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A SEPARATION PROPERTY OF PLANE CONVEX SETS

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1. Introduction.

The main result to be proved in this paper is

THEOREM 1. *For every natural number k there is an integer* $K = K_2(k,1)$ *with the following property.*

Whenever $C_1,\ldots,C_K$ *are nonempty convex sets in* $\mathbf{R}^2$, *with pairwise disjoint relative interiors, there is a closed halfplane which contains at least k of the sets, while its complementary closed halfplane contains at least one of the remaining* $K-k$ *sets.*

The well known case $k=1$, and the case $k=2$, are basic to the proof, and we formulate

THEOREM 2. $K_2(1,1)=2$, $K_2(2,1)=5$.

The proof of Theorem 2 requires some detailed geometric considerations, while Theorem 1 follows quite simply from Theorem 2, by means of Carathéodory's and Ramsey's theorems. We use, in fact, a generalization of Carathéodory's theorem (see Theorem 3). We could have used Carathéodory's original theorem instead, but it would then be necessary to compute, or at least prove the existence of, $K_2(3,1)$. This number is at present unknown.

It is natural to ask for numbers $K_d(r,s)$. Examples (to be given in section 5) show that $K_2(2,2)$ and $K_3(2,1)$ do not exist. These examples, due to cand. real. Kåre P. Villanger, are reproduced here with his kind permission.

The nonexistence of $K_2(2,2)$ (and hence of $K_d(2,2)$, for $d \geqq 2$) is somewhat disappointing, in view of the original aim of this study. In [5] I proved that it is always possible to cut a convex d-polytope into finitely many simplices by first splitting it by a hyperplane, then splitting one of the parts by another hyperplane, and so on. Considering the process one finds that when a part P, say, is split into P_1 and P_2, then at least one of P_1 and P_2 has *fewer* facets than

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P. Examples, like a triangular prism in $\mathbb{R}^3$, or a cube in $\mathbb{R}^d$, with its facets slightly perturbed in a suitable way, show that one cannot in general require both P_1 and P_2 to have fewer facets than P.

It might be possible, however, that up to isomorphism, there are only finitely many examples like the prism or the "cube" (for a given d). A possible way to prove this would be: Prove the existence of $K = K_{d-1}(2,2)$. Let now P be a convex polytope in $\mathbb{R}^d$ with at least $1+K$ facets. Consider one of its Schlegel diagrams, obtained by choosing a point C just outside some facet F of P and projecting the remaining facets onto F, with C as the centre of projection. By the definition of K there would now be a hyperplane H in the $(d-1)$-space spanned by F, separating at least two of the projected facets from at least two other projected facets. Extending H to a hyperplane in $\mathbb{R}^d$, through C, one would get a splitting of P as desired.

The non-existence of $K_2(2,2)$ destroys the project just described, and the examples to be given in section 5 can even be modified to give Schlegel-diagrams. The only possibility left, to save the given approach, seems to be use of more than one Schlegel-diagram for the same P.

2. Carathéodory's theorem and its generalization.

CT [1] says, in its standard form, that if S is any set in $\mathbb{R}^d$, then the convex hull of S is the union of all simplices having vertices in S. This can be written as

$$\text{(1)} \qquad \operatorname{conv} S = \bigcup \operatorname{conv} S'; \quad |S'| \leqq d+1 \text{ and } S' \subset S .$$

It turns out that one can restrict oneself to simplices having s_0 as a vertex, where $s_0 \in S$ is any fixed point. Thus

$$\text{(2)} \qquad \operatorname{conv} S = \bigcup \operatorname{conv} S'; \quad |S'| \leqq d+1 \text{ and } s_0 \in S' \subset S .$$

This result, occurring in [2], will be used in the proof of

THEOREM 3. *Let S be a union of convex sets in $\mathbb{R}^d$, all meeting a fixed hyperplane H. Then*

$$\text{(3)} \qquad \operatorname{conv} S = \bigcup \operatorname{conv} S'; \quad |S'| \leqq d \text{ and } S' \subset S .$$

Theorem 3 generalizes CT; if the convex sets are single points, one gets CT in $\mathbb{R}^{d-1}$. For a good survey of other generalizations of CT, see Reay [4]. See also Danzer–Grünbaum–Klee [3] for connections with other basic convexity theorems.

PROOF OF THEOREM 3. Putting $S = \bigcup C_i$; $i \in I$, one can write the RHS of (1) as

$$\bigcup \operatorname{conv}(C_{i_1} \cup \ldots \cup C_{i_{d+1}}); \quad \{i_1,\ldots,i_{d+1}\} \subset I .$$

This means that it is sufficient to prove (3) in the case when S is the union of $d+1$ convex sets, $C_0,\ldots,C_d$, all meeting H.

We first choose a point r_i from $H \cap C_i$, $i=0,\ldots,d$. By Radon's theorem in $\mathbf{R}^{d-1}(=H)$ we may assume, after renumbering, that there is a point s_0 so that

$$s_0 \in \operatorname{conv}\{r_0,\ldots,r_m\} \cap \operatorname{conv}\{r_{m+1},\ldots,r_d\}$$

for some $m, 0 \leqq m \leqq d-1$. Clearly $s_0 \in \operatorname{conv} S$.

Consider now any point q in $\operatorname{conv} S$ $(=\operatorname{conv} S \cup \{s_0\})$. Applying (2), with S replaced by $S \cup \{s_0\}$, we know that

(4) $$q = \mu_0 s_0 + \mu_1 c_1 + \ldots + \mu_d c_d, \quad \mu_i \geqq 0, \ \sum \mu_i = 1 ,$$

where each c_i is in S, i.e. in some $C_{j(i)}$. It is no essential restriction to assume that $d \notin \{j(1),\ldots,j(d)\}$, so that $\{c_1,\ldots,c_d\} \subset \operatorname{conv}(C_0 \cup \ldots \cup C_{d-1})$. As

$$s_0 \in \operatorname{conv}\{r_0,\ldots,r_m\} \subset \operatorname{conv}(C_0 \cup \ldots \cup C_{d-1}) ,$$

we get from (4) that $q \in \operatorname{conv}(C_0 \cup \ldots \cup C_{d-1})$. This shows that the LHS of (3) is a subset of the RHS. The opposite inclusion is trivial.

3. Proof of Theorem 2.

It is well known that $K_d(1,1)=2$ for all d, and we shall make use of this result. In order to see that $K_2(2,1) \geqq 5$, it suffices to consider three congruent circular discs, mutually tangent, surrounding a fourth, smaller, one, tangent to all three. It remains to prove that $K_2(2,1) \leqq 5$.

Assume that $K_2(2,1) \geqq 6$ (or does not exist) i.e. that there are non-empty convex plane sets $C_1,\ldots,C_5$, with mutually disjoint relative interiors, having the following property: Every line separating two of the sets (weakly) meets the relative interior of the three others. This assumption we will denote by A, for short.

Before starting to deduce a contradiction from A, we remark that easy limit arguments show that the C_i can be assumed to be compact and pairwise disjoint, with non-empty interiors.

Let T' and T'' be those two common tangents to C_1 and C_2 which separate C_1 from C_2. Then $T' \setminus (C_1 \cup C_2)$ consists of three components X', Y' and Z', with Y' between X', Z'. Similarly, $T'' \setminus (C_1 \cup C_2)$ decomposes into X'', Y'' and Z''. We choose the lettering so that X' and X'' meet only C_1, while Z' and Z'' meet only C_2. ("Meet" is taken to mean that $\operatorname{cl}(X') \cap C_1 \neq \emptyset$, etc.).

Now $C_3 \cap Y' = \emptyset$. For, if $C_3 \cap Y'$ meets, say, the part of T' between $T' \cap T''$

and C_1, then any line separating C_3 from C_1 avoids C_2, which contradicts our assumption A. Hence we can assume that, say, $C_3 \cap X' \neq \emptyset = C_3 \cap Z'$. Similarly, we find that either $C_3 \cap X'' \neq \emptyset = C_3 \cap Z''$ or $C_3 \cap Z'' \neq \emptyset = C_3 \cap X''$. The former alternative is excluded by assumption A (any line separating C_1 from C_3 shows this) and so we're left with the latter.

The considerations above apply to C_4 and C_5, too, and so we get, up to symmetry, two cases:

1. C_3, C_4 and C_5 meet X' and Z''.
2. C_3 and C_4 meet X' and Z'', while C_5 meets X'' and Z'.

In case 1 we observe that a common separating tangent T for C_3 and C_4 cannot meet the sets C_1, C_2, C_3 and C_4 in the order stated. This means that if we rename C_3 and C_4 by C_1 and C_2, and vice versa, we get case 2, now to be treated.

Let S be a line separating C_3 and C_4. Up to symmetry there are two subcases. The first one, when $C_5 \cap S$ is between $C_1 \cap S$ and $C_2 \cap S$, is easily seen to be impossible.

In the second subcase when $C_2 \cap S$, say, is between $C_1 \cap S$ and $C_5 \cap S$, we choose the numbering so that $C_4 \cap T'$ is between $C_3 \cap T'$ and $C_5 \cap T'$. Then $C_4 \cap T''$ is between $C_3 \cap T''$ and $C_5 \cap T''$, too. It is now clear that a line separating C_2 from C_5 can not meet C_4. This final contradiction to assumption A finishes the proof of theorem 2.

4. Proof of Theorem 1.

Let k be a given natural number. There exists, by Ramsey's theorem, an integer $R = R(k)$ with the following property: Whenever the 3-subsets of an R-set S are split into three families F_1, F_2, and F_3, then for some $i \in \{1,2,3\}$ there is a k_i-subset S_i, $S_i \subset S$, such that all the 3-subsets of S_i belong to the family F_i. Here $k_1 = k_2 = k+1$, $k_3 = 5$.

We assert that $K_2(k,1)$ exists and does not exceed $R(k)+k-1$ (being probably much smaller).

Consider $R+k-1$ convex sets, as described in Theorem 1. We want to prove the existence of a closed halfplane having the desired property. We may assume, by Theorem 2, that $k \geqq 3$, and also, as in the proof of that theorem, that the C_i are compact and pairwise disjoint, with non-empty interiors. We may also assume that no horizontal line is tangent to more than one C_i.

Now let L be the unique horizontal line which is an upper tangent to one C_i, say C_1, while H, the lower closed halfplane bounded by it contains exactly k of the C_i, say $C_1, \ldots, C_k$.

If C_{k+1}, say, did not meet L in an interior point, then H would be the halfplane we're looking for. We may thus assume, upon renumbering of the C_i, that the interiors of $C_1, \ldots, C_R$ all meet L' (a line slightly below L) in the stated order.

Each 3-subset $\{l, m, n\}$ of $\{1, \ldots, R\}$, with $l<m<n$, now has at least one of the following three properties.

1. Some line separates C_l from C_m and C_n.
2. Some line separates C_n from C_l and C_m.
3. No line separates one of the three sets in question from the two others.

By the definition of R there is either a $(k+1)$-subset of $\{1, \ldots, R\}$, all 3-subsets of which have the same property (1 or 2), or a 5-subset, all 3-subsets of which have property 3. The last-mentioned possibility is, however, excluded by Theorem 2, and so we may assume, by symmetry, that all 3-subsets of $\{1, \ldots, k+1\}$ have property 1. In particular, the interior of C_1 is disjoint from $\operatorname{conv}(C_m \cup C_n)$ whenever $1<m<n\leqq k+1$. But this implies by Theorem 3 (for $d=2$) that the interior of C_1 is disjoint from $\operatorname{conv}(C_2 \cup \ldots \cup C_{k+1})$, which finishes the proof of Theorem 1.

5. The non-existence of $K_2(2,2)$ and $K_3(2,1)$.

Our first example consists of infinitely many line segments in $\mathbf{R}^2$, with disjoint relative interiors, chosen such that the convex hull of any two of them has the point $(1,1)$ in its interior. This example shows the non-existence of $K_2(2,2)$, and thus of $K_2(r,s)$ when $r\geqq 2$, $s\geqq 2$.

The second example is a set of infinitely many disjoint lines in $\mathbf{R}^3$, chosen such that no line in $\mathbf{R}^3$ is orthogonal to more than two of them. (One can for instance choose a non-horizontal and non-vertical tangent T to the circle $x^2+y^2=1$, $z=0$ and let the set consist of all the lines obtained from T by rotation around the z-axis.) This example shows the non-existence of $K_3(2,1)$, which implies the non-existence of $K_d(r,s)$ whenever $d\geqq 3$ and $r+s\geqq 3$.

We return to a more detailed description of the first example. The segments, to be constructed one by one, are $S_1, S_2, \ldots$, where S_n has endpoints (x_n, y_n) and $(x_n', 0)$, with $x_n<1<x_n'$, $y_n>1$. Furthermore the point $(1,1)$ lies above S_n. The endpoint (x_{n+1}, y_{n+1}) of S_{n+1} belongs to S_n, with $y_{n+1}<y_n$, and x_{n+1}' is chosen so large as to ensure that $(1,1)$ is an interior point of

$$\operatorname{conv}\{(x_n, y_n), (x_n', 0), (x_{n+1}', 0)\} .$$

Thus the x_i' form an increasing sequence, so that $(1,1)$ will also be an interior point of $\operatorname{conv}(S_n \cup S_m)$ for every $m>n$, as needed.

There is a difference between the two examples, as the one just described consists of only countably many sets. Must this be so? Another question which arises naturally is that of the computation of numbers $K_d(r,s;C)$, defined in the obvious way by considering, instead of the class of all convex sets in $\mathbf{R}^d$, some suitably restricted class C for which such numbers would exist. Even the computation of $K_2(k, 1)$ in general seems to offer a challenge, however.

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