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HYPERBOLIC HARDY CLASS H^1

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1. Introduction.

A meromorphic function f in $D=\{|z|<1\}$ is known to be of bounded Nevanlinna characteristic if and only if the Shimizu–Ahlfors characteristic function of f,

$$\int_0^r \pi^{-1}t^{-1}\left[\iint_{|z|<t} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2}\,dx\,dy\right]dt\,,$$

is bounded for $0<r<1$, where $z=x+iy$; see [3, p. 13]. Similarly, a holomorphic function f in D is of Hardy class H^2 if and only if

$$\int_0^r \pi^{-1}t^{-1}\left[\iint_{|z|<t} |f'(z)|^2\,dx\,dy\right]dt$$

is bounded for $0<r<1$. This follows on integrating an obvious version of the Hardy–Stein equality [2, Theorem 3.1, p. 42] applied to f and $\lambda=2$.

Let now B be the family of f holomorphic and bounded, $|f|<1$, in D. Set for $f\in B$,

$$T^*(r,f) = \int_0^r \pi^{-1}t^{-1}\left[\iint_{|z|<t} \frac{|f'(z)|^2}{(1-|f(z)|^2)^2}\,dx\,dy\right]dt\,,$$

where $0<r<1$. The main purpose of the present paper is to study f whose hyperbolic Shimizu–Ahlfors characteristic function $T^*(r,f)$ remains bounded for $0<r<1$.

It is well known that the disk D is endowed with the non-Euclidean hyperbolic distance

$$\sigma(z,w) = \tfrac{1}{2}\log\frac{|1-\bar{z}w|+|z-w|}{|1-\bar{z}w|-|z-w|},\qquad z,\,w\in D\,.$$

As will be shown in Lemma 1, for each $f\in B$ and for each constant $a\in D$, the function $\log\sigma(f,a)$ is subharmonic in D. Therefore, $\sigma(f,a)^p=\exp[p\log\sigma(f,a)]$ is also subharmonic for each $p>0$. In particular,

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$$M_1^*(r, f, a) = \frac{1}{2\pi}\int_0^{2\pi} \sigma(f(re^{i\theta}), a)\, d\theta$$

is bounded for $0 \leqq r < 1$ if and only if the subharmonic function $\sigma(f, a)$ admits a harmonic majorant U in D, that is, U is harmonic and $\sigma(f, a) \leqq U$ in D. Therefore, it is reasonable to say that $f \in B$ is of hyperbolic H^1 if $\sigma(f, 0)$ admits a harmonic majorant in D. Let H_h^1 be the family of $f \in B$, being of hyperbolic H^1. Our main result is

THEOREM 1. *For $f \in B$ to be of H_h^1 it is necessary and sufficient that $T^*(r, f)$ is bounded for $0 < r < 1$.*

Theorem 1 follows from the next theorem on setting $a = 0$.

THEOREM 2. *For each $f \in B$, each $a \in D$, and each $r, 0 < r < 1$, the following inequality holds.*

$$T^*(r, f) \leqq M_1^*(r, f, a) + \tfrac{1}{2}\log[1 - |(f(0) - a)/(1 - \bar{a}f(0))|^2] < T^*(r, f) + \log 2\,. \tag{1.1}$$

The inequality (1.1) is sharp. First, $T^*(r, 0) = M_1^*(r, 0, 0) = 0$ for each $0 < r < 1$. Next, the constant $\log 2$ in (1.1) cannot be replaced by a smaller positive constant. Actually, for $f(z) = z$,

$$M_1^*(r, f, 0) - T^*(r, f) = \log(1 + r) \to \log 2 \text{ as } r \to 1\,.$$

Let G be a subdomain of D such that the boundary of G has the only one point 1 in common with the unit circle. Assume that there exist $r_0, 0 < r_0 < 1$, and a function $A(r)$, $r_0 < r < 1$, both depending on G, such that the intersection of G with each circle $\{|z| = r\}$, $r_0 < r < 1$, is of linear measure $rA(r)$, where

$$0 < \inf_{r_0 < r < 1} (1 - r)^{-1} A(r) \text{ and } \sup_{r_0 < r < 1} (1 - r)^{-1} A(r) < +\infty\,.$$

Let $\mathscr{G}$ be the family of all domains G of the described type. A typical example of $G \in \mathscr{G}$ is a triangular domain in D with one vertex at 1 and the other vertices in D. Set $G(\theta) = \{e^{i\theta}z\,;\, z \in G\}$ for $G \in \mathscr{G}$ and $\theta \in [0, 2\pi]$, and set for $f \in B$,

$$S(f, G(\theta)) = \iint_{G(\theta)} \frac{|f'(z)|^2}{(1 - |f(z)|^2)^2}\, dx\, dy\,,$$

being the non-Euclidean area of the Riemannian image of $G(\theta)$ by f. We next propose a criterion for $f \in B$ to belong to H_h^1.

THEOREM 3. *Assume that $f \in B$, and assume that, for a certain $G \in \mathscr{G}$,*

$$\int_0^{2\pi} S(f, G(\theta))\, d\theta < +\infty . \tag{1.2}$$

Then $f \in H_h^1$. Conversely, if $f \in H_h^1$, then (1.2) *holds for each $G \in \mathscr{G}$.*

Finally, we assert that H_h^1 is closed for the multiplication and that H_h^1 is convex.

THEOREM 4. *For each $f \in H_h^1$, each $g \in H_h^1$, and each constant $t, 0<t<1$,*

$$fg \in H_h^1 \text{ and } tf + (1-t)g \in H_h^1 .$$

2. Proof of Theorem 2.

The following lemma is fundamental to deduce Theorem 1 from Theorem 2.

LEMMA 1. *For each $f \in B$ and for each constant $a \in D$, the function* $\log \sigma(f, a)$ *is subharmonic in D.*

Setting $u(z) = \sigma(z, 0)$, $z \in D$, one obtains for $z \neq 0$, the identity

$$\Delta \log u(z) = (1-|z|^2)^{-2} u(z)^{-2} [|z|^{-1}(1+|z|^2)u(z) - 1] ,$$

which, together with the inequality

$$\frac{1+x^2}{2x} \log \frac{1+x}{1-x} - 1 \geqq 0 \quad \text{for} \quad 0 < x < 1 ,$$

shows that $\Delta \log u \geqq 0$ in $D - \{0\}$. Since $\log u(0) = -\infty$, the function $\log u$ is subharmonic in the whole D. Since

$$T_a(f) = (f-a)/(1-\bar{a}f)$$

is holomorphic in D, it follows that the composed function

$$\log \sigma(T_a(f), 0) = \log \sigma(f, a)$$

is subharmonic in D.

For the proof of Theorem 2 we note first that, for $f \in B$ and $a \in D$, the following inequality holds in D.

$$-\tfrac{1}{2}\log(1-|T_a(f)|^2) \leqq \sigma(f, a) < -\tfrac{1}{2}\log(1-|T_a(f)|^2) + \log 2 . \tag{2.1}$$

In effect, for $0 \leqq x < 1$,

$$-\tfrac{1}{2}\log(1-x^2) \leqq \sigma(x, 0) < -\tfrac{1}{2}\log(1-x^2) + \log 2 ,$$

which, together with $\sigma(f, a)=\sigma(|T_a(f)|, 0)$, proves (2.1). We next note that, in D.

$$-\Delta \log(1-|f|^2) = 4f^{*2}, \quad f \in B,$$

where $f^* = |f'|/(1-|f|^2)$. Since $(T_a(f))^* = f^*$, it follows that

$$-\Delta \log(1-|T_a(f)|^2) = 4f^{*2} \tag{2.2}$$

in D.

Setting

$$I(r, f, a) = -\frac{1}{2\pi}\int_0^{2\pi} \log(1-|T_a(f(re^{i\theta}))|^2)\, d\theta$$

for $f \in B$, $a \in D$, and $0<r<1$, one observes by the Green formula, together with (2.2), that

$$r\frac{d}{dr} I(r, f, a) = \frac{2}{\pi}\iint_{|z|<r} f^*(z)^2\, dx\, dy, \quad 0<r<1. \tag{2.3}$$

Since $\lim_{r\to 0} I(r, f, a) = -\log(1-|T_a(f(0))|^2)$, the integration of (2.3) yields $\frac{1}{2}[I(r, f, a)+\log(1-|T_a(f(0))|^2)] = T^*(r, f)$, $0<r<1$. On the other hand, it follows from (2.1) that

$$\tfrac{1}{2}I(r, f, a) \leqq M_1^*(r, f, a) < \tfrac{1}{2}I(r, f, a)+\log 2,$$

whence follows (1.1) of Theorem 2.

REMARK. For $f \in B$, the function $\log[-\log(1-|f|^2)]$ is subharmonic in D. In effect, except for the zeros of f, one obtains in D,

$$\Delta \log[-\log(1-|f|^2)] = 4\frac{\partial^2}{\partial z \partial \bar{z}}\log[-\log(1-f\bar{f})]$$

$$= 4[\log(1-|f|^2)]^{-2} f^{*2}[-\log(1-|f|^2)-|f|^2] \geqq 0.$$

By (2.1) for $a=0$ and by Theorem 1 one observes that $f \in B$ is of H_h^1 if and only if $-\log(1-|f|^2)$ has a harmonic majorant in D. Let $f(e^{i\theta})$ be the angular limit of $f \in B$, whose existence at almost every $e^{i\theta}$, $\theta \in [0, 2\pi]$, is well known. Assume that $f \in H_h^1$. It then follows from [1, Theorem] (see also [4]) that

$$\sigma(f, 0) = \exp[\log \sigma(f, 0)]$$

and

$$-\log(1-|f|^2) = \exp[\log[-\log(1-|f|^2)]]$$

both have the least harmonic majorants in D, being the Poisson integrals of $\sigma(f(e^{i\theta}), 0)$ and $-\log(1-|f(e^{i\theta})|^2)$ on $[0, 2\pi]$, respectively.

3. Proof of Theorem 3.

We shall make use of

LEMMA 2. *Let $f \in B$. Then, $f \in H_h^1$ if and only if*

$$E_1 = \iint_D (1-|z|) f^*(z)^2 \, dx\, dy < +\infty .$$

First of all, by Theorem 1, f is a member of H_h^1 if and only if

$$E_2 = \int_0^1 \left[\iint_{|z|<r} f^*(z)^2 \, dx\, dy \right] dr < +\infty .$$

Let X_r be the function in D, being one on $\{|z|<r\}$, and zero otherwise. Then

$$\begin{aligned} E_2 &= \int_0^1 \left[\iint_D X_r(z) f^*(z)^2 \, dx\, dy \right] dr \\ &= \iint_D \left[\int_0^1 X_r(z)\, dr \right] f^*(z)^2 \, dx\, dy = E_1 , \end{aligned}$$

which proves lemma 2.

For the proof of Theorem 3, let r_0 and A be as described in the definition of $G \in \mathscr{G}$. Let $X(z,\theta)$ be the function of z in D, being one on $G(\theta)$ and zero otherwise, θ ranging on $[0, 2\pi]$. Then

$$A(|z|) = \int_0^{2\pi} X(z;\theta)\, d\theta, \qquad r_0 < |z| < 1 . \tag{3.1}$$

On the other hand,

$$\int_0^{2\pi} S(f, G(\theta))\, d\theta = \iint_D \left[\int_0^{2\pi} X(z,\theta)\, d\theta \right] f^*(z)^2 \, dx\, dy .$$

It then follows from (3.1) that (1.2) holds if and only if

$$\iint_{r_0<|z|<1} (1-|z|) f^*(z)^2 \, dx\, dy < +\infty .$$

Our Theorem 3 now follows from Lemma 2.

4. Proof of Theorem 4.

According to the remark at the end of Section 2, $F \in H_h^1$ if and only if $-\log(1-|F|^2)$ has a harmonic majorant in D.

For $0 \leqq P < 1$ and $0 \leqq Q < 1$,

$$-\log(1-P)-\log(1-Q) \geqq -\log(1-PQ)\,, \tag{4.1}$$

being a consequence of $2PQ \leqq (2\sqrt{PQ} \leqq) P+Q$. Now, to prove that $fg \in H_h^1$, we have only to apply (4.1) to $P=|f|^2$ and $Q=|g|^2$.

Since the function $-\log(1-x^2)$ is convex for $0 \leqq x < 1$, it follows that

$$\begin{aligned} -\log[1-(tP+(1-t)Q)^2] \\ \leqq -t\log(1-P^2)-(1-t)\log(1-Q^2) \end{aligned} \tag{4.2}$$

for $0 \leqq P < 1$ and $0 \leqq Q < 1$. Therefore,

$$-\log(1-|tf+(1-t)g|^2) \leqq -\log[1-(t|f|+(1-t)|g|)^2]\,,$$

together with (4.2), where $P=|f|$, $Q=|g|$, proves that $tf+(1-t)g \in H_h^1$.

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