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THE FAILURE OF THE TOPOLOGICAL FROBENIUS PROPERTY FOR NILPOTENT LIE GROUPS

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The object of this paper is to demonstrate the failure of Fell's topological (or weak) Frobenius reciprocity property — usually called (FP) — for all connected and simply-connected non-abelian nilpotent Lie groups. Moscovici ([7]) has shown that if G is such a group and if H is a closed connected normal subgroup then the pair (G,H) satisfies (FP) (the definition of which is given below). This paper grew out of the desire to investigate the more general case in which H is not required to be normal in G. It will be shown that if H is a closed connected subgroup of G (where G is as above) and if the pair (G,H) satisfies (FP) then H must in fact be normal in G. The proof of this will be by way of the Kirillov correspondence which relates the unitary representations of G to the orbits of the co-adjoint representation of G and whose effect here is to reduce problems concerning (FP) to ones concerning Lie algebras.

Consider a locally compact group G and a closed subgroup H of G. The pair (G,H) is said to satisfy (FP) if, for any two continuous irreducible unitary representations π and ϱ of G and H, resp., it is the case that ϱ is weakly contained in $\pi|_H$, the restriction of π to H, if and only if π is weakly contained in $\mathrm{ind}_H^G\,\varrho$, the representation of G obtained by inducing ϱ up from H to G; and G itself is said to satisfy (FP) if (G,K) satisfies (FP) for all closed subgroups K of G (this definition is due to Fell — see [3; p. 442]). It is not hard to see that G satisfies (FP) if it is either compact or abelian (see [3; Theorem 5.1]) and that G is amenable if and only if $(G,\{e\})$ satisfies (FP). In general, however, the task of deciding whether a given pair (G,H) satisfies (FP) is a difficult one. Gootman ([4]) has developed some criteria for when (G,H) satisfies (FP) under the assumptions that G be second countable and that H be normal in G (in fact, the theorem of Moscovici referred to above could be deduced from these criteria), and Henrichs ([5]) has obtained a necessary and sufficient condition that a discrete group satisfies (FP). The main result of the present paper can now be stated:

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THEOREM *Let G be a connected and simply-connected nilpotent Lie group, and suppose that H is a closed connected subgroup of G which is not normal in G. Then there is a continuous irreducible unitary representation* π *of G such that the trivial representation of H is not weakly contained in* $\pi|_H$ *and yet* π *is weakly contained in the quasi-regular representation of G on* $L^2(G/H)$.

COROLLARY. *If G is as in the theorem and if H is a closed connected subgroup of G then the pair* (G, H) *satisfies* (FP) *if and only if H is normal in G, and G itself satisfies* (FP) *if and only if it is abelian.*

One negative consequence of this corollary is that it is possible for the semi-direct product of two vector groups to fail to satisfy (FP)—cf. the Heisenberg group. The corollary, of course, is an immediate consequence of the theorem preceeding it together with the theorem of Moscovici mentioned earlier, and the proof of the above theorem can be reduced to a problem concerning Lie algebras. To describe this reduction, let G and H be as in the corollary, let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathfrak{h}$ be the subalgebra of $\mathfrak{g}$ corresponding to H. Then one knows that H, too, is simply-connected. For any element f [resp., g] in $\mathfrak{g}'$ [resp., $\mathfrak{h}'$], let$\Omega_G(f)$ [resp., $\Omega_H(g)$] denote the orbit of f [resp., g] under the co-adjoint representation of G on $\mathfrak{g}'$ [resp., of H on $\mathfrak{h}'$]. Next, let p denote the restriction mapping of $\mathfrak{g}'$ onto $\mathfrak{h}'$ and put

$$P(\mathfrak{g};\, \mathfrak{h}) = \{(f, g) \in \mathfrak{g}' \times \mathfrak{h}' :\ g \in p(\Omega_G(f))\}\,.$$

Finally, let π [resp., ϱ] be a continuous irreducible unitary representation of G [resp., H], let f [resp., g] be an element in $\mathfrak{g}'$ [resp., $\mathfrak{h}'$] such that $\Omega_G(f)$ [resp., $\Omega_H(g)$] is the orbit corresponding to π [resp., ϱ] under the Kirillov correspondence (cf. [6] or [8]). Then one knows from the work of Moscovici and Brown ([7; Propositions 1 and 2] and [1], resp.) that ϱ is weakly contained in $\pi|_H$ [resp., π is weakly contained in $\mathrm{ind}_H^G\, \varrho$] if and only if g [resp., f] is contained in the closure $Q(\mathfrak{g}, f;\, \mathfrak{h})$ [resp., $Q(\mathfrak{g};\, \mathfrak{h}, g)$] of the image of the set $(\{f\} \times \mathfrak{h}') \cap P(\mathfrak{g};\, \mathfrak{h})$ [resp., $(\mathfrak{g}' \times \{g\}) \cap P(\mathfrak{g};\, \mathfrak{h})$] under the projection of $\mathfrak{g}' \times \mathfrak{h}'$ onto its second [resp., first] factor. It is not hard to see that if H is normal in G then both of these images are already closed, and hence the pair (G, H) satisfies (FP) (cf. [7; proof of Theorem 1]). In order to prove the above theorem, then, it will be sufficient to establish the following result:

PROPOSITION. *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra, and suppose that* $\mathfrak{h}$ *is a subalgebra of* $\mathfrak{g}$ *which is not an ideal in* $\mathfrak{g}$. *Then there is an element f of* $\mathfrak{g}'$ *satisfying* $f \in Q(\mathfrak{g};\, \mathfrak{h}, 0)$ *and* $0 \notin Q(\mathfrak{g}, f;\, \mathfrak{h})$, *and, moreover, f may be chosen so that its kernel contains any given one-dimensional subspace of the center of* $\mathfrak{g}$.

The remainder of this paper will be devoted to the proof of this proposition. Accordingly, let $\mathfrak{g}$ and $\mathfrak{h}$ be as in the proposition, let $\mathfrak{z}$ be the center of $\mathfrak{g}$, and let G be a connected and simply-connected Lie group whose Lie algebra is isomorphic to $\mathfrak{g}$.

It will be convenient to begin the proof by considering the special case in which $\mathfrak{h}$ is of co-dimension two in $\mathfrak{g}$, $\dim \mathfrak{z}=1$, $\mathfrak{h} \cap \mathfrak{z}=\{0\}$, and $[\mathfrak{g},\mathfrak{h}] \subset \mathfrak{h}+\mathfrak{z}$. The restriction D of $\operatorname{ad} X$ to $\mathfrak{h}+\mathfrak{z}$, where X is some element of $\mathfrak{g}$ which does not belong to $\mathfrak{h}+\mathfrak{z}$, is then a nilpotent derivation of $\mathfrak{h}+\mathfrak{z}$. If $\mathfrak{s}$ denotes the center of $\mathfrak{h}$ then the centers of $\mathfrak{h}+\mathfrak{z}$ and $\mathfrak{g}$ are evidently $\mathfrak{s}+\mathfrak{z}$ and $(\mathfrak{s}+\mathfrak{z}) \cap \ker D$, resp., and thus

$$\mathfrak{z} = (\mathfrak{s}+\mathfrak{z}) \cap \ker D . \tag{1}$$

Now if $\mathfrak{h}$ is not abelian then there will be a non-zero element Y in $\mathfrak{s} \cap [\mathfrak{h},\mathfrak{h}]$ and a non-negative integer k with $0 \neq D^k Y \in \ker D$. But then (as both the center and the derived algebra of $\mathfrak{h}+\mathfrak{z}$ are characteristic ideals in $\mathfrak{h}+\mathfrak{z}$) it follows from (1) that

$$D^k Y \in (\mathfrak{s}+\mathfrak{z}) \cap [\mathfrak{h},\mathfrak{h}] \cap \ker D = \{0\} ,$$

which is a contradiction. This shows that $\mathfrak{h}$ is abelian, and consequently (again by (1)) that $\ker D$ is one-dimensional.

As D is a nilpotent linear transformation of $\mathfrak{h}+\mathfrak{z}$ with a one-dimensional kernel one knows from linear algebra that there is a basis $Y_1,\ldots,Y_{n-1}$, where $n=\dim \mathfrak{g}$, satisfying $DY_1=0$ and $DY_k=Y_{k-1}$ for $2 \leqq k \leqq n-1$. If one now puts $X_1=Y_1$ and inductively defines real numbers $a_2,\ldots,a_{n-1}$ and elements $X_2,\ldots,X_{n-1}$ of $\mathfrak{h}$ by the equations

$$Y_k = X_k+a_2X_{k-1}+a_3X_{k-2}+\ldots+a_kX_1 ,$$

where $2 \leqq k \leqq n-1$, then $0 \neq X_1 \in \mathfrak{z}$ and $DX_1=0$ and $X_2,\ldots,X_{n-1}$ is a basis for $\mathfrak{h}$ with $DX_k=X_{k-1}$, $2 \leqq k \leqq n-1$. Put $X_n=X$ and let $h_1,\ldots,h_n$ be the basis for $\mathfrak{g}'$ which is dual to the basis $X_1,\ldots,X_n$ for $\mathfrak{g}$. Then for any real number t the quantity

$$\langle X_k, \mathrm{Ad}'_G(\exp(-tX))h_1\rangle$$

is $t^{k-1}/(k-1)!$ for $1 \leqq k \leqq n-1$ and is 0 for $k=n$. So if one puts

$$f_m = \sum_{k=1}^{n-1} \frac{(n-2)!}{m^{(n-k-1)/(n-2)}(k-1)!} h_k$$

for $m=1,2,\ldots$ then

$$\mathrm{Ad}'_G(\exp(m^{1/(n-2)}X))f_m = \frac{(n-2)!}{m} h_1 ,$$

and hence $(f_m, 0) \in P(\mathfrak{g}; \mathfrak{h})$, for each m. At the same time, $\lim_{m\to\infty} f_m = h_{n-1}$ and $\Omega_G(h_{n-1}) = \{h_{n-1}\}$ as $[\mathfrak{g}, \mathfrak{g}] \subset \ker h_{n-1}$, and thus $h_{n-1} \in Q(\mathfrak{g}; \mathfrak{h}, 0)$ and $0 \notin Q(\mathfrak{g}, h_{n-1}; \mathfrak{h})$. This completes the proof of the proposition in the special case under consideration.

The proof of the proposition in the general case will be by induction on the dimension of $\mathfrak{g}$. Now $\mathfrak{g}$ must be non-abelian and hence be of dimension at least three, and if $\mathfrak{g}$ is three-dimensional it must be the Heisenberg algebra ([2; Proposition 1]). But if $\mathfrak{g}$ is the Heisenberg algebra then $\mathfrak{h}$ will have to be one-dimensional and satisfy $[\mathfrak{g}, \mathfrak{h}] = \mathfrak{z}$, and then the pair $(\mathfrak{g}, \mathfrak{h})$ will satisfy the conclusion of the proposition by what was just proven.

Now assume that the proposition is true whenever $\dim \mathfrak{g} = n-1$, where n is an integer satisfying $n \geqq 4$, and suppose that $\dim \mathfrak{g} = n$. Then either (1) there is a one-dimensional subspace $\mathfrak{a}$ of $\mathfrak{z}$ with $[\mathfrak{g}, \mathfrak{h}] \not\subset \mathfrak{h} + \mathfrak{a}$, or (2) $\dim \mathfrak{z} = 1$, $\mathfrak{h} \cap \mathfrak{z} = \{0\}$, and $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h} + \mathfrak{z}$.

Suppose first that (1) holds, and let θ be the canonical mapping of $\mathfrak{g}$ onto $\mathfrak{g}/\mathfrak{a}$. Then $\theta(\mathfrak{h})$ is a subalgebra of but not an ideal in $\mathfrak{g}/\mathfrak{a}$, and therefore the pair $(\mathfrak{g}/\mathfrak{a}, \theta(\mathfrak{h}))$ satisfies the conclusion of the proposition by the inductive hypothesis. This means that there is a convergent sequence (f_m) in $(\mathfrak{g}/\mathfrak{a})'$ such that $(f_m, 0) \in P(\mathfrak{g}/\mathfrak{a}; \theta(\mathfrak{h}))$ for each m and such that $0 \notin Q(\mathfrak{g}/\mathfrak{a}, f; \theta(\mathfrak{h}))$, where $f = \lim_{m\to\infty} f_m$. The connected subgroup A of G corresponding to $\mathfrak{a}$ is closed and normal in G, and if φ denotes the canonical mapping of G onto G/A then $d\varphi = \theta$ and

$$\mathrm{Ad}'_G(x)(h \circ \theta) = [\mathrm{Ad}'_{G/A}(\varphi(x))h] \circ \theta \tag{2}$$

for all $x \in G$ and all $h \in (\mathfrak{g}/\mathfrak{a})'$. It follows easily from this formula that $(f_m \circ \theta, 0) \in P(\mathfrak{g}; \mathfrak{h})$ for all m, and consequently that $f \circ \theta \in Q(\mathfrak{g}; \mathfrak{h}, 0)$. On the other hand, if the origin in $\mathfrak{h}'$ belonged to $Q(\mathfrak{g}, f \circ \theta; \mathfrak{h})$ there would be a sequence (g_m) in $\mathfrak{h}'$ with $\lim_{m\to\infty} g_m = 0$ and with $(f \circ \theta, g_m) \in P(\mathfrak{g}; \mathfrak{h})$ for each m. Let p and q denote the restriction mappings of $\mathfrak{g}'$ and $(\mathfrak{g}/\mathfrak{a})'$ onto $\mathfrak{h}'$ and $\theta(\mathfrak{h})'$, resp. Then for every fixed m there is an element x_m in G with $p(\mathrm{Ad}'_G(x_m)(f \circ \theta)) = g_m$, and thus $g_m = 0$ on $\mathfrak{a} \cap \mathfrak{h}$ by (2). There is therefore a sequence (g'_m) in $\theta(\mathfrak{h})'$ with $\lim_{m\to\infty} g'_m = 0$ and with $g'_m \circ \theta = g_m$ for each m, and hence (by (2)) with $g'_m = q(\mathrm{Ad}'_{G/A}(\varphi(x_m))f)$ for each m. But this contradicts the condition that $0 \notin Q(\mathfrak{g}/\mathfrak{a}, f; \theta(\mathfrak{h}))$, and so one must have $0 \notin Q(\mathfrak{g}, f \circ \theta; \mathfrak{h})$. Finally, if $\mathfrak{s}$ is a given one-dimensional subspace of $\mathfrak{z}$ one may (by the inductive hypothesis) assume that $\theta(\mathfrak{s}) \subset \ker f$, and therefore that $\mathfrak{s} \subset \ker (f \circ \theta)$.

Now suppose that (2) holds. Then one may as well assume that the co-dimension of $\mathfrak{h}$ in $\mathfrak{g}$ is at least three. Indeed, the case in which the co-dimension is two has already been disposed of, and the co-dimension cannot be one as then $\mathfrak{h}$ would be an ideal in $\mathfrak{g}$. Since both $\mathfrak{z}$ and $\mathfrak{h} + \mathfrak{z}$ are ideals in $\mathfrak{g}$ there is a Jordan–Hölder basis $X_1, \ldots, X_n$ for $\mathfrak{g}$ with $X_1 \in \mathfrak{z}$ and with $X_2, \ldots, X_{r+1} \in \mathfrak{h}$, where $r = \dim \mathfrak{h}$ (cf. [8; p. 48]). Then neither X_{n-1} nor X_n belong to $\mathfrak{h}$, and (by interchanging these two elements if necessary) one may assume that $\mathfrak{h}$ is not an ideal in $\mathfrak{k} = \mathsf{R}X_1 + \ldots + \mathsf{R}X_{n-1}$. But then (by the inductive hypothesis) there must be an element f in $\mathfrak{k}'$ with $f \in Q(\mathfrak{k}; \mathfrak{h}, 0)$, $0 \notin Q(\mathfrak{k}, f; \mathfrak{h})$ and $\mathfrak{z} \subset \ker f$. Put

$$k = \min\{j : 1 \leqq j \leqq n-1 \text{ and } \langle X_j, f\rangle \neq 0\}$$

and let $\tilde{f}$ be any element of $\mathfrak{g}'$ which agrees with f on $\mathfrak{k}$. Then it is easy to see that $\tilde{f} \in Q(\mathfrak{g}; \mathfrak{h}, 0)$, that

$$\begin{aligned} \langle X_k, \mathrm{Ad}'_G(\exp Y)\tilde{f}\rangle &= \langle \exp(-\mathrm{ad}\, Y)X_k, f\rangle \\ &= \langle X_k, f\rangle \end{aligned}$$

for all $Y \in \mathfrak{g}$, and hence (as $k \geqq 2$) that $0 \notin Q(\mathfrak{g}, \tilde{f}; \mathfrak{h})$.

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Shortly after this paper was submitted for publication, the author became aware of a recently-published paper dealing with the topological Frobenius property (viz., R. Felix, R. W. Henrichs, H. L. Skudlarek, *Topological Frobenius reciprocity for projective limits of Lie groups*, Math. Z. 165 (1979), 19–28). Here, among other things, it is shown that a large number of Lie groups, including all those which are connected, simply-connected, solvable, and non-abelian, fail to satisfy (FP); more specifically, it is shown that for a large number of Lie groups G one can find a closed connected one-dimensional subgroup H of G and a non-trivial character ϱ of H such that the trivial representation of G is weakly contained in $\mathrm{ind}_H^G \varrho$. In fact, if G is a connected and simply-connected nilpotent Lie group then one can use the methods of the present paper to deduce that if H is any closed, connected, one-dimensional, non-central subgroup of G and if ϱ is any non-trivial character of H then the trivial representation of G is weakly contained in $\mathrm{ind}_H^G \varrho$.

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