$ which is orthogonal to $H_p(a)$. However, if $\eta'$ is sufficiently close to $\eta$, this is still a contradiction. Hence (2) is not violated.

STEP II. Let $D=S-B$. Since $B$ is open in $S$, $D$ is compact and we may therefore find a constant $K>0$ such that $\|p(x)\|\geqq K$ for all $x\in D$. It follows that given $x\in D$, $|p_i(x)|\geqq K/\sqrt{m}$ for some index $i$. Since $g_i=o(r^{k_i})$ as $r\to 0$, there is a positive number $r\leqq r_1$ such that $|g_i(\varrho x)|<\varrho^{k_i}K/\sqrt{m}$ for all $x\in S$, $0<\varrho\leqq r$ and $1\leqq i\leqq m$. Set

$$f_{it}(\varrho x) = p_i(\varrho x)+tg_i(\varrho x) .$$

For each $x\in D$, $0<\varrho\leqq r$, $0\leqq t\leqq 1$ and some index $i$,

$$|f_{it}(\varrho x)| = |p_i(\varrho x)+tg_i(\varrho x)| \geqq \varrho^{k_i}|p_i(x)|-t|g_i(\varrho x)| > \varrho^{k_i}\frac{K}{\sqrt{m}}-t\varrho^{k_i}\frac{K}{\sqrt{m}} \geqq 0 .$$

Hence

$$\|f_t(\varrho x)\| > 0 \quad \text{for all } x\in D,\ 0<\varrho\leqq r \text{ and } 0\leqq t\leqq 1 . \tag{3}$$

STEP III. Recall that $B'_a$ is an oriented $m$-dimensional manifold and $B_a$ is an open subset of $B'_a$ with compact closure. Let $C_a$ be the boundary of $B_a$ in $B'_a$. Given $a\in A$, let us compute $\deg(p,B_a,0)$. Since $Z_p\cap B_a=\{a\}$ and $(TB_a)_a=H_p(a)$, we find from (1) that 0 is a regular value of $p\,|\,B_a$. Also, $0\notin p(C_a)$. Hence $\deg(p,B_a,0)=\operatorname{sgn} dp_a\,|\,H_p(a)=1$ or $-1$.

Obviously, $\deg(p,\varrho B_a,0)=\deg(p,B_a,0)$ for any $\varrho>0$. By virtue of (3), $f_t(x)\neq 0$ for any $x\in\varrho C_a$ and $0<\varrho\leqq r$. It follows therefore from the homotopy invariance property of degree that $\deg(f,\varrho B_a,0)=1$ or $-1$. Consequently, $f(y)=0$ for some $y\in\varrho B_a$. Moreover, this $y$ is unique. To show this, recall that by (2), $f\,|\,\varrho B_a$ has no singular values. Since $B_a$ (and hence also $\varrho B_a$) is homeomorphic to an open ball, and is therefore connected, $\operatorname{sgn} df_x|T(\varrho B_a)_x$ is constant on $\varrho B_a$. Thus, since $\deg(f,\varrho B_a,0)=1$ or $-1$, $Z_f\cap\varrho B_a$ consists of precisely one point, $y$.

STEP IV. Now we are ready to define a homeomorphism

$$\alpha\colon Z_p\cap B(0,r)\to Z_f\cap B(0,r) .$$

Set $\alpha(0)=0$. If $x=\varrho a$ for some $a\in A$ and $0<\varrho\leqq r$, set $\alpha(x)=y$; where $y$ is the unique point of $\varrho B_a$ satisfying $f(y)=0$. To verify that $\alpha$ is continuous, choose $x_0=\varrho a$ with $0<\varrho\leqq r$ and $a\in A$. Let $y_0=\alpha(x_0)$ and suppose that $\{x_k\}=\{\varrho_k a_k\}\subset Z_p\cap B(0,r)$ is a sequence of points converging to $x_0$. Since $\alpha(\varrho_k a_k)\in\varrho_k B_{a_k}$, $\varrho_k\to\varrho$ and $a_k\to a$, we see that every cluster point $y'$ of $\{\alpha(x_k)\}$ must necessarily be in $\varrho B_a$. By continuity of $f$, $f(y')=0$. But since $Z_f\cap\varrho B_a=\{y_0\}$, $y'=y_0$ and $\lim\alpha(x_k)=y_0$. Now suppose $x_0=0$ and $x_k=\varrho_k a_k\to 0$, where $a_k\in A$. Then $\varrho_k\to 0$ and since $\alpha(\varrho_k a_k)\in\varrho_k B_{a_k}$, $\alpha(\varrho_k a_k)\to 0$. This completes the proof of continuity.

Let $x=\varrho a$ and $x'=\varrho' a'$ be two distinct points in $Z_p$. Then $\alpha(x)\in\varrho B_a$ and

$\alpha(x') \in \varrho' B_{a'}$. Since the sets $\varrho B_a$ and $\varrho' B_{a'}$ are disjoint unless $\varrho = \varrho'$ and $a = a'$ (i.e. $x = x'$), $\alpha$ is one-to-one. By (3) (with $t=1$), $\alpha(Z_p \cap B(0,r)) = Z_f \cap B(0,r)$.

STEP V. Suppose that $A = \emptyset$. Then $\|p(x)\| \geqq K > 0$ on $S$. So, for some positive number $r$, $f(x) \neq 0$ for any $x$ satisfying $0 < \|x\| \leqq r$ (compare with the proof of (3)). Hence $Z_p \cap B(0,r) = Z_f \cap B(0,r) = \{0\}$.

## 4. Remarks and examples.

(4.1). One could expect that if all $p_i$ are nonhomogeneous polynomials of degree $k_i$, then the conclusion of Theorem (1.2) still holds. That this is not the case can be demonstrated by the following example (taken from [4, p. 147]): Let

$$p(x,y) = x^3 - 3xy^7 \quad \text{and} \quad r(x,y) = x^3 - 3xy^7 + 2|y|^{21/2}.$$

Then $\operatorname{grad} p = (3x^2 - 3y^7, -21xy^6) \neq (0,0)$ if $(x,y) \neq (0,0)$ and $\operatorname{grad} r = (3x^2 - 3y^7, -21xy^6 + 21 \operatorname{sgn}(y)|y|^{19/2}) = (0,0)$ if $x^2 = y^7$ and $x \geqq 0$. By [3, Theorem 1], there is a function $g(x,y) \in C^9$ such that $g(x,y) = o((x^2+y^2)^4)$ at the origin but the sets $Z_p$ and $Z_{p+g}$ are not homeomorphic in any neighbourhood of the origin.

(4.2). If we replace the assumption that $g$ be a $C^1$ mapping by the weaker one, that $g$ be continuous, the formulas (1) and (3) of Section 3 are still true. So, $\deg(f, \varrho B_a, 0) = 1$ or $-1$ and $f(y) = 0$ for some $y \in \varrho B_a$. But now we have no reason to expect that this $y$ be unique (see (4.6) below). We may, however, define a continuous mapping $\beta\colon Z_f \cap B(0,r) \to Z_p \cap B(0,r)$ in the following way: if $x = \varrho a$, $a \in A$ and $0 < \varrho \leqq r$, set $\beta(y) = x$ for all $y \in \varrho B_a \cap Z_f$; if $y = 0$, set $\beta(y) = 0$. Note that $\beta$ is one-to-one and $\alpha = \beta^{-1}$ if $g$ satisfies all assumptions of Theorem (1.2).

(4.3). In Theorem (1.2), assume that $\operatorname{rank} J_p(x) = m$ for some, instead of for all $x \in Z_p - \{0\}$. Then $\operatorname{rank} J_p(a) = m$ for some $a \in A$. Let $H'_p(a)$ be the hyperplane parallel with $H_p(a)$ and passing through $a$. Replace $B_a$ in the proof of the theorem by a small neighbourhood $F_a$ of $a$ in $H'_p(a)$. Since $Z_p \cap F_a = \{a\}$ whenever $F_a$ is sufficiently small, it is easy to show that for some $r > 0$ (which may depend on $a$), the line segment joining $ra$ to the origin and the set $Z_f \cap \bigcup_{0 \leqq \varrho \leqq r} \varrho F_a$ are homeomorphic. If $g$ is continuous but not necessarily continuously differentiable, there is a continuous function $\beta$ from $Z_f \cap \bigcup_{0 \leqq \varrho \leqq r} \varrho F_a$ onto this line segment such that $\beta(Z_f \cap \varrho F_a) = \varrho a$. In both cases the equation $f(x) = 0$ has a nontrivial solution in every neighbourhood of the origin.

(4.4). The assumption that $p$ have maximal rank on $Z_p - \{0\}$ cannot be dropped. This can be shown by a very simple example: let $p(x,y) = x^2$, $g(x,y) = y^4$ and $f(x,y) = p(x,y) + g(x,y)$. Then $Z_p = \{(x,y) \in \mathbf{R}^2 : x = 0\}$ and $Z_f = \{0\}$.

Actually, if $p$ does not have maximal rank for some $x \in Z_p - \{0\}$ and if all the $p_i$ are polynomials, it follows from Theorem 1 of [3] that we may always find a map $g = (g_1, \ldots, g_m)$ satisfying assumptions (i) and (ii) of Theorem (1.2) and such that the conclusion does not hold for $p$ and $f = p + g$.

(4.5). If $m \geqq n$ and $Z_p - \{0\} \neq \varnothing$, it follows from the implicit function theorem that $J_p$ cannot have maximal rank for any $x \in Z_p - \{0\}$. For otherwise $x$ would be an isolated zero of $p$; but this is impossible because $p$ is homogeneous.

(4.6). Let $p(x, y) = x$ and $g(x, y) = -\sqrt{|x|(x^2 + y^2)}$. Then $g \notin C^1$, all other assumptions of Theorem (1.2) being satisfied. An easy computation shows that

$$Z_f = \{(x, y) \in \mathbf{R}^2 : x = 0 \text{ or } (x - \tfrac{1}{2})^2 + y^2 = \tfrac{1}{4}\} .$$

This set is obviously not homeomorphic to $Z_p = \{(x, y) \in \mathbf{R}^2 ; x = 0\}$ in any neighbourhood of the origin.

## 5. Applications to bifurcation theory.

Let $X$ and $Y$ be two real Banach spaces and let $f$ be a $C^k$ mapping, $k \geqq 1$, of a neighbourhood of the origin in $X$ into $Y$ such that $f(0) = 0$. Assume that the Fréchet derivative of $f$ at the origin (denoted by $f_x(0)$) is a Fredholm operator, i.e. the null space $Nf_x(0)$ is of finite dimension and the range $Rf_x(0)$ is closed in $Y$ and of finite codimension. Let $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$ (direct sums) with $X_1 = Nf_x(0)$ and $Y_1 = Rf_x(0)$. For $y = y_1 + y_2$ ($y_i \in Y_i$) write $y_1 = Qy$, $y_2 = Py$. It is known ([6, pp. 64–65]) that the equation $Qf(x_1 + x_2) = 0$ has a unique $C^k$ solution $x_2 = u(x_1)$ in a suitably small neighbourhood of the origin in $X_1$ and the equation $f(x_1 + x_2) = 0$ is equivalent to

$$F(x_1) = Pf(x_1 + u(x_1)) = 0 .$$

This is the well-known Lyapunov–Schmidt procedure of reducing an infinite dimensional problem to a finite dimensional one. $F(x_1) = 0$ is called the bifurcation equation.

Now we present some possible applications of Theorem (1.2) to bifurcation theory. They are essentially based upon methods which may be found in [6, Section 3.2]. For simplicity, we assume throughout that $f \in C^k$ for some suitable $k \geqq 2$, although all we really need are weaker regularity conditions of $f$ which make Theorem (1.2) or Remarks (4.2) and (4.3) applicable.

(5.1) THEOREM. *Let $U$ be an open neighbourhood of the origin in $X$. Suppose that $f \colon U \to Y$ is a $C^k$ mapping, $k \geqq 2$, satisfying the above assumptions and that*

(i) $f(x) = f_x(0)x + B_k(x, \ldots, x) + r_k(x)$, *where $B_k$ is a bounded k-linear form and $r_k$ is $o(\|x\|^k)$;*

(ii) $n = \dim Nf_x(0) > \operatorname{co\,dim} Rf_x(0) = m \geqq 1$;

(iii) *$p$ denotes the homogeneous term of degree $k$ of $Pf\,|\,X_1 \cap U$ and $J_p$ has maximal rank for all $x_1 \in Z_p - \{0\}$ (note that $p$ may be considered as a mapping of* $\mathbf{R}^n$ *into* $\mathbf{R}^m$*).*

*Then, in some neighbourhood of the origin in $X$, $Z_f$ is homeomorphic to $Z_p \cap B(0, 1)$.*

If $f$ is a $C^k$ mapping, we have by the Taylor formula $f(x) = f_x(0)x + B_2(x, x) + \dots + B_k(x, \dots, x) + r_k(x)$. So the assumption (i) merely means that $B_j(x, \dots, x) \equiv 0$ for $2 \leqq j < k$.

We precede the proof with the following

(5.2) Lemma. *Let $F(x_1) = Pf(x_1 + u(x_1)) = 0$ be the bifurcation equation. Then $u_{x_1}(0) = 0$.*

This lemma is a part of the proof of Theorem 3.2.1 of [6]. For the sake of completeness we insert the proof (taken from [6] with minor changes).

Proof. Recall that $Qf(x_1 + u(x_1)) \equiv 0$. Differentiating this with respect to $x_1$ we find that $Qf_x(0)(x_1 + u_{x_1}(0)x_1) = 0$. Since $x_1 \in X_1 = Nf_x(0)$, $f_x(0)x_1 = 0$ and hence $Qf_x(0)u_{x_1}(0)x_1 = 0$. Since $Q$ is a projection into $Y_1 = Rf_x(0)$, $f_x(0)u_{x_1}(0)x_1 = 0$. Observe that $u_{x_1}(0)x_1 \in X_2$ and $f_x(0)\,|\,X_2$ is an isomorphism. Hence $u_{x_1}(0)x_1 = 0$.

Proof of Theorem (5.1). We need to show that $F(x_1) = p(x_1) + o(\|x_1\|^k)$ and that the assumptions of Theorem (1.2) are satisfied. Recall that $P$ is a projection into the complementary space of $Rf_x(0)$. So, $Pf_x(0) = 0$ and $Pf(x) = PB_k(x, \dots, x) + Pr_k(x)$. This, together with the fact that $F \in C^k$, implies that $F(x_1) = q(x_1) + g(x_1)$, where $q(x_1)$ is homogeneous of degree $k$, $g(x_1) = o(\|x_1\|^k)$ and $g_{x_1}(x_1) = o(\|x_1\|^{k-1})$.

To complete the proof it remains to show that $p \equiv q$. For this purpose we only need observe that

$$\begin{aligned} F(x_1) &= Pf(x_1 + u(x_1)) = PB_k(x_1 + u(x_1), \dots, x_1 + u(x_1)) + o(\|x_1\|^k) \\ &= PB_k(x_1, \dots, x_1) + PB_k(x_1, \dots, x_1, u(x_1)) + \dots \\ &\quad + PB_k(u(x_1), \dots, u(x_1)) + o(\|x_1\|^k) \\ &= PB_k(x_1, \dots, x_1) + o(\|x_1\|^k)\,. \end{aligned}$$

The last equality follows from Lemma (5.2). Since $p(x_1) = PB_k(x_1, \dots, x_1)$, $p \equiv q$.

It is easily seen that the above result is a generalization of [6, Theorem 3.2.1].

(5.3) REMARK. Suppose that $J_p$ has maximal rank at some (not necessarily all) $x_1 \in Z_p - \{0\}$, other assumptions being as in Theorem (5.1). Then it follows from (4.3) and from the proof just given that $Z_f$ contains an arc emanating from the origin.

In bifurcation problems the mapping $f$ is usually dependent on a parameter $\lambda \in \mathbf{R}^d$. Let us assume that $\lambda \in \mathbf{R}$ (i.e. $d=1$) and that $f(x,\lambda)$ is a $C^2$ mapping of a neighbourhood of $(0,\lambda_0) \in X \times \mathbf{R}$ into $Y$. Suppose that $f(0,\lambda)\equiv 0$. Then $x=0$ is referred to as the trivial solution of $f(x,\lambda)=0$. If there are nontrivial solutions in every neighbourhood of $(0,\lambda_0)$, then $(0,\lambda_0)$ is said to be a bifurcation point of $f$ (with respect to the trivial solution). It is known ([1, Theorem 1.7]; [6, Theorem 3.2.2]) that the set of nontrivial solutions of $f(x,\lambda)=0$ near $(0,\lambda_0)$ consists of an arc passing through $(0,\lambda_0)$ whenever $\dim Nf_x(0,\lambda_0) = \operatorname{co}\dim Rf_x(0,\lambda_0)=1$ and $f$ satisfies some additional conditions (which may easily be shown to yield applicability of Theorem (5.1) with $k=2$). Now let us turn to the case $\dim Nf_x(0,\lambda_0)=\operatorname{co}\dim Rf_x(0,\lambda_0)=2$.

(5.4) COROLLARY. *Let $f(x,\lambda)$ be a $C^2$ mapping of some neighbourhood of $(0,\lambda_0) \in X \times \mathbf{R}$ into $Y$. Suppose that $f(0,\lambda)\equiv 0$ and that $\dim Nf_x(0,\lambda_0) = \operatorname{co}\dim Rf_x(0,\lambda_0)=2$. Denote by p the quadratic term of $Pf \mid X_1 \times \mathbf{R}$ (recall that $X_1 = Nf_x(0,\lambda_0)$ and $P$ is the projection of $Y$ onto $Y_2$ associated with the decomposition $Y=Y_1 \oplus Y_2$) and suppose that $J_p$ has maximal rank for all $(x_1,\lambda) \in Z_p - \{(0,\lambda_0)\}$. Then $(0,\lambda_0)$ is a bifurcation point of $f$ and the set of nontrivial solutions of $f(x,\lambda)=0$ near $(0,\lambda_0)$ consists of one or tree arcs intersecting only at the origin.*

PROOF. We may obviously assume that $\lambda_0=0$. Set $\hat{X}=X\times \mathbf{R}$ and $\hat{x}=(x,\lambda)$. Then $f_{\hat{x}}(0)=f_x(0,0)\oplus f_\lambda(0,0)$. Since $f(0,\lambda)\equiv 0$,

$$\dim Nf_{\hat{x}}(0) = \dim Nf_x(0,0)+1 = 3 \quad \text{and}$$

$$\operatorname{co}\dim Rf_{\hat{x}}(0) = \operatorname{co}\dim Rf_x(0,0) = 2\ .$$

Hence we may apply Theorem (5.1) to study the set $Z_f$ near the origin. Note that $p=(p_1,p_2)$ maps $\mathbf{R}^3$ into $\mathbf{R}^2$. Since $p(0,\lambda)\equiv 0$ and $J_p$ has maximal rank on $Z_p-\{0\}$, $Z_{p_1}$ (and $Z_{p_2}$) is either a (geometric) cone or two intersecting planes. Furthermore, there exist tangent planes to $Z_{p_1}$ and $Z_{p_2}$ at any point of $Z_p-\{0\}$ and those planes do not coincide (because $Z_p$ has maximal rank). So it follows from an elementary geometric argument that $Z_p$ consists of 2 or 4 lines passing through the origin. Hence the conclusion.

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