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CROSSED PRODUCTS OF C*-ALGEBRAS BY APPROXIMATELY UNIFORMLY CONTINUOUS ACTIONS

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Abstract.

Using a generalized concept of ground state we show that if a connected abelian group G acts on a unital C*-algebra A in an approximately uniformly continuous manner then the crossed product between G and A is never a simple C*-algebra.

1. Introduction.

Recently a certain amount of progress has been made on the problem of deciding when a crossed product between a C*-algebra A and a locally compact abelian group G of automorphisms of A is simple, i.e. has no non-trivial closed ideals; see e.g. [2], [3], [4], [9] and [11]. Even though the ultimate goal is to describe completely the ideal structure of the crossed product (the inevitable Mackey-Machine providing the industrial power) it may perhaps be worthwhile first to analyse in detail the simple problem above, where more bucolic arguments can be applied to good results. Furthermore, the crossed product construction is at the moment our best source of examples of simple C*-algebras, and simple C*-algebras are in high demand both in the theory and in its applications.

When G is discrete, the problem above has a complete solution in terms of the Connes spectrum $\Gamma(\alpha)$ of the representation $\alpha\colon G \to \mathrm{Aut}(A)$, cf. [9, 6.5]. In the general case only partial results are available, such as A being of type I or $G = \mathbf{R}$ with A non-primitive ([2], [11]). That the conditions for the discrete case do not suffice was first demonstrated by Bratteli [1], who showed that the gauge group (isomorphic to the circle) on the Fermion algebra does not give a simple crossed product. We shall deduce Bratteli's result from the observation (Corollary 2.9) that no approximately inner action of $\mathbf{R}$ or $\mathbf{T}$ on a C*-algebra with unit can give a simple crossed product. Since such actions of $\mathbf{R}$ and $\mathbf{T}$ are

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the only ones known to exist on the Fermion algebra, our result is admittedly a little depressing.

We shall make free use of the Arveson–Olesen theory of spectral subspaces as developed in [12, Chapter 8].

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2. Generalized ground states.

2.1. Let (A, G, α) be a C*-dynamical system where G is a metrizable, locally compact abelian group, and fix a non-empty open set Ω in the dual group $\hat{G}$ of G.

A state φ of A is said to be a *ground state for* Ω if it annihilates the left ideal of A generated by $R^{\alpha}(\Omega)$, i.e. $\varphi(y\alpha_f(x))=0$ for all x, y in A and all f in $L^1(G)$ such that $\hat{f}$ has (compact) support in Ω. Since $\varphi=\varphi^* \geqq 0$ this is equivalent to the demand that φ annihilates the right ideal $R^{\alpha}(-\Omega)A$ or annihilates the hereditary *-algebra $R^{\alpha}(-\Omega)AR^{\alpha}(\Omega)$. Taking $G=\mathsf{R}$ and $\Omega=]-\infty,0[$ our definition describes the usual ground states in C*-physics (cf. [13, Definition 2.1], [10, Remark C] and [12, 8.12.5]).

2.2. If φ is a ground state for Ω it annihilates $R^{\alpha}(\Omega)$. Thus by definition $\varphi \in M^{\alpha'}(\hat{G}\setminus \Omega)$. Therefore by [12, 8.1.4]

$$\varphi = \varphi^* \in M^{\alpha'}(\hat{G}\setminus \Omega) \cap M^{\alpha'}(\hat{G}\setminus (-\Omega)) = M^{\alpha'}(\hat{G}\setminus (\Omega \cup (-\Omega))) .$$

If $\Omega \cup (-\Omega)=\hat{G}\setminus \{0\}$, which is the case for ordinary ground states, then $\varphi \in M^{\alpha'}(\{0\})$, so that φ is G-invariant. In the more general case a ground state for Ω need not be G-invariant. However, if $1 \in A$, so that the state space is compact, we can choose an invariant mean m on G (abelian groups are amenable) and define

$$\tilde{\varphi} = m\{t \to \varphi(\alpha_t(\cdot))\} .$$

Since $R^{\alpha}(\Omega)$ is G-invariant each $\varphi(\alpha_t(\cdot))$ is a ground state so that $\tilde{\varphi}$ is a G-invariant ground state for Ω.

2.3. Proposition. *Let (A, G, α) be a* C*-*dynamical system and let φ be a G-invariant ground state for some Ω in $\hat{G}$. Then the covariant representation of (A, G, α) associated with φ via the* GNS-*construction does not give a faithful representation of the crossed product* C*-*algebra $G \times_{\alpha} A$.*

Proof. Let (π, u, H) denote the covariant representation associated with φ (cf. [12, 7.4.12]). The corresponding representation $(\pi \times u, H)$ of $G \times_{\alpha} A$ is defined by

$$(\pi \times u)(y) = \int \pi(y(t))u_t\,dt$$

for each y in $L^1(G, A)$, see [12, 7.6.4]. Take $y = x \otimes f$ (i.e. $y(t) = xf(t)$) for some x in A and some f in $L^1(G)$ such that $\hat{f}$ has compact support in Ω. Clearly $y \neq 0$ in $L^1(G, A)$ if $x \neq 0$ and $f \neq 0$, and thus $y \neq 0$ in the C*-completion $G \times_\alpha A$ of $L^1(G, A)$. However, for all a, b in A we get

$$((\pi \times u)(y)\xi_a \mid \xi_b) = \int (\pi(y(t))u_t\xi_a \mid \xi_b)\,dt$$

$$= \int \varphi(b^*y(t)\alpha_t(a))\,dt = \int \varphi(b^*x\alpha_t(a))f(t)\,dt$$

$$= \varphi(b^*x\alpha_f(a)) = 0\,,$$

since φ annihilates the left ideal generated by $R^\alpha(\Omega)$. Since this holds for the dense set of vectors in H of the form ξ_a, ξ_b we conclude that $y \in \ker(\pi \times u)$.

2.4. PROPOSITION. *Let (A, G, α) be a C*-dynamical system where $1 \in A$, and fix $\Omega \subset \hat{G}$. Then the closed left ideal of A generated by $R^\alpha(\Omega)$ is proper if and only if the same is true for the closed ideal I_Ω of $G \times_\alpha A$ generated by elements of the form $x \otimes f$, $x \in A$, $f \in L^1(G)$ where $\hat{f}$ has compact support in Ω.*

PROOF. Let L_Ω denote the closure of $AR^\alpha(\Omega)$ and assume that $L_\Omega \neq A$. There is then a state φ of A that annihilates L_Ω (or, equivalently, annihilates $L_\Omega^* \cap L_\Omega$, cf. [12, 3.10.7]). Thus φ is a ground state for Ω, and since $1 \in A$ we may apply an invariant mean to φ to obtain a G-invariant ground state, cf. 2.2. By Proposition 2.3 the closed ideal I_Ω of $G \times_\alpha A$ generated by elements $x \otimes f$, $x \in A$, $f \in L^1(G)$ where $\hat{f}$ has compact support in Ω is contained in $\ker(\pi \times u)$ and thus $I_\Omega \neq G \times_\alpha A$.

Conversely, assume that $I_\Omega \neq G \times_\alpha A$ and choose a covariant representation (π, u, H) of (A, G, α) such that $\ker(\pi \times u) = I_\Omega$, see [12, 7.6.4]. Take a G-invariant state φ of $\pi(A)$ and extend it (uniquely) to a state of $(\pi \times u)(G \times_\alpha A)$ by

$$\varphi((\pi \times u)(z)) = \int \varphi(\pi(z(t)))\,dt$$

for every z in $L^1(G, A)$. If now $x, y \in A$ and $f \in L^1(G)$ such that $\hat{f}$ has compact support in Ω then with $z(t) = xf(t)\alpha_t(y)$ we have (identifying y with $y \otimes \delta_0$ in $M(L^1(G, A))$, cf. [12, 7.6.3])

$$\varphi(\pi(x\alpha_f(y))) = \varphi((\pi \times u)(z))$$

$$= \varphi((\pi \times u)(x \otimes f)(\pi \times u)(y \otimes \delta_0)) = 0\,,$$

since $x \otimes f \in I_\Omega$. Consequently $\varphi \circ \pi$ is a ground state for Ω so that $L_\Omega \neq A$.

2.5. Inspired by a condition used by Kraus in [5] we define a semi-group Ω in $\hat{G}$ (i.e. $\Omega+\Omega\subset\Omega$) to be *inner coset free* if its closure contains no set of the form $\{\sigma+\mathbf{Z}\tau\}$ with τ in Ω. Necessarily then $0\notin\Omega$. If Ω is maximal positive in a discrete group $\hat{G}$ (i.e. $-\Omega\cap\Omega=\{0\}$ and $-\Omega\cup\Omega=\hat{G}$) then Ω is coset free if and only if $\Omega\setminus\{0\}$ is inner coset free. As pointed out in [5, 6.7] this happens if and only if the order in $\hat{G}$ induced by Ω is Archimedian.

2.6. LEMMA. *Let (A, G, α) be a C*-dynamical system and assume that α is uniformly continuous. If Ω is a non-empty, open, inner coset free semi-group in $\hat{G}$ then there is a ground state for $\Omega+\Omega$.*

PROOF. Take $\tau_1,\ldots,\tau_n$ in Ω and for m in $\mathbf{N}$ consider the sets

$$\Lambda_m = \bigcup_{k=1}^{n} (m\tau_k+\bar{\Omega})\cap \mathrm{Sp}(\alpha)\,.$$

Since $\bar{\Omega}$ is a semi-group, $\Lambda_{m+1}\subset\Lambda_m$. Suppose that $\sigma\in\bigcap\Lambda_m$. Then

$$\sigma\in\bigcap_m (m\tau_k+\bar{\Omega})\cap\mathrm{Sp}(\alpha)$$

for some $k\leqq n$. Thus $\sigma-\mathbf{N}\tau_k\subset\bar{\Omega}$. But since $\mathbf{N}\tau_k\subset\Omega$ this implies that $\sigma+\mathbf{Z}\tau_k\subset\bar{\Omega}$ in contradiction with $\tau_k\in\Omega$ and Ω being inner coset free. Consequently $\bigcap\Lambda_m=\emptyset$. However, since α is uniformly continuous $\mathrm{Sp}(\alpha)$ is compact by [7, 2.1] or [12, 8.1.12], and thus we conclude that $\Lambda_m=\emptyset$ for some m.

We claim that $AR^\alpha(\bigcup(\tau_k+\Omega))$ is not dense in A. If it were then since by [12, 8.3.3]

$$AR^\alpha(\bigcup(\tau_k+\Omega))R^\alpha(\bigcup(\tau_k+\Omega)) \subset AR^\alpha(\bigcup(\tau_k+\tau_j+\Omega))\,,$$

we conclude that the latter set is also dense in A. Repeating this argument mn times shows that $AR^\alpha(\bigcup(m\tau_k+\Omega))$ is dense in A. As $\bigcup(m\tau_k+\Omega)\cap\mathrm{Sp}(\alpha)=\emptyset$ for m sufficiently large, whence $R^\alpha(\bigcup(m\tau_k+\Omega))=0$, we have reached a contradiction. Thus the closure L_n of $AR^\alpha(\bigcup(\tau_k+\Omega))$ is a proper closed left ideal of A.

Now let $\{\tau_k\}$ be a dense sequence in Ω. Since Ω is open this implies that $\Omega+\Omega=\bigcup(\tau_k+\Omega)$. Thus

$$AR^\alpha(\Omega+\Omega)\subset\left(\bigcup_n AR^\alpha\left(\bigcup_1^n(\tau_k+\Omega)\right)\right)\subset\left(\bigcup_n L_n\right)\,.$$

Since $L_n\subset L_{n+1}$ and each L_n has empty interior (being a proper subspace) we see from the Baire category theorem that the closure L of $AR^\alpha(\Omega+\Omega)$ is a proper left ideal of A. But then there is a state φ of A that annihilates L (or, equivalently, annihilates $L\cap L^*$, cf. [12, 3.10.7]), and φ is therefore a ground state for $\Omega+\Omega$.

2.7. If (A, G, α) is a C*-dynamical system we say that α is *approximately uniformly continuous* if there is an increasing net $\{A_\lambda\}$ of C*-subalgebras of A with dense union, together with a net $\{\alpha^\lambda\}$ of uniformly continuous representations $\alpha^\lambda: G \to \operatorname{Aut}(A_\lambda)$ such that $\|\alpha_t(x) - \alpha_t^\lambda(x)\| \to 0$ for each x in $A_\mu (\mu < \lambda)$ uniformly on compact subsets of G.

This concept of approximate uniform continuity is slightly weaker than the one used in [10] (but it is not hard to check that even so, Theorem 1′ of [10] is still valid with the new concept). The advantage is that in many examples there is an increasing net $\{A_\lambda\}$ of G-invariant C*-subalgebras of A with dense union, such that $\alpha \mid A_\lambda$ is uniformly continuous for every λ (the A_λ's may be finite-dimensional). These examples are approximately uniformly continuous in our terminology.

As seen from the proof of [12, 8.12.7] approximate uniform continuity of a representation $\alpha: \mathbf{R} \to \operatorname{Aut}(A)$ is the same as the well-known condition of approximate innerness.

2.8. THEOREM. *Let (A, G, α) be a C*-dynamical system where $1 \in A$ and α is approximately uniformly continuous. If $\hat{G}$ contains a non-empty, open, inner coset free semi-group Ω then the crossed product $G \times_\alpha A$ is not simple.*

PROOF. Let $\{(A_\lambda, G, \alpha^\lambda)\}$ be a net of C*-dynamical systems which approximate (A, G, α) as prescribed in 2.7. By Lemma 2.6 there is for each λ a state φ_λ of A such that $\varphi_\lambda \mid A_\lambda$ is a ground state for $\Omega + \Omega$ relative to the system $(A_\lambda, G, \alpha^\lambda)$.

We now argue as in [13, 2.3]. Since $1 \in A$ the state space of A is compact so that, passing if necessary to a subnet, we may assume that $\varphi_\lambda \to \varphi$, weak*, for some state φ of A. Take x, y in A_μ and f in $L^1(G)$ such that $\hat{f}$ has compact support in $\Omega + \Omega$. Since $\alpha_t^\lambda(x) \to \alpha_t(x)$ (for $\lambda > \mu$) uniformly on compact subsets of G we see that $\alpha_f^\lambda(x) \to \alpha_f(x)$ in norm. As $\varphi_\lambda \to \varphi$, weak*, we conclude that

$$0 = \varphi_\lambda(y\alpha_f^\lambda(x)) \to \varphi(y\alpha_f(x)) .$$

Since $\bigcup A_\mu$ is dense in A this shows that φ is a ground state for $\Omega + \Omega$ (relative to the system (A, G, α)).

Applying an invariant mean to φ we may assume that φ is G-invariant (cf. 2.2), and thus Proposition 2.3 applies to show that $G \times_\alpha A$ is not simple.

2.9. COROLLARY. *If (A, G, α) is a C*-dynamical system where $1 \in A$, G is not totally disconnected and α is approximately uniformly continuous then $G \times_\alpha A$ is not simple.*

PROOF. Let G_0 be the connected component of 0 in G. By assumption G_0 is a non-zero, closed subgroup of G. Applying van Kampen's theorem we have G_0

$=\mathsf{R}^n \times H$, where $n \geqq 0$ and H is a compact connected group. Thus $\hat{G}_0 = \hat{\mathsf{R}}^n \times \hat{H}$ where $\hat{H}$ is discrete and torsionfree. It is elementary to check that $\Omega_1 = (]0, \infty[)^n$ is an open coset free semi-group in $\hat{\mathsf{R}}^n$. It is equally simple to show that if $\tau \in \hat{H} \setminus \{0\}$ (assuming that $H \neq 0$) then $\Omega_2 = \mathsf{N}\tau$ is a coset free semi-group in $\hat{H}$. Consequently $\Omega_0 = \Omega_1 + \Omega_2$ is a non-empty, open coset free semi-group in $\hat{G}_0$. Let $\Omega = q^{-1}(\Omega_0)$, where $q \colon \hat{G} \to \hat{G}_0$ is the quotient map with kernel $G_0^\perp$. If $\sigma + \mathsf{Z}\tau \subset \bar{\Omega}$ for some τ in Ω then $q(\sigma) + \mathsf{Z}q(\tau) \subset \bar{\Omega}_0$ and $q(\tau) \in \Omega_0$. Since Ω_0 is coset free this implies that $\tau \in \ker q$, whence $0 = q(\tau) \in \Omega_0$, a contradiction. Thus Ω is inner coset free in $\hat{G}$ and $G \times_\alpha A$ is non-simple by Theorem 2.8.

2.10. Remark. The condition of approximate uniform continuity in 2.9 can not be deleted. Indeed, in [10] a C*-dynamical system $(\mathcal{O}_n, \mathsf{T}, \alpha)$ is exhibited ($\mathcal{O}_n$ being Cuntz's C*-algebra generated by n isometries) such that the fixed-point algebra in $\mathcal{O}_n$ under α is $\mathcal{F}_n$—the infinite C*-tensor product of $n \times n$-matrices. Since T is compact and $\mathcal{F}_n$ is simple with unit, $\mathsf{T} \times_\alpha \mathcal{O}_n$ is stably isomorphic to $\mathcal{F}_n$ by [4, Theorem 2], and consequently simple.

In [10] there is also a system $(\mathcal{O}_\infty, \mathsf{T}, \alpha)$ which has a ground state without α being approximately inner. The condition of approximate uniform continuity is therefore not necessary for the crossed product to be non-simple.

2.11. Remark. The condition that G is not totally disconnected in 2.9. can not be deleted. For in [1, Proposition 3.4] there is an example of a C*-dynamical system $(\mathcal{F}, \mathsf{D}_2, \alpha)$ where $\mathcal{F}$ is the Fermion algebra, D_2 is the Cantor group $\prod_1^\infty \mathsf{Z}_2$ and α implements D_2 as product type automorphisms (whence α is approximately uniformly continuous). Nevertheless $\mathsf{D}_2 \times_\alpha \mathcal{F}$ is simple.

For good measure there is also an example $(B, \mathsf{D}_2, \alpha)$ in [1, Proposition 3.4] where B is a Glimm algebra (UHF), α is approximately uniformly continuous and $\mathsf{D}_2 \times_\alpha B$ is non-simple.

2.12. Remark. The result in [6, Theorem 4] gives a large class of one-parameter groups of quasi-free automorphisms of the Fermion algebra which are approximately inner. By 2.9 the corresponding crossed products are not simple.

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