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CLUSTER SETS AND BOUNDARY BEHAVIOR OF QUASIREGULAR MAPPINGS

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1. Introduction.

Let B^n, $n \geqq 2$, be the n-dimensional unit ball and G' a domain in $\mathbf{R}^n$, and let $f\colon B^n \to G'$ be a quasiregular mapping in the sense of Martio, Rickman, and Väisälä [5]. Then f is said to be *boundary-preserving at* $b \in \partial B^n$ if there is a neighborhood U of b such that the cluster set $C(f,b) \subset \partial G'$ for all $b \in \partial B^n \cap U$ and *boundary-preserving* if $C(f,b) \subset \partial G'$ for all $b \in \partial B^n$.

It follows from a result of Srebro [11, 4.2] that a boundary-preserving quasiregular mapping $f\colon B^n \to G' = fB^n$ can be continuously extended to $\bar{B}^n$ if the boundary of G' is sufficiently nice. Related results for injective quasiregular mappings, usually called quasiconformal ones, have been proved by Väisälä and Näkki (cf. [14, Ch. 17]). Generalizing a result of Väisälä about homeomorphic extension of a quasiconformal mapping the author sharpened Srebro's result in [16, 4.7] by showing that if G' is locally connected on the boundary, then the extended mapping is discrete.

The results in [11, 4.2] and [16, 4.7] are of *global* character, because extension to the whole boundary is investigated. The purpose of this paper is to study the following problem, which is of *local* character: Let $f\colon B^n \to B^n$ be quasi-regular and boundary-preserving at $b \in \partial B^n$. Is it true that f has a continuous extension to $\bar{B}^n \cap U$ where U is a neighborhood of b? Is the extended mapping discrete? We shall prove the following facts. Firstly, the continuous extension exists. Secondly, the extended mapping is discrete under the additional assumption that f be locally homeomorphic. The first result has as a consequence an alternative proof for the reflection principle of boundary-preserving quasiregular mappings (cf. Martio and Rickman [4, 5.2]). It is an open problem, whether the assumption that f be locally homeomorphic can be omitted in the second result (cf. 3.14).

We shall also study the behavior of a quasiregular mapping at such boundary points where it is not boundary-preserving, provided that it is boundary-preserving at "almost all" boundary points. For this purpose we consider a quasiregular mapping $f\colon B^n \to B^n$, boundary-preserving at all points

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of $\partial B^n \setminus E$, where $E \subset \partial B^n$ is a compact set of capacity zero. We shall prove that if f is not boundary-preserving at $b \in E$, then f attains in every neighborhood of b infinitely many times all values of B^n except for a possible subset of B^n of capacity zero. The proof of this result is based on the use of Poleckii's modulus inequality (cf. Väisälä [15, p. 3]) in the same manner as the proof of the quasiregular version of the Iversen–Tsuji theorem [17, 3.2]. The results in this paper are, in general, new only for dimensions $n \geqq 3$ (for $n=2$, cf. Collingwood–Lohwater [2], Noshiro [10]).

2. Cluster sets and boundary cluster sets.

2.1. NOTATION AND TERMINOLOGY. We shall use throughout the paper mainly the same terminology and notation as in [5–6] and [16–17] and we shall list here some frequently occurring concepts.

All topological operations are performed with respect to $\bar{\mathsf{R}}^n = \mathsf{R}^n \cup \{\infty\}$, the one-point compactification of R^n, and we assume that $n \geqq 2$ in the sequel. The coordinate unit vectors are $e_1, \ldots, e_n$. We let the notation $f\colon G \to A$, $A \subset \mathsf{R}^n$ ($A \subset \bar{\mathsf{R}}^n$) include the assumptions that f is continuous and that G is a domain in R^n ($\bar{\mathsf{R}}^n$). The *cluster set* of $f\colon G \to \bar{\mathsf{R}}^n$ at $b \in \partial G$ is defined by

$$C(f,b) = \bigcap_U \overline{f(G \cap U)}$$

where U runs through all neighborhoods of b. If $F \subset \partial G$ is non-empty we denote by $C(f,F)$ the union of the cluster sets $C(f,b)$, $b \in F$. We shall also consider the *boundary cluster set* of f at $b \in \partial G$ with respect to a non-empty set $F \subset \partial G$ defined by

$$C_F(f,b) = \bigcap_U \overline{C(f, U \cap (F \setminus \{b\}))}$$

where U runs through all neighborhoods of b. The *multiplicity* of $y \in \bar{\mathsf{R}}^n$ in $D \subset \bar{\mathsf{R}}^n$ is

$$N(y,f,D) = \operatorname{card}\{f^{-1}(y) \cap D\} .$$

The *maximal multiplicity* of f in D is

$$N(f,D) = \sup\{N(y,f,D) : y \in \bar{\mathsf{R}}^n\} .$$

We abbreviate $N(f) = N(f,G)$. Balls and spheres centered at $x \in \mathsf{R}^n$ and with radius $r > 0$ are denoted, respectively, by

$$B^n(x,r) = \{z \in \mathsf{R}^n : |z-x| < r\} ,$$

$$S^{n-1}(x,r) = \{z \in \mathsf{R}^n : |z-x| = r\} .$$

We adopt the abbreviations $B^n(r)=B^n(0,r)$, $B^n=B^n(1)$, $S^{n-1}(r)=S^{n-1}(0,r)$, and $S^{n-1}=S^{n-1}(1)$.

2.2. Modulus of a path family and sets of capacity zero. A *path* in this paper is a continuous nonconstant mapping $\gamma\colon [0,c) \to A$, $c>0$, where $A\subset \bar{\mathsf{R}}^n$, and we let $|\gamma|$ denote the set $\gamma[0,c)$. If $E,F,G\subset \bar{\mathsf{R}}^n$ we let $\Delta(E,F;\ G)$ denote the family of all paths $\gamma\colon [0,1) \to \bar{\mathsf{R}}^n$ with $\gamma(0)\in E$, $\gamma(0,1)\subset G$ and $\gamma(t)\to F$ as $t \to 1$. The *(n-)modulus* of a path family Γ is denoted by $M(\Gamma)$, see [14, Section 6]. A compact proper subset F of $\bar{\mathsf{R}}^n$ is said to be of (conformal) *capacity zero* if the modulus of the family consisting of all paths with one endpoint in F, is zero. An arbitrary subset of $\bar{\mathsf{R}}^n$ is said to be of *capacity zero* if every compact subset is of capacity zero. Otherwise a set is of positive capacity. We write $\operatorname{cap} F=0$ and $\operatorname{cap} F>0$, respectively. For an alternative definition of these concepts, involving n-capacities of condensers, the reader is referred to Ziemer [19].

The following lemma yields an important lower bound for the modulus of a path family. It follows from a result of Martio, Rickman, and Väisälä [6, 3.11] and from a symmetry principle for the modulus (see e.g. [18, Section 4]).

2.3. Lemma. *Let $F\subset \bar{\mathsf{R}}^n$ be a compact set and let $r>0$ be such that* $\operatorname{cap} F_r>0$, *where $F_r=F\cap B^n(r)$. Then for every $r_0>0$ there exists $\delta>0$ such that for every continuum $C\subset B^n(r)\setminus F$ with the euclidean diameter $d(C)\geqq r_0$ the following estimate holds*: $M(\Delta(C,F;\ B^n(r)))\geqq \delta$.

The proof of the following theorem is based on the modulus inequality of Poleckii (cf. Väisälä [15, p. 3]) and on Lemma 2.3 (cf. [17, 3.2], [18, 4.2]). This results is a counterpart to a well known theorem about analytic functions (cf. Noshiro [10, p. 19 and p. 116]).

2.4. Theorem. *Let $f\colon G \to \bar{\mathsf{R}}^n$ be a quasimeromorphic mapping and let $E\subset \partial G$ be a compact set of capacity zero. If $b\in E\cap \overline{(\partial G\setminus E)}$ and if the set $A=C(f,b)\setminus C_{\partial G\setminus E}(f,b)$ is not empty, then f assumes every value of A infinitely many times in every neighborhood of b except for a possible subset of A of capacity zero.*

Proof. By performing a preliminary Möbius transformation if necessary we may assume that $b\neq\infty$. Suppose that the theorem is not true. Then there exists $r_0>0$ such that the assertion does not hold for the neighborhood $B^n(b,r_0)$ of b. Let

$$N_k = \{y \in \bar{R}^n : N(y, f, B^n(b, r_0) \cap G) = k\}, \quad k = 0, 1, 2, \ldots$$
$$N = \{y \in \bar{R}^n : N(y, f, B^n(b, r_0) \cap G) < \infty\} .$$

Then apparently $N \subset \bigcup N_k$. By the choice of r_0 cap $A \cap N > 0$ and hence we may fix an interger k such that cap $A \cap N_k > 0$ (cf. [19]). Choose a compact set $F \subset A \cap N_k$ with cap $F > 0$. There exists a point $z \in F \setminus \{\infty\}$ with cap $\bar{B}^n(z, r) \cap F > 0$ for every $r > 0$. In the sequel we shall assume that $k \geqq 1$; the slightly easier case $k = 0$ requires minor modifications. Let

$$f^{-1}(z) \cap B^n(b, r_0) = \{x_1, \ldots, x_k\} .$$

Write $\sigma_0 = \min\{\sigma_{x_1}, \ldots, \sigma_{x_k}\}$ where σ_{x_j} has the same meaning as in [5, 2.9]. For $\sigma \in (0, \sigma_0]$ denote by $U_j(\sigma)$ the x_j-component of $f^{-1}B^n(z, \sigma)$, $j = 1, \ldots, k$. Then $U_j(\sigma)$ is a normal neighborhood of x_j and, in particular, $\overline{U_j(\sigma)}$ is a compact subset of G for $\sigma \in (0, \sigma_0]$ and $j = 1, \ldots, k$. See [5, 2.4] for terminology. There exist arbitrarily small numbers $r \in (0, r_0)$ such that the following conditions are satisfied:

$$1) \quad \bar{B}^n(b, r) \cap \left(\bigcup_{j=1}^{k} \overline{U_j(\sigma_0)} \right) = \varnothing .$$

$$2) \quad S^{n-1}(b, r) \cap E = \varnothing .$$

$$3) \quad z \notin \overline{C(f, (\partial G \setminus E) \cap \bar{B}^n(b, r))} .$$

Condition 1) is clearly satisfied for all sufficiently small $r > 0$, since the sets $\overline{U_j(\sigma_0)}$ are compact. Since cap $E = 0$ implies that the one-dimensional Hausdorff measure $\Lambda_1(E) = 0$, there are arbitrarily small $r > 0$ which satisfy 2). Condition 3) holds for all sufficiently small $r > 0$ by the definition of the boundary cluster set and by the choice of z. Fix $r_1 \in (0, r_0)$ satisfying 1)–3). Fix $\sigma_1 \in (0, \sigma_0]$ with

$$(2.5) \qquad \bar{B}^n(z, \sigma_1) \cap \overline{C(f, (\partial G \setminus E) \cap \bar{B}^n(b, r_1))} = \varnothing .$$

Choose a sequence (a_j) in G with $|a_j - b| < r_1/j$, $j = 1, 2, \ldots$ and with $f(a_j) \to z$ as $j \to \infty$. This is possible, because $z \in C(f, b)$. Since E is compact, since $b \in E \cap \overline{(\partial G \setminus E)}$, and since $\Lambda_1(E) = 0$ we may choose for $j = 1, 2, \ldots$ a path

$$\gamma_j \colon [0, 1) \to G \cap \bar{B}^n(b, r_1/j)$$

with $\gamma_j[0, 1) \subset G$, $\gamma_j(0) = a_j$, and with

$$\gamma_j(t) \to b_j \in (\partial G \setminus E) \cap \bar{B}^n(b, r_1/j) \quad \text{as } t \to 1$$

(cf. [4, 3.3]). After relabeling if necessary we may assume that $f(a_j) \in B^n(z, \sigma_1/3)$, $j = 1, 2, \ldots$. By taking into account (2.5) we may now choose a continuum $C_j \subset |\gamma_j| \cap G$, $a_j \in C_j$ with $fC_j \subset B^n(z, \sigma_1)$ and with

$$fC_j \cap S^{n-1}(z, \sigma_1/3) \neq \emptyset, \quad fC_j \cap S^{n-1}(z, 2\sigma_1/3) \neq \emptyset,$$

for every interger j. Let $\Gamma'_j = \Delta(fC_j, F; B^n(z, \sigma_1))$. By Lemma 2.3 there is a $\delta > 0$ such that

$$M(\Gamma'_j) \geqq \delta > 0 \quad \text{for all } j = 1, 2, \ldots .$$

Denote by Γ_j the family of the maximal liftings of the elements of Γ'_j, starting at C_j. For terminology, we refer the reader to [7, 3.11]. Fix $j \geqq 2$ and α: $[0, c) \to G$ in Γ_j. From [7, 3.12] and from (2.5) it follows that either $\overline{|\alpha|} \cap E \neq \emptyset$ or $|\alpha| \cap S^{n-1}(b, r_1) \neq \emptyset$. Since cap $E = 0$, the subfamily of Γ_j consisting of paths of the former kind has zero modulus and we obtain by [14, 7.5]

$$M(\Gamma_j) \leqq \omega_{n-1}(\log j)^{1-n},$$

for $j = 2, 3, \ldots$. The modulus inequality [15, 3.1] yields

$$M(\Gamma'_j) \leqq M(f\Gamma_j) \leqq K_I(f) M(\Gamma_j),$$

for $j = 2, 3, \ldots$. For large j this inequality contradicts the above inequalities. The proof is complete.

The capacity used in Noshiro's book [10]—the so called logarithmic capacity—is defined in a way different from the definition of the conformal capacity used by us (cf. [10, p. 6]). From the point of view of Theorem 2.4, which generalizes Theorem 5 on p. 19 of [10], it deserves mentioning that, when the dimension $n = 2$, the compact null-sets of these capacities coincide. We refer the reader to the papers of Wallin and Ziemer quoted in [4, 2.3 and 3.2].

3. Extension to boundary and reflection principle.

In the present section we shall first study some properties of discrete open mappings. These results will be used in the latter part of the section, where we examine boundary behavior of quasiregular mappings. The main theorem of this section gives us a reflection principle for locally homeomorphic quasiregular mappings.

Recall that a mapping f: $G \to \mathbf{R}^n$ is *open* if fA is open in $\mathbf{R}^n$ whenever $A \subset G$ is open in G and *discrete* if $f^{-1}(y)$ is discrete for every $y \in fG$.

3.1. DEFINITION. Let f: $G \to G'$ be a discrete open mapping. Then f is *boundary-preserving at* $b \in \partial G$ if there is a neighborhood U of b such that $C(f, U \cap \partial G) \subset \partial G'$, and *boundary-preserving* if it is boundary-preserving at every point $b \in \partial G$.

3.2. REMARKS. (1) If $f: G \to \mathbf{R}^n$ is open, then it is always true that $\partial fG \subset C(f, \partial G)$.

(2) If $f: G \to \mathbf{R}^n$ is injective, then $f: G \to fG$ is boundary-preserving. As the example $g: B^2 \to B^2$, $g(z)=z^2$ shows, boundary-preserving mappings need not be injective. For further examples we refer the reader to Section 4.

The following lemma gives some characterizations for the fact that a discrete open mapping is boundary-preserving. For the implication (2) $\Rightarrow$ (4) we refer the reader to Väisälä [13, 5.5] and Martio–Srebro [8, 3.3]. Only the implication (4) $\Rightarrow$ (1) will be proved here, because the others were proved in [16, 3.3].

3.3 LEMMA. *Let $f: G \to fG$ be a discrete open mapping. Then the following conditions are equivalent*:

(1) *f is boundary-preserving.*
(2) *f is closed, i.e. fA is closed in fG whenever A is closed in G.*
(3) *f is proper, i.e. $f^{-1}K$ is compact whenever $K \subset fG$ is compact.*
(4) *For every $y \in fG$*

$$N(f) = \sum_{z \in f^{-1}(y)} |i(z, f)| < \infty .$$

PROOF. By [16, 3.3] it suffices to prove (4) $\Rightarrow$ (1). Assume that (1) does not hold. Then there is a point $b \in \partial G$ and a sequence (b_k) in G so that $b_k \to b$ and $f(b_k) \to y_0 \in fG$ as $k \to \infty$. Suppose that (4) holds. Then there is an integer $p \geqq 1$ such that $f^{-1}(y_0)=\{x_1,\ldots,x_p\}$ and

$$N(f) = \sum_{j=1}^{p} |i(x_j, f)| < \infty .$$

Fix $r>0$ such that the x_j-component $U_j(r)$ of $f^{-1}B^n(y_0, r)$ is a normal neighborhood of x_j, $1 \leqq j \leqq p$, and $\bar{U}_j(r) \cap \bar{U}_k(r) = \emptyset$ whenever $j \neq k$. This is possible by [5, 2.9]. For terminology we refer the reader to [5, 2.4]. Since $b_k \to b \in \partial G$ and $f(b_k) \to y_0$ as $k \to \infty$ there is an integer k_0 such that $b_k \in G \setminus \bigcup_{j=1}^{p} \bar{U}_j(r)$ and $f(b_k) \in B^n(y_0, r)$ for $k \geqq k_0$. Let z_h^j, $1 \leqq h \leqq a_j$, be the points of $U_j(r) \cap f^{-1}(f(b_{k_0}))$, $1 \leqq j \leqq p$. Since $U_j(r)$ is a normal neighborhood of x_j we get by [5, 2.12] and (4)

$$\begin{aligned} N(f) &\geqq \sum_{j=1}^{p} \sum_{h=1}^{a_j} |i(z_h^j, f)| + |i(b_{k_0}, f)| \\ &\geqq \sum_{j=1}^{p} |i(x_j, f)| + 1 = N(f)+1 . \end{aligned}$$

This contradiction completes the proof.

3.4. CONDITION $N(f,b)<\infty$. Let $f\colon G \to \mathbf{R}^n$ be discrete open and let $b \in \partial G$. Then we use the notation $N(f,b)<\infty$ if there is a neighborhood U of b such that $N(f,U)<\infty$. Otherwise $N(f,b)=\infty$.

According to Lemma 3.3 a discrete open boundary-preserving mapping f has finite maximal multiplicity $N(f)$. One may ask whether $N(f,b)<\infty$ if $f\colon G \to fG$ is a discrete open mapping, which is boundary-preserving at $b \in \partial G$. It is easy to show that then $N(y,f,U)<\infty$ for all $y \in \mathbf{R}^n$ if U is a neighborhood of b so that $C(f, U \cap \partial G) \subset \partial fG$. However, the following example by Väisälä shows that the condition $N(f,b)<\infty$ may fail to hold.

3.5. EXAMPLE (due to J. Väisälä). (1) We shall construct a discrete open mapping $f\colon H \to fH$, $H=\{(x,y) \in \mathbf{R}^2 : x>0\}$ which is boundary-preserving at every point of $\partial H \setminus \{\infty\}$ and for which $N(f,0)=\infty$. To this end, let the value of f at $(x,y) \in H$ be defined in the polar coordinates (r,φ) by $r=x$, $\varphi=y/x$. Then f is a local homeomorphism, hence a discrete open mapping, and $C(f,b)=\{0\} \subset \partial fH=\{0,\infty\}$ for all $b \in \partial H \setminus \{\infty\}$. It is seen that $N(f,0)=\infty$.

(2) We shall now slightly modify the above example so that we get a locally homeomorphic mapping $g\colon B^2 \to B^2$ with the properties $C(g, \partial B^2 \setminus \{-e_1\}) \subset \partial B^2$ and $N(g,e_1)=\infty$. Let $h_1\colon B^2 \to H=h_1 B^2$ be a conformal homeomorphism with $C(h_1, -e_1)=\{\infty\}$ and $C(h_1,e_1)=\{0\}$ and let $h_2\colon \mathbf{R}^2 \setminus \{0\}=fH \to B^2 \setminus \{0\}$ be a homeomorphism so that $C(h_2,0)=\partial B^2$ and $C(h_2,\infty)=\{0\}$. Then $g=h_2 \circ f \circ h_1\colon B^2 \to B^2$ is the desired local homeomorphism.

The next lemma will not be needed in the sequel and it is included mainly because it completes some earlier results in [16, Section 3]. We remark that the assumption $n \geqq 3$ of the next lemma is natural, because any domain with more than one boundary component is not simply connected when $n=2$. The *branch set* B_f of a mapping $f\colon G \to \mathbf{R}^n$ is the set of all points of G at which f fails to be a local homeomorphism. It is clear that B_f is closed in G (cf. [1, 1.1]).

3.6. LEMMA. *Let $f\colon B^n \to fB^n \subset \mathbf{R}^n$, $n \geqq 3$, be a discrete open closed mapping and suppose that there exists a domain V containing fB_f such that $\bar{V} \subset fB^n$ and $fB^n \setminus \bar{V}$ is simply connected. Then f is injective.*

PROOF. Since f is closed

$$C(f,\partial B^n) = \bigcap_{r>0} \overline{f(B^n \setminus \bar{B}^n(r))} \subset \partial fB^n$$

and there is $\varrho \in (0,1)$ such that $f(B^n \setminus \bar{B}^n(\varrho)) \subset fB^n \setminus \bar{V} = A$. This inference shows that the open set $f^{-1}A$ has exactly one component D with $\partial B^n \subset \partial D$. Since f is discrete open closed, the mapping $f \mid D\colon D \to A$ is surjective by [16, 3.6] and [5, 2.5]. The mapping $f \mid D$ is locally homeomorphic and since A is simply connected, $f \nmid D$ is injective by a monodromy theorem (cf. [1, 5.1] and [16, 3.8]). Then by Lemma 3.3 $N(f) = N(f \mid D) = 1$ and hence f is injective.

3.7. Remarks. (1) It seems to be an open problem whether a discrete open closed mapping $f\colon B^n \to fB^n \subset \mathbb{R}^n$ must be injective if $n \geqq 3$ and B_f is compact. Observe that by Lemma 3.6 the answer is in the positive if $fB^n = B^n$ and $n \geqq 3$. For $n=2$ such a result fails to hold as the mapping $g\colon B^2 \to B^2$, $g(z) = z^2$ shows.

(2) Answering to a question of the author Prof. Leon Greenberg has proved (unpublished) that if $f\colon B^n \to fB^n \subset \mathbb{R}^n$, $n \geqq 2$, is discrete open closed and $B_f = \emptyset$, then f is a homeomorphism. The proof makes use of Smith's fixed point theorem for periodic homeomorphisms.

3.8 Remark. V. M. Goldštein has proved in [3] some results related to Lemma 3.6 under the additional assumption that the mapping be quasiregular. From [1, 5.1] and [16, 3.8] it follows that Theorem 3 in [3, p. 1007] holds for general discrete open closed mappings as well.

In the remaining part of this section we shall consider the behavior of a quasiregular mapping $f\colon G \to B^n$, $G \subset B^n$, close to a set $A \subset \partial G$. We begin with the following result concerning continuous extendability of f to A. A related result for boundary-preserving f was proved by Srebro [11, 4.2] (cf. also [16, 4.10]).

3.9. Theorem. *Let G be a domain in B^n and let $A \subset \partial G \cap \partial B^n$ be a non-empty set and suppose that for every $b \in A$ there exists a neighborhood U for which $U \cap \partial G = U \cap A$. Let $f\colon G \to B^n$ be a quasiregular mapping with $C(f, A) \subset \partial B^n$. Then f has a limit at every point of A.*

Proof. Fix $b \in A$. Choose a neighborhood U of b such that $U \cap \partial G = U \cap A$. Since $C(f, A) \cap \bar{B}^n(1/2) = \emptyset$, there is a number $r > 0$ such that $f^{-1}\bar{B}^n(1/2) \cap \bar{B}^n(b,r) = \emptyset$ and $\bar{B}^n(b,r) \subset U$. Then $B^n(b,r) \cap \partial G \subset B^n(b,r) \cap \partial B^n$. Suppose that there is no limit at b. Then there are sequences (a_k) and (b_k) in B^n with $a_k, b_k \in B^n(b, r/k)$, $k = 1, 2, \ldots$ and $f(a_k) \to a'$, $f(b_k) \to b' \neq a'$ as $k \to \infty$ and so that $|f(a_k) - f(b_k)| > |a' - b'|/2$ for all $k = 1, 2, \ldots$. Join a_k and b_k in $B^n(b, r/k) \cap B^n$ by a continuum C_k and denote by Γ_k' the family of all paths $\gamma\colon [0,1) \to B^n$ with $\gamma(0) \in fC_k$ and $\gamma(t) \to \bar{B}^n(1/2)$ as $t \to 1$. By Lemma 2.3 there is a $\delta > 0$ such that $M(\Gamma_k') \geqq \delta$ for all k. Let Γ_k be the family of the maximal liftings of the elements of Γ'_k starting at C_k, $k=1,2,\ldots$. The terminology is as in [7, 3.11]. Since $f^{-1}\bar{B}^n(1/2)\cap\bar{B}^n(b,r)=\varnothing$ and $C(f,A)\subset\partial B^n$ it follows from [7, 3.12] and [14, 7.5] that for $k=2,3,\ldots$

$$M(\Gamma_k) \leqq \omega_{n-1}(\log k)^{1-n} .$$

From [15, 3.1] it follows that for all k

$$0 < \delta \leqq M(\Gamma'_k) \leqq K_I(f)M(\Gamma_k)$$

which for large k contradicts the above upper bound. The proof is complete.

By Lemma 3.3 (4) the next theorem provides us with a new proof for the reflection principle of boundary-preserving quasiregular mappings, which was proved by Martio and Rickman in [4, 5.2]. Our proof is based on Theorem 3.9 and some results in [16], and it is completely different from the proof in [4].

3.10. THEOREM. *Let $G\subset B^n$ and $A\subset\partial G\cap\partial B^n$ be as in Theorem 3.9 and let $f\colon G\to B^n$ be a quasiregular mapping with $C(f,A)\subset\partial B^n$. Then there is a quasimeromorphic mapping $f_1\colon G\cup A\cup\alpha G\to\bar{R}^n$ with $f_1|G=f$, where α is the reflection in S^{n-1}, if and only if $N(f,b)<\infty$ for all $b\in A$.*

PROOF. Observe that $G\cup A\cup\alpha G$ is a domain by assumptions. The necessity of the condition follows from the fact that the maximal multiplicity of a discrete open mapping is finite in a relatively compact set (cf. [5, 2.12]). To prove the sufficiency we observe that by Theorem 3.9 there is a continuous mapping $\bar{f}\colon G\cup A\to\bar{B}^n$, which by reflection (cf. the proofs of [16, 5.10] and [4, 5.2]) can be extended to a continuous mapping $f_1\colon G\cup A\cup\alpha G\to\bar{R}^n$, quasimeromorphic in G and αG. Since $N(f,b)<\infty$ for $b\in A$ it follows from the proof of [16, 4.5] that f_1 is light. Since $f_1 A\subset\partial B^n$ one can show that f_1 is open and sense-preserving (cf. Titus–Young [12]). Thus f_1 is discrete and open by [12, p. 333]. The proof now follows from [16, 2.2].

3.11. REMARK. Let $f_1\colon G_1\to\bar{R}^n$, $G_1=G\cup A\cup\alpha G$, be the (continuous) mapping in the proof of Theorem 3.10, which was obtained from f by reflection. Then a sufficient condition for $N(f,b)<\infty$, $b\in A$, is the requirement that the mapping f_1 be *quasilight* (i.e. the components of $f_1^{-1}(y)$ are compact when $y\in f_1G_1$). One can verify this claim by estimating the multiplicity function in terms of the topological index as in [16, 2.10, 2.16]. Thus the condition $N(f,b)<\infty$, $b\in A$, can fail to hold only if f_1 maps onto a point a connected set $C\subset A$ with $\bar{C}\cap\partial G_1\neq\varnothing$.

Let us continue to examine the situation of Theorem 3.10. According to

Lemma 3.3 (4) the condition $N(f, b)<\infty$ for $b \in A$ is satisfied if $C(f, \partial G)\subset\partial fG$. Since only the behavior of f near A is considered in 3.10, one might expect that the weaker condition $C(f, A)\subset\partial B^n$ would imply $N(f, b)<\infty$ for $b \in A$. Observe that by Väisälä's example 3.5 (2) this is not true for general discrete open mappings, not even if $B_f=\varnothing$. Here we shall show that this actually is the case in the situation of 3.10, under the additional assumption that $B_f=\varnothing$.

3.12. THEOREM. *Let $G\subset B^n$ and $A\subset\partial G\cap\partial B^n$ be as in Theorem 3.9 and let $f\colon G \to B^n$ be a locally homeomorphic quasiregular mapping with $C(f, A)\subset\partial B^n$. For $x \in A$ and $r>0$ let $U(x,r)$ denote the component of $f^{-1}B^n(f(x),r)$ with $x \in \partial U(x,r)$, where $f(x)$ denotes the limit of f at x (cf. Theorem 3.9). Then for each $x \in A$ there is $\sigma_x>0$ such that the mapping $f\,|\,U(x,\sigma_x)\colon U(x,\sigma_x) \to B^n(f(x),\sigma_x)\cap B^n$ is a quasiconformal homeomorphism.*

PROOF. By Theorem 3.9 f has a limit at each point $x \in A$, denoted by $f(x)$. If $x \in A$ and $r>0$, then there exists a $\delta>0$ such that $f(y) \in B^n(f(x),r)\cap \bar{B}^n$ for $y \in B^n(x,\delta)\cap \bar{B}^n$ and hence there exists exactly one component $U(x,r)$ of $f^{-1}B^n(f(x),r)$ with $x \in \partial U(x,r)$.

Fix $x \in A$. We shall now show that there exists a number $\sigma_x>0$ with the desired property.

Choose $r_1>0$ such that $B^n(x,3r_1)\cap\partial G\subset\partial B^n$ and $B^n(x,3r_1)\cap\partial B^n\subset A$ and a path $\alpha\colon [0,1) \to B^n(x,r_1)\cap B^n$ with $\alpha(s) \to x$ when $s \to 1$. By continuity $f(\alpha(s)) \to f(x)$ when $s \to 1$. Let $t_0=|f(\alpha(0))-f(x)|>0$. For $t \in (0,t_0)$ write $C(t) =S^{n-1}(f(x),t)\cap B^n$ and define

$$s(t) = \max\{w \in (0,1) : |f(\alpha(w))-f(x)| = t\}.$$

Denote by ϱ the hyperbolic metric in B^n. For each $t \in (0,t_0)$ let

$$C(t,\varphi) = \{x \in C(t) : \varrho(x, f(\alpha(s(t)))) < \tan\varphi\}, \qquad \varphi \in (0,\pi/2),$$

$$C(t, 0) = \{f(\alpha(s(t)))\},$$

$$C(t,\pi/2) = C(t).$$

Let $C^*(t,\varphi)$ be the $\alpha(s(t))$-component of $f^{-1}C(t,\varphi)$ when $t \in (0,t_0)$ and $\varphi \in [0,\pi/2]$. For $t \in (0,t_0)$ let

$$\varphi_t = \sup\{\varphi \in [0,\pi/2] : f\,|\,C^*(t,\varphi) \text{ is injective}\}.$$

Since f is locally homeomorphic, it is obvious that $\varphi_t>0$ for all $t \in (0,t_0)$.

We shall now show that there are numbers $s(t) \in [0,1)$ arbitrarily close to 1 such that $C^*(t,\varphi_t)\subset B^n\cap\bar{B}^n(x,2r_1)$ and $\varphi_t=\pi/2$.

We show first that if $t \in (0,t_0)$ and $C^*(t,\varphi_t)\subset B^n\cap\bar{B}^n(x,2r_1)$, then $\varphi_t=\pi/2$.

Assume that $\varphi_t \in (0, \pi/2)$. Then $\overline{C^*(t, \varphi_t)}$ is a compact subset of $B^n \cap \bar{B}^n(x, 2r_1)$, since otherwise

$$\overline{C^*(t, \varphi_t)} \cap (\partial B^n \cap \bar{B}^n(x, 2r_1)) \neq \varnothing$$

which yields a contradiction, because $C(f, \partial B^n \cap B^n(x, 3r_1)) \subset \partial B^n$ and $\overline{C(t, \varphi_t)} \subset B^n$ for $\varphi_t \in (0, \pi/2)$. Since $\overline{C^*(t, \varphi_t)}$ is a compact subset of $B^n \cap \bar{B}^n(x, 2r_1)$, it follows from a result of Martio, Rickman, and Väisälä [7, 2.2] that f maps $\overline{C^*(t, \varphi_t)}$ homeomorphically onto $\overline{C(t, \varphi_t)}$. From a result of Zorič [20, p. 422], given also in [7, 3.8], it follows that f is injective in a neighborhood of $\overline{C^*(t, \varphi_t)}$. This fact contradicts the maximality of φ_t. Hence we have shown that if $t \in (0, t_0)$ and $C^*(t, \varphi_t) \subset B^n \cap \bar{B}^n(x, 2r_1)$ then $\varphi_t = \pi/2$.

We shall now show that there are numbers $s(t) \in [0, 1)$, arbitrarily close to 1, such that $C^*(t, \varphi_t) \subset B^n \cap \bar{B}^n(x, 2r_1)$. Suppose that this is not the case. Then there is a number $s_0 \in [0, 1)$ such that if $s(t) \in [s_0, 1)$ then

$$C^*(t, \varphi_t) \cap S^{n-1}(x, 2r_1) \neq \varnothing .$$

For $s(t) \in [s_0, 1)$ fix $x_t \in C^*(t, \varphi_t) \cap S^{n-1}(x, 2r_1)$. There is $\psi_t \in (0, \varphi_t)$ such that $x_t \in C^*(t, \psi_t)$. Write

$$\Gamma_t = \Delta(\{\alpha(s(t))\}, \{x_t\} ; C^*(t, \psi_t)), \; s(t) \in [s_0, 1) ,$$

and

$$\Gamma = \bigcup \{\Gamma_t : s(t) \in [s_0, 1)\} .$$

Since f is injective in $C^*(t, \psi_t)$,

$$f\Gamma_t = \Delta(\{f(\alpha(s(t)))\}, \{f(x_t)\}; C(t, \psi_t)) .$$

By [14, 10.2] there is a constant $b_n > 0$ depending only on n such that for $s(t) \in [s_0, 1)$

$$M(f\Gamma_t) \geqq \frac{b_n}{|f(\alpha(s(t))) - f(x)|} = \frac{b_n}{t} .$$

By integrating with respect to t and by using the modulus inequality [15, 3.1] or [7, 2.14] we get by [14, 7.5]

$$\infty = M(f\Gamma) \leqq K_I(f) M(\Gamma) \leqq K_I(f) \omega_{n-1} (\log 2)^{1-n}$$

The last estimate follows since paths in Γ intersect both $S^{n-1}(x, r_1)$ and $S^{n-1}(x, 2r_1)$. This contradiction shows that there are numbers $s(t) \in (0, 1)$ arbitrarily close to 1 such that $C^*(t, \varphi_t) \subset B^n \cap \bar{B}^n(x, 2r_1)$.

For each $k = 1, 2, \ldots$ let $s_k \in (0, 1)$ be such that

$$\alpha(s_k, 1) \subset B^n(x, 2^{-k} r_1) \cap B^n .$$

Reasoning as above one can prove that for each k there are numbers $s(t) \in (s_k, 1)$, arbitrarily close to 1, such that

$$C^*(t, \varphi_t) \subset B^n \cap \bar{B}^n(x, 2^{-k+1} r_1)$$

and $\varphi_t = \pi/2$. This observation, together with a diagonal argument, shows that there is a sequence (t_k) in $(0, t_0)$ with $\lim t_k = 0$ and $t_{k+1} \in (0, t_k)$ for all k such that $\varphi_{t_k} = \pi/2$ and $C^*(t_k, \varphi_k) \to x$ when $k \to \infty$.

Let $t = t_k$ for some $k = 1, 2, \ldots$. Then $\varphi_t = \pi/2$. For $\varphi \in (0, \pi/2)$ let

$$A_\varphi = \{d(y, \partial B^n) : y \in \overline{C(t, \varphi)} \setminus C(t, \varphi)\} .$$

Because $\sup A_\varphi \to 0$ as $\varphi \to \pi/2$ and because f is open

$$\sup\{d(z, \partial B^n) : z \in \overline{C^*(t, \varphi)} \setminus C^*(t, \varphi)\} \to 0 \tag{3.13}$$

when $\varphi \to \pi/2$. (Otherwise there exist a number $u \in (0, 1)$, a sequence (φ_j) in $(0, \pi/2)$ with $\varphi_j \to \pi/2$ and points $x_j \in \overline{C^*(t, \varphi_j)} \setminus C^*(t, \varphi_j)$ with $|x_j| \leqq u$ for all $j = 1, 2, \ldots$. Choose a subsequence, denoted again by (x_j), with $x_j \to x_0 \in \bar{B}^n(u)$. Since $f(x_j) \to \partial B^n$ it follows that $f(x_0) \in \partial B^n$, which is impossible because f is open.) Since for each φ, $0 < \varphi < \varphi_t = \pi/2$, $C^*(t, \varphi)$ is topologically equivalent to $C(t, \varphi)$, there is by (3.13) a component $D = D_k$ of $B^n \setminus C^*(t, \varphi_t)$ such that $D \subset B^n(x, 2^{-k+1} r_1)$ where k is determined by $t = t_k$. We shall now show that the mapping $f \mid D_k$: $D_k \to fD_k$ is closed and $fD_k = A_k$ except for finitely many k, where $A_k = B^n \cap B^n(f(x), t_k)$. From 3.2 (1) and from the definition of D_k it follows that $\partial f D_k \subset f \partial D_k \subset C(t_k) \cup \partial B^n$. Suppose that $B^n \setminus A_k \subset fD_k$ for infinitely many k. Together with the above inclusions this implies that $C(f, x) = \bar{B}^n$, since $D_k \to x$. This is not possible, since f has a limit at x. Hence the above inclusions show that $fD_k = A_k$ for all large k, say for $k \geqq k_0$. By Lemma 3.3 $f \mid D_k$ is closed for $k \geqq k_0$.

Fix $k \geqq k_0$. Choose $\varrho_k > 0$ such that the closure of $f(\bar{B}^n \cap \bar{B}^n(x, \varrho_k))$ is a subset of $\bar{A}_k$. Let U_k be the unique component of $f^{-1} A_k$ with $x \in \partial U_k$ and $B^n \cap B^n(x, \varrho_k) \subset U_k$.

Suppose that $U_k \neq D_k$. Fix $v > s(t_k)$ with $\alpha(v) \in B^n \cap B^n(x, \varrho_k)$. Let β: $[0, 1) \to A_k$ be the path

$$\beta(w) = f\big(\alpha(v - (v - s(t_k))w)\big), \qquad w \in [0, 1) .$$

Since f is a local homeomorphism, there is by [7, 3.12] a unique maximal $(f \mid U_k)$-lifting of β starting at $\alpha(v)$, let it be γ: $[0, c) \to U_k \cap |\alpha|$. According to [7, 3.12] $\gamma(w) \to f^{-1} C(t_k)$ when $w \to c$. From the definition of $s(t_k)$ it follows that $\gamma(w) \to C^*(t_k, \pi/2)$ when $w \to c$ and that $c = 1$. Then $\gamma(w) \to x_0 = \alpha(s(t_k))$ when $w \to 1$. Since $fD_k = A_k$ and U_k is a component of $f^{-1} A_k$, it follows that $N(f, B^n(x_0, \varepsilon)) \geqq 2$ for all $\varepsilon > 0$, which is a contradiction, since f is a local homeomorphism.

Hence $U_k = D_k$. Since $f \mid D_k$ is closed, $B_f = \emptyset$, and $fD_k = A_k$ is simply connected, it follows from a monodromy theorem (cf. [1, 5.1] and [16, 3.8]) that f is injective in D_k. We have proved the theorem with $\sigma_x = t_k$.

3.14. REMARKS. (1) Let $f\colon G \to B^n$, $n \geqq 3$, be a quasiregular mapping as in Theorem 3.9. It is an open problem, whether $C(f, A) \subset \partial B^n$ implies $N(f, b) < \infty$ for $b \in A$ if $B_f \neq \emptyset$. Observe that by Theorem 3.12 this is the case if $B_f = \emptyset$.

(2) For analytic functions in the plane results stronger than Theorem 3.12 are known (cf. Collingwood–Lohwater [2, p. 94]).

Theorems 3.12 and 3.10 together yield

3.15. COROLLARY. *Let $G \subset B^n$ and $A \subset \partial G \cap \partial B^n$ be as in Theorem* 3.9 *and let $f\colon G \to B^n$ be a locally homeomorphic quasiregular mapping with $C(f, A) \subset \partial B^n$. Then there exists a quasimeromorphic mapping $f_1\colon G \cup A \cup \alpha G \to \bar{\mathsf{R}}^n$ with $f_1 \mid G = f$ where α is the reflection in S^{n-1}.*

4. Almost boundary-preserving quasiregular mappings.

It was shown in [16, Ch. II] that many boundary properties of quasiconformal mappings have their natural generalizations for boundary-preserving quasiregular mappings. Among the results proved in [16, Section 4] are those concerning the possibility of extending a boundary-preserving quasiregular mapping of B^n continuously to ∂B^n. Theorem 3.9 of the previous section can be regarded as a local counterpart of Theorem 4.10 in [16], since in 3.9 one assumes that the mapping in question is boundary-preserving only on a part of the boundary. In the same manner Theorem 3.12 corresponds to Theorem 4.7 in [16] under the additional assumption that $B_f = \emptyset$.

In this section we shall complete the above mentioned results by studying the behavior of a quasiregular mapping of B^n which is boundary-preserving at the points of $\partial B^n \setminus E$, where $E \subset \partial B^n$ is a small set. We introduce the following definition.

4.1. DEFINITION. Let G' be a domain in R^n and let $f\colon B^n \to G'$ be a nonconstant quasiregular mapping. If there exists a compact set $E_f \subset \partial B^n$ of capacity zero such that $C(f, \partial B^n \setminus E_f) \subset \partial G'$ and $C(f, b) \not\subset \partial G'$ for $b \in E_f$, then f is said to be *almost boundary-preserving*. The (possibly empty) set E_f is called the exceptional set of f.

We now give examples of almost boundary-preserving quasiregular mappings.

4.2. EXAMPLES. (1) Let $g: B^2 \to B^2$ be the analytic function

$$g(z) = \exp\left(\frac{z+1}{z-1}\right), \quad z \in B^2 .$$

Then $C(g, \partial B^2 \setminus \{e_1\}) \subset \partial B^2$ and we see that g is almost boundary-preserving with $E_g = \{e_1\}$.

(2) By slightly modifying the function of Zorič [20] we get a quasiregular mapping $f: B^3 \to B^3$ such that $C(f, \partial B^3 \setminus \{e_1\}) \subset \partial B^3$, $fB^3 = B^3 \setminus \{0\}$, $N(y, f, B^n) = \infty$ for every $y \in B^3 \setminus \{0\}$ and such that $B_f \neq \emptyset$ and $e_1 \in \bar{B}_f$. In many respects, this function resembles the above function g. An n-dimensional version of this example was given by Martio and Srebro in [8, 8.1].

(3) From the example of Martio and Srebro in [9, 4.1] we get a locally homeomorphic quasiregular almost boundary-preserving mapping $f: B^3 \to T$, where $T = fB^3 \subset R^3$ is an open solid torus. In this example the exceptional set consists of two points.

The next result follows from Theorem 3.9 and 3.12.

4.3. COROLLARY. *Let $f: B^n \to B^n$ be an almost boundary-preserving quasiregular mapping. Then f has a limit at every point of $\partial B^n \setminus E_f$. If $B_f = \emptyset$, then there is a quasimeromorphic mapping $f_1: \bar{R}^n \setminus E_f \to \bar{R}^n$ with $f_1 | B^n = f$.*

The following result gives information about the behavior of an almost boundary-preserving quasiregular mapping near the exceptional set. For results related to Theorem 4.4 in the case of analytic functions the reader is referred to [2, pp. 107–109] and to [10, III § 1, pp. 116–120].

4.4. THEOREM. *Let $f: B^n \to B^n$ be an almost boundary-preserving quasiregular mapping and let $b \in E_f$. Then f assumes in every neighborhood of b infinitely many times every value of B^n except for a possible subset of B^n of capacity zero. In particular, $C(f, b) = \bar{B}^n$ and $C_{\partial B^n \setminus E_f}(f, b) = \partial B^n$.*

PROOF. Since $C(f, \partial B^n \setminus E_f) \subset \partial B^n$ it follows that $C_{\partial B^n \setminus E_f}(f, b) \subset \partial B^n$. From the Iversen–Tsuji theorem for quasiregular mappings it follows that $\partial C(f, b) \subset \partial C_{\partial B^n \setminus E_f}(f, b)$ (see [4, 3.10]). Since $C(f, b) \subset \bar{B}^n$ that inclusion implies that $C(f, b) = \bar{B}^n$ and $C_{\partial B^n \setminus E_f}(f, b) = \partial B^n$. The proof follows from Theorem 2.4.

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REFERENCES

1. P. T. Church and E. Hemmingsen, *Light open maps on n-manifolds*, Duke Math. J. 27 (1960), 527–536.
2. E. F. Collingwood and A. J. Lohwater, *The theory of cluster sets*, Cambridge University Press, Cambridge, 1966.
3. V. M. Goldštein, *A certain homotopy property of mappings with bounded distortion* (Russian), Sibirsk. Mat. Ž. 11 (1970), 999–1008.
4. O. Martio and S. Rickman, *Boundary behavior of quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I 507 (1972), 1–17.
5. O. Martio, S. Rickman and J. Väisälä, *Definitions for quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I 448 (1969), 1–40.
6. O. Martio, S. Rickman and J. Väisälä, *Distortion and singularities of quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I 465 (1970), 1–13.
7. O. Martio, S. Rickman and J. Väisälä, *Topological and metric properties of quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. AI 488 (1971), 1–31.
8. O. Martio and U. Srebro, *Periodic quasimeromorphic mappings*, J. Analyse Math. 28 (1975), 20–40.
9. O. Martio and U. Srebro, *Automorphic quasimeromorphic mappings in* $\mathbf{R}^n$, Acta Math. 135 (1975), 221–247.
10. K. Noshiro, *Cluster sets* (Ergebnisse der Mathematik und ihrer Grenzgebiete 28), Springer-Verlag, Berlin - Göttingen - Heidelberg, 1960.
11. U. Srebro, *Conformal capacity and quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I 529 (1973), 1–8.
12. C. J. Titus and G. S. Young, *The extension of interiority, with some applications*, Trans. Amer. Math. Soc. 103 (1962), 329–340.
13. J. Väisälä, *Discrete open mappings on n-manifolds*, Ann. Acad. Sci. Fenn. Ser. A I 392 (1966), 1–10.
14. J. Väisälä, *Lectures on n-dimensional quasiconformal mappings* (Lecture Notes in Mathematics 229), Springer-Verlag, Berlin - Heidelberg - New York, 1971.
15. J. Väisälä, *Modulus and capacity inequalities for quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I 509 (1972), 1–14.
16. M. Vuorinen, *Exceptional sets and boundary behavior of quasiregular mappings in n-space*, Ann. Acad. Sci. Fenn. Ser. A I, Math. Dissertationes 11 (1976), 1–44.
17. M. Vuorinen, *On the Iversen–Tsuji theorem for quasiregular mappings*, Math. Scand. 41 (1977), 90–98.
18. M. Vuorinen, *On the existence of angular limits of n-dimensional quasiconformal mappings*, Ark. Mat. (To appear).
19. W. P. Ziemer, *Extremal length as a capacity*, Michigan Math. J. 17 (1970), 117–128.
20. V. A. Zorič, *A theorem of M. A. Lavrentiev on quasiconformal mappings of space* (Russian), Mat. Sb. 74 (1967), 417–433.

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