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THE LAPLACE TRANSFORM OF MEASURES ON THE CONE OF A VECTOR LATTICE

EGBERT DETTWEILER

Introduction

For bounded positive measures on R which are concentrated on the positive halfline $\mathbf{R}_+$ the Fourier transform can be replaced by the Laplace transform, which is somewhat easier to handle. It therefore seems to be a natural question, whether this can be done also in the infinite dimensional case if one wants to study measures concentrated on the cone of an ordered (topological) vector space. Clearly, there is no difficulty to define the Laplace transform of such measures if the dual space (or more precisely the dual cone) is large enough. The main problem however—just as in the classical case—consists in getting a theorem of Bochner type providing via the Laplace transform a bijective mapping between a certain specified class of functions on the dual cone and the measures on the original cone.

Based on the paper [1] of Berg, Christensen and Ressel, who studied this problem in the general case of Abelian semigroups, Hoffmann-Jørgensen and Ressel [3] proved such a theorem for the cone $C^b_+(X)$ of the lattice $C^b(X)$ of all bounded continuous functions on a completely regular space X furnished with the strict topology. Every continuous positive definite function on $C^b_+(X)$ (in the sense of [1]) is the Laplace transform of a Radon measure on the dual cone $M^b_+(X)$ of all positive bounded Radon measures on X, where $M^b_+(X)$ carries the weak topology $\sigma(M^b(X), C^b(X))$.

In the present paper we investigate the relationship between positive definite functions on a cone and measures on the dual cone in the more general context of locally convex vector lattices. More precisely, we study the following question: Let E be a locally convex vector lattice with positive cone E_+ and let φ be a continuous positive definite function on E_+ in the sense of [1]. When does there exist a measure μ on the dual cone E'_+ (with the weak topology $\sigma(E',E)$) such that φ is the Laplace transform of μ? As the main result (theorems 2.13 and 3.9) we get that there is a "good" Bochner theorem (as it was proved by Hoffmann-Jørgensen and Ressel for the space $C^b(X)$) if and only

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if the topology of E is generated by a family of (AM)-seminorms, or equivalently if and only if E is the projective limit of Banach lattices isomorphic to spaces $C(X)$ with compact X. This characterization is obtained in section 3 with the aid of the theory of (cone) absolutely summing maps (see [9, ch. IV]).

1. Preliminaries.

First we recall some notations and a few definitions from the theory of ordered topological vector spaces (see [8] and [9] as a general reference).

Let E be a vector lattice over field $\mathbf{R}$ of real numbers. A function p: $E \to \mathbf{R}_+$ is called a *lattice seminorm*, if p is a seminorm and if $|x| \leqq |y|$ $(x, y \in E)$ always implies $p(x) \leqq p(y)$. If in addition E is also a topological vector space, E is called a *locally convex vector lattice* if the topology of E is generated by a family of lattice seminorms. E_+ always denotes the positive cone of E, and if F is a second vector lattice such that (E, F) is a dual pair, we call (E, F) a *dual pair of vector lattices* if E_+ and F_+ separate the points of F_+ and E_+ resp. For any dual pair (E, F), $\sigma(E, F)$ and $\tau(E, F)$ denote the weak and the Mackey topology resp. on E with respect to the duality (E, F). $\beta(E)$ denotes the *order topology* on E, which by definition is the finest locally convex topology on E for which every order interval is bounded. E^b, C^b, and E^{boc} denote the space of all order bounded, positive, and order continuous linear forms on E respectively.

A *Fréchet lattice* is a locally convex vector lattice E which is complete and whose topology is generated by a countable family of lattice seminorms. If E is complete and if the topology of E is generated by a single lattice norm, then E is called a *Banach lattice.* For a closed, convex, circled subset $B \subset E$ the subspace $E_B := \bigcup_{n \in \mathbf{N}} nB$ is normed space with respect to the norm p_B defined by

$$p_B(x) := \inf\{t > 0 : x \in tB\} .$$

The spaces $E_a := E_{[-a, a]}$ $(a \in E_+)$ are normed lattices, which are Banach lattices, if E is quasi-complete and if the topology of E is coarser than the order topology. A Banach lattice E is called an (AM)-*space* if the norm fulfills the condition $\|x \vee y\| = \|x\| \vee \|y\|$ for all $x, y \in E_+$, and an (AL)-*space* if the norm is additive on E_+.

We now give the basic notations and definitions needed from the probability theory on vector spaces. For any locally convex space E with Borel σ-algebra $\mathscr{B}(E)$ we denote by $M^b_+(E)$ and $P(E)$ resp. the set of all bounded positive Radon measures and the set of all (Radon) probability measures on E. If E is in addition a locally convex vector lattice, then $M^b_+(E_+)$ and $P(E_+)$ denote the corresponding subsets of all measures which are concentrated on the cone E_+. If (E, F) is a dual pair of vector lattices, and if $\mathscr{T}(E, F)$ is a consistent topology on E, we sometimes write $M^b_+(E_+, \mathscr{T}(E,F))$ or $P(E_+, \mathscr{T}(E,F))$ whenever we wish to emphasize the special induced topology on E_+.

For a dual pair (E,F) of vector spaces let $F^{(\mathbf{N})}$ denote the set of all finite sequences in F, and for every $\bar{y}=(y_1,\ldots,y_n) \in F^{(\mathbf{N})}$ let $\pi_{\bar{y}}: E \to \mathbf{R}$ be the canonical projection defined by

$$\pi_{\bar{y}}(x) := (\langle x, y_i \rangle)_{1 \leqq i \leqq n} \quad \text{for all } x \in E .$$

Let us write $\bar{y} < \bar{z} = (z_1,\ldots,z_m) \in F^{(\mathbf{N})}$ if there is a linear map $\pi_{\bar{y},\bar{z}}: \mathbf{R}^m \to \mathbf{R}^n$ such that $\pi_{\bar{y},\bar{z}} \circ \pi_{\bar{z}} = \pi_{\bar{y}}$. A family $(\mu_{\bar{y}})_{\bar{y} \in F^{(\mathbf{N})}}$ of measures $\mu_{\bar{y}}$ on the finite dimensional spaces $\mathbf{R}^{d(\bar{y})}$ (where $d(\bar{y})$ denotes the length of the sequence $\bar{y}$) is called a *cylindrical measure* if for all $\bar{y}, \bar{z} \in F^{(\mathbf{N})}$ with $\bar{y} < \bar{z}$ one has $\pi_{\bar{y},\bar{z}}(\mu_{\bar{z}}) = \mu_{\bar{y}}$. $\check{M}^b_+(E)$ and $\check{P}(E)$ denote the sets of all bounded cylindrical measures and all probability cylindrical measures respectively. The one-dimensional marginal distributions of a cylindrical measure μ are simply denoted by μ_y $(y \in F)$. A sequence (μ_n) of bounded cylindrical measures is said to *converge cylindrically* towards a cylindrical measure if for all $\bar{y} \in F^{(\mathbf{N})}$ the sequence $(\pi_{\bar{y}}(\mu_n))$ converges weakly towards $\pi_{\bar{y}}(\mu)$. If (E,F) is a dual pair of vector lattices, then we say that a cylindrical measure on E is a *cylindrical measure on the cone* E_+ if for all $\bar{y} \in F_+^{(\mathbf{N})}$ the measure $\mu_{\bar{y}}$ is concentrated on the natural cone $\mathbf{R}_+^{d(\bar{y})}$ of $\mathbf{R}^{d(\bar{y})}$, and we use the notation $\check{P}(E_+)$ for the set of all probability cylindrical measures on E_+. For further notations we refer the reader to [11].

2. The Laplace transform for measures on the positive cone of a topological vector lattice.

Let (E,F) be a dual pair of real vector lattices. For any cylindrical measure μ on E_+ one can define the *Laplace transform* of μ as the function $L\mu: F_+ \to \mathbf{R}_+$ defined by

$$L\mu(y) := \int_{\mathbf{R}_+} e^{-t}\, d\mu_y(t) \quad \text{for all } y \in F_+ .$$

If E is a topological vector lattice with a consistent topology with respect to the duality (E,F), and if μ is a Radon measure on E_+, then of course $L\mu$ is given by

$$L\mu(y) = \int_{E_+} e^{-\langle x,y \rangle}\, d\mu(x) \quad \text{for all } y \in F_+ .$$

For every $\mu \in \check{P}(E_+)$ the Laplace transform $L\mu$ is *positive definite* in the following sense (see [1]): For every $n \in \mathbf{N}$ and every choice of n elements $y_1,\ldots,y_n \in F_+$ and n real numbers $a_1,\ldots,a_n$ we have

$$\sum_{i,j=1}^{n} a_i a_j L\mu(y_i + y_j) \geqq 0 .$$

Let $\hat{F}_+$ denote the set of all mappings $\lambda\colon F_+ \to [0,\infty]$, such that $\lambda(0)=0$ and $\lambda(x+y)=\lambda(x)+\lambda(y)$ hold for all $x, y \in F_+$. $\hat{F}_+$ is again a semigroup which is compact with respect to the topology of pointwise convergence, and the following theorem (which is proved in [1] for general Abelian semigroups) holds:

2.1. THEOREM (of Berg, Christensen and Ressel). *For every positive definite function φ on F_+ there exists a bounded positive Radon measure μ on $\hat{F}_+$, such that*

$$\varphi(y) = \int_{\hat{F}_+} e^{-\lambda(y)}\, d\mu(\lambda) \quad \textit{for all } y \in F_+ .$$

In the following the main problem consists in finding conditions under which the measure μ on $\hat{F}_+$ associated with a given positive definite function on F_+ by (2.1) is concentrated on the subsemigroup E_+ of $\hat{F}_+$ where E_+ is furnished with a suitable topology. First we need some further properties of positive definite functions on the positive cone of a vector lattice, which we collect in the next proposition.

2.2. PROPOSITION. *Let φ be a positive definite function on $C := F_+$. Then φ has the following properties*:

(i) $0 \leqq \varphi(y) \leqq \varphi(0)$ *and* $(\varphi(\tfrac{1}{2}x+\tfrac{1}{2}y))^2 \leqq \varphi(x)\varphi(y)$ *for all* $x, y \in C$.

(ii) $|\varphi(x)-\varphi(y)| \leqq \varphi(0)-\varphi(|x-y|)$ *for all* $x, y \in C$.

(iii) φ *is always a decreasing function and for every* $a \in C$ *one has* $\varphi(x)-\varphi(y) \geqq \varphi(x+a)-\varphi(y+a)$.

PROOF. Property (i) is proved in [1] for general Abelian semigroups. The properties (ii) and (iii) follow by integration from the corresponding properties of the positive definite functions $x \to e^{-\lambda(x)}$.

For every $A \in \mathscr{F}(C)$ (the family of all finite subsets of C) we set

$$\hat{C}_A := \{\lambda \in \hat{C} : \lambda(y) < \infty \text{ for all } y \in A\} .$$

$\hat{C}_A$ is open and therefore a Borel subset of $\hat{C}$. If φ is a positive definite function on C with representing measure μ, then it is not difficult to show (see [3] for the idea of the proof) that μ is concentrated on $\hat{C}_A$ if and only if $\lim_{n\to\infty} \varphi(y/n) = \varphi(0)$ for all $y \in A$.

2.3. PROPOSITION. *Let φ be a positive definite function on C with representing* *measure μ. Then μ is a cylindrical measure on C^b with respect to the duality (F, F^b) if and only if $\lim_{n\to\infty} \varphi(y/n) = \varphi(0)$ for all $y \in C$. If φ is the Laplace transform of a cylindrical measure μ with $\varphi^{-1}(\varphi(0)) = \{0\}$, then for all $x, y \in C$ with $x < y$ one has $\varphi(y) < \varphi(x)$ and $\lim_{t\to\infty} \varphi(tx) = \mu(\{\lambda : \lambda(x) = 0\})$.*

PROOF. If $\lim_{n\to\infty} \varphi(y/n) = \varphi(0)$ holds for all $y \in C$ and if for all $A = \{y_1, \ldots, y_n\} \in \mathscr{F}(C)$, π_A denotes the natural projection on $\hat{C}$ defined by

$$\pi_A(\lambda) := (\lambda(y_1), \ldots, \lambda(y_n)),$$

then the above remark shows that the measures $\mu_A := \pi_A(\mu)$ are concentrated on $\mathbf{R}^n_+$. Now it is easy to show that the system $(\mu_A)_{A \in \mathscr{F}(C)}$ defines in fact a cylindrical measure on C^b. If conversely $(\mu_A)_{A \in \mathscr{F}(C)}$ is a cylindrical measure on C^b, then from $\varphi(y) = L\mu_y(1)$ and $\varphi(y/n) = L\mu_y(1/n)$ for all $y \in C$ one gets $\lim_{n\to\infty} \varphi(y/n) = \varphi(0)$.

For the proof of the second assertion suppose that there exist $x, y \in C$ with $x < y$ and $\varphi(x) = \varphi(y) =: c$. We set $z := y - x$. Then from (2.2) one gets $\varphi(x + 2z) = c$. Repeating this procedure one gets finally $\varphi(x + tz) = c$ for all $t \in \mathbf{R}_+$. Now for all $n \in \mathbf{N}$ we have

$$c = \varphi(x + nz) = \int_{[\lambda(z) = 0]} e^{-\lambda(x)} \, d\mu(\lambda) + \int_{[\lambda(z) \neq 0]} e^{-\lambda(x + nz)} \, d\mu(\lambda).$$

From this equality we get with $n \to \infty$

$$c = \int_{[\lambda(z) = 0]} e^{-\lambda(x)} \, d\mu(\lambda) = \varphi(x) = \int e^{-\lambda(x)} \, d\mu(\lambda).$$

By assumption μ is concentrated on $\hat{C}_{\{x\}}$. Therefore $\mu([\lambda(z) = 0]) = \|\mu\|$. But then $\varphi(z) = \|\mu\| = \varphi(0)$, i.e. $z = 0$ by assumption about φ. This is a contradiction. Now if $\lim_{t\to\infty} \varphi(tx) =: b$ $(x \in C)$, then from

$$b \leqq \mu([\lambda(x) = 0]) + \int_{[\lambda(x) \neq 0]} e^{-\lambda(nx)} \, d\mu(\lambda) \qquad (n \in \mathbf{N})$$

we get $\mu([\lambda(x) = 0]) = b$.

If (E, F) is a dual pair of ordered vector lattices with cones E_+ and F_+, it seems to be quite natural to ask for the relationship between the positive definite functions on F_+ on one side and the (bounded) Radon measures on E_+ (for a suitable topology) on the other side given by the Laplace transformation. In particular we are interested in determining those locally convex vector lattices for which this connection is of Bochner type (with Laplace transform instead of Fourier transform). In this context we introduce a topology which already in [3] turned out to be very useful. This topology is the topology of uniform convergence on the order intervals of E and we will call this topology on F the *L^1-topology* on F (with respect to the duality (E,F)). We use the notation $\lambda^1(F,E)$ or only λ^1 if there is no danger of confusion. The L^1-topology is generated by the family $(p_x)_{x\in E_+}$ of seminorms defined by $p_x(y) := \langle x, |y| \rangle$ for all $y \in F$. One has $\sigma(F,E) \leqq \lambda^1(F,E)$ and $\lambda^1(F,F') \leqq \tau(F,F')$ if F is a locally convex vector lattice. If the order intervals of F are weakly compact, then also $\lambda^1(F',F) \leqq \tau(F',F)$ holds, and if the topology of F is the order topology (this is the case for example, if F is a Fréchet lattice), then $\lambda^1(F',F) \leqq \beta(F',F)$.

2.4. PROPOSITION. *Let E be a quasi-complete locally convex vector lattice. Then for all $\mu \in M_+^b(E_+)$ the Laplace transform $L\mu$ on E'_+ is continuous in the L^1-topology $\lambda^1(E',E)$.*

PROOF. Because of (2.2) (ii) it is sufficient to prove the continuity of $L\mu$ at the origin. For every $\varepsilon > 0$ we choose a compact set $K \subset E_+$ with $\mu(K^c) < \varepsilon$. Because of the quasi-completeness of E we can suppose that K is convex. Then we have for every $x' \in E'_+$

$$L\mu(0) - L\mu(x') \leqq \int_K (1 - e^{-\langle x, x' \rangle})\, d\mu(x) + \varepsilon \leqq \int_K \langle x, x' \rangle\, d\mu(x) + \varepsilon .$$

From that inequality now the assertion follows because $\int_K x\, d\mu(x) \in E_+$.

A semi-norm p on E will be called an (AM)-*seminorm* if p is a lattice seminorm with the property

$$\sup(p(x), p(y)) = p(x \vee y) \quad \text{for all } x, y \in E_+ .$$

Now we prove our first version of "Bochner's theorem".

2.5 THEOREM. *Let E be a Fréchet lattice whose topology is generated by a sequence of* (AM)*-seminorms (p_n), and let $\varphi: E_+ \to \mathbf{R}_+$ be a positive definite function. Then the following assertions are equivalent:*

(i) *φ is continuous for the topology $\lambda^1(E,E')$,*

(ii) *φ is continuous for the given topology on E,*

(iii) *there is a $\mu \in M_+^b(E'_+, \sigma(E',E))$ with $L\mu = \varphi$.*

PROOF. We have to show the implication (ii) $\Rightarrow$ (iii). Without loss of generality let us assume that the sequence (p_n) is increasing, and that the sets

$$B_n := \{x \in E_+ \ : \ p_n(x) \leqq 1\}$$

form a neighborhood basis of the origin in E_+. Let μ denote the representing measure of φ on the space $\hat{E}_+$. For all $n \in \mathbf{N}$ we denote by $\hat{p}_n$ the functional on $\hat{E}_+$ defined by

$$\hat{p}_n(\lambda) := \sup\{\lambda(x) \ : \ x \in B_n\} .$$

One gets immediately that $\hat{p}_n(\lambda) < \infty$ implies $\lambda \in E'_+$ (see [8, ch. V, theorem 5.5]). Furthermore the functionals $\hat{p}_n$ are lower semi-continuous and one easily checks the relation

$$\inf(\hat{p}_n(\lambda), 1_{\hat{E}_+}(\lambda)) = \sup_{x \in B_n} \inf(\lambda(x), 1_{\hat{E}_+}(\lambda))$$

and observes that the family of functions

$$\lambda \mapsto \inf(\lambda(x), 1_{\hat{E}_+}(\lambda)) \qquad (x \in B_n)$$

is directed for every $n \in \mathbf{N}$.

Now we choose an arbitrary $\varepsilon > 0$ and an $n \in \mathbf{N}$ such that $x \in B_n$ implies $\varphi(0) - \varphi(x) < \varepsilon/2$. From the above remarks and the inequality $\frac{1}{2}\inf(1,t) \leqq 1 - e^{-t}$ (valid for all $t \geqq 0$) we get

$$\begin{aligned} \int \inf(\hat{p}_n, 1_{\hat{E}_+})\, d\mu &= \sup_{x \in B_n} \int \inf(\lambda(x), 1_{\hat{E}_+}(\lambda))\, d\mu(\lambda) \\ &\leqq 2 \sup_{x \in B_n} \int (1 - e^{-\lambda(x)})\, d\mu(\lambda) \\ &= 2 \sup_{x \in B_n} (\varphi(0) - \varphi(x)) < \varepsilon . \end{aligned}$$

Especially we have $\mu\{\lambda \in \hat{E}_+ : \hat{p}_n(\lambda) > 1\} < \varepsilon$. Now the set

$$K_n := \{x' \in E'_+ \ : \ \hat{p}_n(x') \leqq 1\} = \{\lambda \in \hat{E}_+ \ : \ \hat{p}_n(\lambda) \leqq 1\}$$

is a compact subset of $\hat{E}_+$ and of E'. Therefore μ is even a measure on E'_+ with respect to the weak topology as asserted.

2.6. Corollary. *Let E be a Banach lattice which is an* (AM)*-space. Then a positive definite function φ on E_+ is the Laplace transform of a measure $\mu \in M^b_+(E'_+, \sigma(E', E))$ if and only if φ is continuous for a topology $\mathcal{T}$ between the L^1-topology and the norm topology. Especially any norm continuous positive definite function is already L^1-continuous.*

Theorem 2.5 admits a generalization to locally convex vector lattices for which the topology is generated by an arbitrary family of (AM)-seminorms.

For this more general Bochner theorem (and essentially the most general as is proven in the next paragraph) we need some preparations concerning lifting properties of positive definite functions.

2.7. PROPOSITION. *Let C denote the positive cone of a locally convex vector lattice E. If φ is a positive definite and L^1-continuous function on C, then*

$$C_0 := \{x \in C : \varphi(0) = \varphi(x)\}$$

is a closed solid subcone of C, and if π denotes the canonical projection from E onto $F := E/(C_0 - C_0)$, then there exists a positive definite function ψ on F_+ with $\varphi = \psi \circ \pi$, and ψ is continuous with respect to the quotient topology. If in addition E is ordercomplete with dual $E' = E^{boc}$, then $I_0 := C_0 - C_0$ is a band in E and ψ is L^1-continuous.

PROOF. For $0 \leqq y \leqq z$ one concludes from the relation

$$\varphi(z)^2 \leqq \varphi(2y)\varphi(2(z-y))$$

that $z \in C_0$ implies $y \in C_0$. From the same relation one gets $\varphi(nz) = \varphi(0)$ for all $n \in \mathbb{N}$ if $z \in C_0$, and therefore $\varphi(tz) = \varphi(0)$ for all $t \in \mathbb{R}_+$. From

$$|\varphi(x+y) - \varphi(x)| \leqq \varphi(0) - \varphi(y)$$

one gets further $\varphi(x+y) = \varphi(x) = \varphi(0)$ if $x, y \in C_0$. All this together implies that C_0 is a solid subcone of C. Of course C_0 is closed because $\lambda^1(E, E')$ is consistent with the duality (E, E').

Now for every $z = z_1 - z_2 \in I_0 = C_0 - C_0$ one has $|z| \in C_0$ because of $|z| \leqq z_1 + z_2$. Now (2.2) (ii) implies $\varphi(x) = \varphi(y)$ for all $x, y \in C$ with $x - y \in C_0 - C_0$. Therefore ψ given by $\psi(\pi(x)) := \varphi(x)$ is a well defined function on F_+, and it is easy to see that ψ is again positive definite and continuous with respect to the quotient topology.

If now E is order complete with $E' = E^{boc}$ then every increasing directed net $(x_i)_{i \in I}$ with $x := \sup_{i \in I} x_i$ converges towards x with respect to $\lambda^1(E, E')$. From this observation it follows that I_0 is a band. We set $I_1 := I_0^{\perp}$, $J_0 := I_0^{0}$, $J_1 := J_0^{\perp}$. Then one easily deduces from the band decompositions $E = I_0 + I_1$ and $E' = J_0 + J_1$ and the natural isomorphisms $E/I_0 \cong I_1$ and $(E/I_0)' \cong J_0$ that ψ is continuous with respect to $\lambda^1(\mathrm{E}/\mathrm{I}_0, (\mathrm{E}/\mathrm{I}_0)')$.

2.8. PROPOSITION. *Let φ be a continuous positive definite function on an arbitrary locally convex vector lattice E. Then there always exist a Fréchet lattice F, a positive continuous linear map u from E into F, and a continuous positive definite function ψ such that $\varphi = \psi \circ u$ holds. Moreover this factorizing can always be done in such a way that $\psi(x) < \psi(0)$ for all $x \in F_+$ with $x > 0$. If ψ*

is $\lambda^1(F,F')$-continuous, then ψ can further be factorized in the following way. There exists an (AL)*-space G, a positive linear map v from F into G, and a positive definite and $\lambda^1(G,G')$-continuous function χ on G_+ with $\psi=\chi\circ v$.*

PROOF. Let $\mathscr{P}$ denote a family of lattice seminorms generating the topology of E. By the continuity of φ there exists for a fixed sequence (ε_n) of positive numbers converging to zero a sequence (p_n) of seminorms in $\mathscr{P}$ such that $\varphi(0)-\varphi(x)\leqq\varepsilon_n$ whenever $p_n(x)\leqq\delta_n$ (for a suitable δ_n). We now set $N:=\bigcap_{n\in\mathbb{N}}p_n^{-1}(0)$ and define F to be the completion of E/N with respect to the quotient topology coming from the topology on E generated by the sequence (p_n). The seminorms p_n induce in a natural way seminorms $\tilde{p}_n$ on F such that F is a Fréchet lattice whose topology is generated by $(\tilde{p}_n)$. Now let u denote the canonical projection from E into F. By definition of N we have $N\subset I_0$. Therefore $\psi(u(x)):=\varphi(x)$ $(x\in E_+)$ defines a positive definite function ψ on E/N. ψ has the property $\psi(0)-\psi(y)\leqq\varepsilon_n$ whenever $\tilde{p}_n(y)\leqq\delta_n$ for $y\in(E/N)_+$. Thus ψ is continuous on E/N and it is easy to see now that ψ possesses a unique positive definite and continuous extension, which we again denote by ψ. By factorizing F if necessary as in (2.7) we can even assume $\psi(y)<\psi(0)$ for $y>0$.

Now let us suppose that ψ is $\lambda^1(F,F')$-continuous. Then there exists a sequence (z_n) in F'_+ such that $\langle y,z_n\rangle\leqq 1$ implies $\psi(0)-\psi(y)\leqq\varepsilon_n$. We choose a suitable sequence (a_n) of positive numbers such that $z:=\sum_{n\in\mathbb{N}}a_nz_n$ is an element of F'_+. Then there is a sequence (η_n) in $\mathbb{R}_+$ such that $\langle y,z\rangle\leqq\eta_n$ always implies $\psi(0)-\psi(y)\leqq\varepsilon_n$. Now we make the same procedure of factorizing as in the first step of the proof with respect to the single seminorm p_z $(p_z(y):=\langle|y|,z\rangle)$. As p_z is an (AL)-seminorm, the resulting quotient space G is an (AL)-space. In fact we have $G=(F,z)$ in the terminology of [9]. If v denotes the projection from F into G, it is now clear how to define the desired positive definite function χ.

2.9. DEFINITION. We call a locally convex vector lattice E a *C^b-vector lattice* if the family of all continuous (AM)-seminorms on E generates a topology consistent with the duality (E,E'). In this case we call this topology the *C^b-topology* on E and denote it by $c^b(E,E')$.

If E is a C^b-vector lattice, then by definition $\lambda^1(E,E')\leqq c^b(E,E')\leqq\mathscr{T}$, if $\mathscr{T}$ denotes the given topology on E. A prime example of a C^b-lattice is the space $C^b(X)$ of all bounded continuous functions on a completely regular space X, whose topology is given by a family $(p_B)_{B\in\mathscr{B}}$ of seminorms, where $\mathscr{B}$ is a suitable family of subsets of X and the p_B are defined by $p_B(f):=\sup_{x\in B}f(x)$. Another example one gets, if one considers the so-called *strict topology* on $C^b(X)$. This topology, giving as dual of $C^b(X)$ exactly the bounded Radon measures on X, is generated by the seminorms of the form $p := \sup_{n \in \mathbb{N}} a_n p_{K_n}$, where (a_n) is a null sequence and (K_n) is a sequence of compact subsets of X (see [2]).

2.10. THEOREM. *Let E be a C^b-lattice. Then every $c^b(E, E')$-continuous positive definite function φ on E_+ is the Laplace transform of a $\mu \in M^b_+(E'_+, \sigma(E', E))$ with the property*

(CR) *for every $\varepsilon > 0$ there exists a weakly compact and convex set K with $\mu(K^c) < \varepsilon$.*

Conversely, for any measure $\mu \in M^b_+(E'_+, \sigma(E', E))$ with the property (CR) *the Laplace transform $\varphi = L\mu$ is $\lambda^1(E, E')$-continuous and therefore also $c^b(E, E')$-continuous.*

PROOF. Let φ be $c^b(E, E')$-continuous. By the method of factorization used in the proof of (2.8), there exists a projection u from E into a Fréchet lattice F, whose topology in the present case is generated by a sequence of (AM)-seminorms, and there exists a positive definite function ψ on F_+, which is continuous and fulfills the relation $\varphi = \psi \circ u$. Now by theorem (2.5) there exists a measure λ on F'_+ (with the weak topology) with $L\lambda = \psi$. We set $\mu := {}^t u(\lambda)$. Then clearly $L\mu = \varphi$. Now λ has property (CR) because F is a Mackey space. Therefore μ has property (CR) too. The last assertion follows as in the proof of proposition (2.4).

REMARK. If E itself is a Mackey space, then it is unnecessary to demand the property (CR) because of the fact that then every weakly compact set of E' is equicontinuous. However the above mentioned example of the space $C^b(X)$ with the strict topology shows that in general one cannot dispense with property (CR), because $C^b(X)$ with the strict topology is in general not a Mackey space.

The above theorem for the just mentioned case $C^b(X)$ with strict topology has been proved by Hoffman-Jørgensen and Ressel in [3]. The strict topology has the following completeness property: If $\mathscr{P}$ is the generating family of (AM)-seminorms, then for any sequence (p_n) in $\mathscr{P}$ there is a $p \in \mathscr{P}$ and a sequence (c_n) of positive numbers such that $p_n \leqq c_n p$ for all $n \in \mathbb{N}$. Let us call a C^b-topology with that additional property *strict*. In this case it is not difficult to prove the following slightly stronger result than (2.10).

2.11. COROLLARY. *Let E be a C^b-lattice with strict C^b-topology. Then in addition to the statement of* (2.10) *to every c^b-continuous positive definite function*

φ on E_+ there exists an (AM)*-space G and a positive injection $v: G' \to E'$ (weakly continuous) such that φ is the Laplace transform of a measure $v(\nu)$ with $\nu \in M^b_+(G'_+, \sigma(G', G))$.*

Now we want to investigate the relation between measures and positive definite functions for vector lattices which have a more general structure than a C^b-lattice. First we give two results about the support of measures on a cone.

2.12. Proposition. *Let E be a quasi-complete locally convex lattice and $\mu \in M^b_+(E_+)$. As in* (2.7) *we define*

$$I_0 := \{x' \in E' : L\mu(0) = L\mu(|x'|)\} .$$

Then μ is concentrated on the solid subcone $T_0 := I_0^0 \cap E_+$, and T_0 is the smallest solid subcone of E_+ with supp $(\mu) \subset T_0$.

Proof. Suppose that there exists an $x_0 \in \operatorname{supp}(\mu)$ with $x_0 \notin T_0$. By the second separation theorem then there is a $y \in E'$ and a neighborhood $U(x_0)$ with $U(x_0) \cap T_0 = \emptyset$ such that $\langle x, y \rangle = 0$ for all $x \in I_0^0$ and $\langle x, y \rangle \geqq \delta > 0$ for all $x \in U(x_0)$ (for a suitable $\delta > 0$). From this we get $y \in C_0 := I_0 \cap E'_+$. By assumption we have $\mu(U(x_0)) > 0$. But this gives a contradiction by the inequality

$$L\mu(0) - L\mu(y) \geqq \int_{U(x_0)} (1 - e^{-\langle x, y \rangle} d\mu(x)$$
$$\geqq (1 - e^{-\delta})\mu(U(x_0)) > 0 .$$

Therefore μ is concentrated on T_0, and it is not hard to prove that T_0 is the smallest solid subcone with this property.

2.13. Proposition. *If E is a Fréchet lattice, then for all $\mu \in M^b_+(E_+)$ there exists an $a \in E_+$ such that* supp $(\mu) \subset \overline{E_a}$.

Proof. We first prove the following assertion:

(*) for all $\varepsilon > 0$ there exists an $a \in E_+$ such that $\mu(\overline{E_a}^c) < \varepsilon$.

Suppose that (*) is not true. Then there is an $\varepsilon > 0$ such that for all $a \in E_+$ one has $\mu(\overline{E_a}^c) > \varepsilon$. This implies that for all $a \in E_+$ there is a compact set $K_a \subset \overline{E_a}^c \cap E_+$ such that $\mu(K_a) > \varepsilon$. We now show $\overline{\text{co}}\,(K_a) \cap (\overline{E_a})_+ = \emptyset$ ($\overline{\text{co}}$ denotes the closed convex hull), so that we can suppose without loss of generality that all sets K_a are convex. For all $x \in K_a$ we can choose an element $y_x \in E'_+$ such that $\langle x, y_x \rangle > c_x > 0$ and $\langle z, y_x \rangle \leqq 0$ hold for all $z \in (\overline{E_a})_+$. Now

$$A_x := \{z \in E : \langle z, y_x \rangle \geqq c_x\}$$

is a closed convex neighborhood of x. Therefore $C_x := \overline{\text{co}}\,(A_x \cap K_a) \subset A_x$ and $C_x \cap (\overline{E_a})_+ = \emptyset$. Because K_a is compact, there exists a finite number of elements in K_a, say $x_1, \ldots, x_n$, such that $K_a \subset \bigcup_{i=1}^n C_{x_i}$, and therefore $\overline{\text{co}}\,(K_a) \subset \text{co}\,(\bigcup_{i=1}^n C_{x_i})$. Now suppose that there is a $z \in \text{co}\,(\bigcup_{i=1}^n C_{x_i}) \cap (\overline{E_a})_+$. This means without loss of generality $z = su + tv$ with $s + t = 1$, $0 < s < 1$, $u \in C_{x_1}$, and $v \in \text{co}\,(\bigcup_{i=2}^n C_{x_i})$. But this would imply $s^{-1}z = u + s^{-1}tv \in (\overline{E_a})_+$ and therefore $u \in (\overline{E_a})_+$, which is impossible because of $C_{x_1} \cap \overline{E_a} = \emptyset$. By this contradiction we now can assume that for all $a \in E_+$ there exists a compact and even convex set K_a such that $\mu(K_a) > \varepsilon$ and $K_a \cap \overline{E_a} = \emptyset$. By the second separation theorem one has for all $a \in E_+$ an element $x'_a \in E'_+$ with $\langle x, x'_a \rangle \geqq 1$ for all $x \in K_a$ and $\langle x, x'_a \rangle = 0$ for all $x \in \overline{E_a}$. This yields for the Laplace transform $\varphi = L\mu$:

$$\begin{aligned} \varphi(0) - \varphi(x'_a) &= \int (1 - e^{-\langle x, x'_a \rangle})\, d\mu(x) \\ &\geqq \int_{K_a} (1 - e^{-\langle x, x'_a \rangle})\, d\mu(x) \geqq (1 - e^{-1})\mu(K_a) > \tfrac{1}{2}\varepsilon\,. \end{aligned}$$

On the other side the net $(x'_a)_{a \in E_+}$ has the limit 0 in the L^1-topology, because for all $a \geqq b \in E_+$ one has $\langle x'_a, b \rangle = 0$. Therefore the above inequality is a contradiction to the L^1-continuity of φ. Thus (*) is proven.

By (*) we get for a sequence (a_n) in E_+ $\mu((\bigcup_{n \in \mathbb{N}} \overline{E_{a_n}})^c) = 0$. By the assumption that E is a Fréchet lattice, there exists an element $a \in E_+$ such that $\bigcup_{n \in \mathbb{N}} \overline{E_{a_n}} \subset \overline{E_a}$. Therefore $\mu(\overline{E_a}^c) = 0$.

If E is a locally convex lattice whose topology is coarser than the order topology, then for all $a \in E_+$ the canonical injections $u_a \colon E_a \to E$ are continuous. This implies that for any continuous positive definite function φ on E_+ the composed function $\varphi \circ u_a$ is the Laplace transform of a measure μ_a on the (AL)-space E'_a (with the weak topology $\sigma(E'_a, E_a)$). Furthermore the system $(\mu_a)_{a \in E_+}$ of measures on the positive cones C'_a of E'_a is a projective system. These projective systems seem to be a good substitute for the cylindrical measures in case one is interested in measures on the cone of a vector lattice. The next proposition gives a criterium, when a projective system $(\mu_a)_{a \in E_+}$ defines in fact a measure on the cone of the original space E'.

2.14. Proposition. *Let φ be a continuous positive definite function on the positive cone C of a Fréchet lattice E and let (μ_a) be the associated projective family of measures on the cones C'_a $(a \in C)$. If for all $a \in C$ the measure μ_a is concentrated on $v_a(C')$, where v_a denotes the adjoint of u_a, then (μ_a) can be identified with an abstract measure on the cone C' of E' furnished with the cylinder σ-algebra with respect to the duality (E', E).*

PROOF. E is the inductive limit of the (AM)-spaces E_a (see [8]). Therefore E' is the projective limit of the (AL)-spaces E'_a with the weak topology $\sigma(E'_a, E_a)$ ($a \in C$). Now $(\mu_a)_{a\in C}$ defines in an obvious way a finite additive set function on the algebra

$$\bigcup_{a\in C} v_a^{-1}(\mathscr{B}(\sigma(E'_a, E_a))) ,$$

and it is sufficient to show that μ is σ-additive. To prove that let (a_n) denote any without loss of generality increasing sequence in C and let (B_n) be a sequence of sets $B_n \in \mathscr{B}(\sigma(E'_{a_n}, E_{a_n}))$ such that

$$\bigcap_{n\in \mathbf{N}} v_{a_n}^{-1}(B_n) = \varnothing .$$

There is an $a \in C$ with $a_n \in E_a$ for all n. This implies

$$C_n := v_{a_n,a}^{-1}(B_n) \in \mathscr{B}(\sigma(E'_a, E_a)) \quad \text{and} \quad \bigcap_{n\in \mathbf{N}} v_a^{-1}(C_n) = \varnothing$$

($v_{a,b}$ denotes the adjoint of the canonical map $u_{a,b}$: $E_a \to E_b$ where $a, b \in C$ and $a \in E_b$). Now by supposition μ_a is concentrated on $v_a(E')$ and we have with D: $=\bigcap_{n\in\mathbf{N}} C_n$ that $D \in v_a(E')^c$. Therefore

$$\lim_{n\to\infty} \mu(v_{a_n}^{-1}(B_n)) = \mu_a(D) = 0 ,$$

and μ is σ-additive.

REMARK. If E is a Banach lattice, then for all $a \in C$ the measure μ_a is concentrated on $v_a(C')$ if and only if for all $\varepsilon > 0$ there exists an $n = n(\varepsilon, a)$ such that $\mu_a(v_a(B_n)^c) \leq \varepsilon$, where B_n denotes the ball in E' with radius n.

If all the projections v_a are surjective, then essentially the same proof shows that (2.14) is valid without the restriction that E is a Fréchet lattice, and one gets the following result:

2.15. PROPOSITION (see [4] and [7]). *If E is the strict inductive limit of a family (E_i) of (AM)-spaces such that E induces the given topology on E_i for all i, then every L^1-continuous positive definite function on E_+ is the Laplace transform of an abstract measure μ on E'_+ (with topology $\sigma(E', E)$). μ is a Radon measure on E'_+ if E' is a Suslin space. This is the case for example, if E is the space $\mathscr{K}(S)$ of all continuous functions with compact support on a locally compact second countable space S, and therefore $E' = M(S)$ is the space of all Radon measures on S with the vague topology.*

3. Cone radonifying maps and the validity of the Bochner theorem.

In the following let E always denote a Banach lattice with positive cone C. A cylindrical measure μ on E is called to be *of type p* $(0<p<\infty)$, if one has for a certain constant $k \geqq 0$ and for all $x' \in E'$

$$\int |t|^p \, d\mu_{x'}(t) \leqq k\|x'\|^p .$$

We set

$$\tau^p(\mu) := \sup_{\|x'\| \leqq 1} \left(\int |t|^p \, d\mu_{x'}(t) \right)^{1/p} .$$

μ is said to be *approximately of type p* if μ is the cylindrical limit of a net $(\mu_i)_{i \in I}$ of measures μ_i with finite support and uniformly bounded type, i.e. $\sup_{i \in I} \tau^p(\mu_i) < \infty$. $\mu \in P(E_\sigma)$ is said to be *of order p*, if

$$\omega^p(\mu) := \left(\int \|x\|^p \, d\mu(x) \right)^{1/p} < \infty .$$

For a cylindrical measure on the cone C the inequality

$$\int t^p \, d\mu_{|x'|}(t) = \int (t_1 + t_2)^p \, d\mu_{x'_+, x'_-}(t_1, t_2)$$

$$\geqq \int |t_1 - t_2|^p \, d\mu_{x'_+, x'_-}(t_1, t_2) = \int |t|^p \, d\mu_{x'}(t)$$

shows that

$$\tau^p(\mu) = \sup_{x' \in B'_+} \left(\int_{\mathbf{R}_+} t^p \, d\mu_{x'}(t) \right)^{1/p}$$

($B'_+ :=$ the positive part of the unit ball B' of E').

Finally, $\mu \in \check{P}(C)$ is said to be *cone approximately of type p* if μ is approximately of type p and if the approximating measures are all concentrated on C.

A weakly continuous linear map u from E into a Banach space F is called (*approximately*) *cone- p-radonifying* (or in the case $p=1$ simply (approximately) cone radonifying) if u maps every cylindrical measure on C, which is (cone approximately) of type p, onto a Radon measure on F_σ of order p. Suppose that E' possesses the positive metric approximation property, i.e. the identity $\mathrm{id}_{E'}$ can be approximated in the topology of compact convergence by *positive* linear maps u_i: $E' \to E'$ of finite rank and norm $\leqq 1$. Then every $\mu \in \check{P}(C)$ of type p is also cone approximately of type p. Consequently, every approximately cone p-radonifying linear map is cone p-radonifying. The proof of this fact is essentially the same as the proof of the analogous result for general cylindrical measures on Banach spaces with the metric approximation property (see [6] or [10] as general references on this topic).

For a sequence $a=(a_n)$ in C and a sequence $b=(b_n)$ in a Banach space F we set

$$\|a\|_{s,p} := \sup\left\{\left(\sum_{n\in\mathbb{N}} \langle a_n, x'\rangle^p\right)^{1/p} : x' \in B'_+\right\}$$

$$\|b\|_p := \left(\sum_{n\in\mathbb{N}} \|b_n\|^p\right)^{1/p},$$

and call a linear map $u\colon E \to F$ *cone p-absolutely summing* if there is a constant $k \geqq 0$ such that for all sequences $a=(a_n)$ in C one has $\|u(a)\|_p \leqq k\|a\|_{s,p}$. One has the following connection between cone radonifying and cone absolutely summing maps.

3.1. THEOREM. *Let E be a Banach lattice and F be a Banach space with weakly compact unit ball. Let further u be a weakly continuous linear map from E into F. Then u is approximately cone p-radonifying* $(0<p<\infty)$ *if and only if u is cone p-absolutely summing.*

We omit the proof, since it follows essentially the lines of the proof of L. Schwartz of the analogous theorem about the connection between radonifying and absolutely summing maps (see [10]).

3.2. COROLLARY. *Let E be a Banach lattice and F be a Banach space. Then one has the following assertions*:

(i) *If F is the dual of a Banach space G and if $u \in \mathscr{L}(E_\sigma, (F, \sigma(F,G)))$, then u is approximately cone p-radonifying if and only if u is cone p-absolutely summing.*
(ii) *If $u \in \mathscr{L}(E_\sigma, F_\sigma)$, then u is cone p-absolutely summing if and only if u as a mapping into F'' is approximately cone p-radonifying.*
(iii) *If E' possesses the positive metric approximation property, then $u \in \mathscr{L}(E_\sigma, F_\sigma)$ is cone p-absolutely summing if and only if $u\colon E \to F''$ is cone p-radonifying.*

3.3. COROLLARY. *The following assertions are equivalent*:

(i) *E is an* (AL)*-space,*
(ii) *every cylindrical measure on C of type* 1 *is already a Radon measure on C'' (with topology $\sigma(E'', E')$) of order* 1.

Proof. By [9; ch. IV, theorem 2.7] the identity on a Banach lattice E is cone absolutely summing if and only if E is an (AL)-space. Now the assertion follows from the above corollary.

3.4. Corollary. *E is an* (AM)*-space if and only if every $\mu \in \check{P}(C')$ of type* 1 *is already a Radon measure of order* 1 *on C' with respect to the topology $\sigma(E', E)$.*

The idea of the next theorem is due to Kwapien, who proved the analogous result for radonifying maps on Banach spaces (see [10, p. 225]).

3.5. Theorem. *Let $u\colon E \to G$ be a weakly continuous positive linear map from the Banach lattice E into the dual lattice G of a Banach lattice F. If u transforms every $\mu \in \check{P}(C)$, which is cone approximately of type p, into a Radon measure $u(\mu)$ on the positive cone D of G (with topology $\sigma(G, F)$), then u is already approximately cone p-radonifying.*

Proof. The proof uses the representation of cylindrical measures as linear random functionals. Let us denote by $F(C)$ the set of all probability measures on C with finite support, and let $(\Omega, \mathscr{A}, P)$ be a fixed probability space which is diffuse. Then we can imbed $F(C)$ into $\mathscr{L}(E', L^p(\Omega))$ in the following way:

$$\lambda = \sum_{i=1}^{n} a_i \varepsilon_{x_i} \in F(C)$$

is mapped onto

$$f_\lambda = \sum_{i=1}^{n} 1_{A_i} \otimes x_i ,$$

where $(A_i)_{1 \leq i \leq n}$ is a decomposition of Ω with $P(A_i) = a_i$ and

$$f_\lambda(x')(\omega) := \langle x_i, x' \rangle \qquad \text{for } \omega \in A_i .$$

Every f_λ ($\lambda \in F(C)$) is positive and one has $\tau^p(\lambda) = \|f_\lambda\|$. Thus the image of $F(C)$ in $\mathscr{L}(E', L^p(\Omega))$ can be identified with the set S_+ of all simple functions from Ω into C. Let $\bar{S}_+$ be the closure of S_+ with respect to the norm of $\mathscr{L}(E', L^p(\Omega))$. If $f\colon E' \to L^p(\Omega)$ is in $\bar{S}_+$, then the associated cylindrical measure λ_f is cone approximately of type p. By assumption about u the image $u(\lambda_f)$ is therefore a Radon measure on D. This implies that the linear random function $f \circ {}^t u$: $F \to L^0(\Omega)$ is decomposable (see [10, p. 168]), i.e. there exists a unique $g_f \in L^0(\Omega, D)$ with $\langle g_f, y \rangle = f({}^t u(y))$ for all $y \in F$. If

$$f = \sum_{i=1}^{n} 1_{A_i} \otimes x_i \in S_+$$

then

$$g_f = u \circ f = \sum_{i=1}^{n} 1_{A_i} \otimes u(x_i) .$$

The map $\bar{u}: \bar{S}_+ \to L^0(\Omega, D)$ defined by $\bar{u}(f) := g_f$ is positive homogeneous and additive. We show that $\bar{u}$ is continuous at zero, i.e. we show that for all $\varepsilon > 0$, $\alpha \in]0,1[$ there is a $\delta > 0$ such that $\|f\| < \delta$ $(f \in \bar{S}_+)$ implies

$$J_\alpha(\bar{u}(f)) := \inf\{t > 0 : P[\bar{u}(f) \geqq t] \leqq \alpha\} < \varepsilon .$$

Suppose that $\bar{u}$ is not continuous at zero. Then we can without loss of generality suppose that there is a sequence (f_n) in $\bar{S}_+$ with $\|f_n\| \leqq 1$ and $J_\alpha(\bar{u}(f_n)) \geqq n^3$. We set $f = \sum_{n \in \mathbb{N}} n^{-2} f_n$. Then $f \in \bar{S}_+$ and $n^{-2} f_n \leqq f$ for all $n \in \mathbb{N}$, and therefore $n^{-2}\bar{u}(f_n) \leqq \bar{u}(f)$. This implies $J_\alpha(\bar{u}(f)) \geqq n$ for all $n \in \mathbb{N}$. From this contradiction now the continuity of $\bar{u}$ at 0 follows.

The function Φ on $L^0(\Omega, D)$, defined by $\Phi(g) := \int \inf(1, \|g\|)\, dP$, is also continuous at zero. Therefore there exists a constant $c \geqq 0$ such that for all $f \in \bar{S}_+$ with $\|f\| \leqq c$ one has $\Phi(\bar{u}(f)) \leqq 1/2$.

We now show that u is cone p-absolutely summing. For this let $(a_k)_{1 \leqq k \leqq n}$ be an arbitrary sequence in C with $\|(a_k)\|_{s,p} \leqq c$. Let further $(p_k)_{1 \leqq k \leqq n}$ be a sequence of positive numbers with $\sum_{k=1}^{n} p_k = 1$, and let $(A_k)_{1 \leqq k \leqq n}$ denote a partition of Ω with $P(A_k) = p_k$. We set

$$f := \sum_{k=1}^{n} 1_{A_k} \otimes p_k^{-1/p} a_k .$$

Then $\|f\| = \|(a_k)\|_{s,p}$ and therefore

$$\Phi(\bar{u}(f)) = \sum_{k=1}^{n} p_k \inf(1, p_k^{-1/p} \|u(a_k)\|) \leqq 1/2 .$$

Now we set $p_k := \|u(a_k)\|^p / \|(u(a_k))\|_p^p$. Then one gets $\|(u(a_k))\|_p \leqq 1/2$, and hence u is cone p-absolutely summing.

From the above theorem we get the following characterization of (AM)-spaces.

3.6. THEOREM. *The following assertions are equivalent*:

(i) *E is an* (AM)-*space.*

(ii) *Every continuous positive definite function on C is the Laplace transform of a Radon measure on* $(C', \sigma(E', E))$.

PROOF. In the foregoing section we have already shown that (i) $\Rightarrow$ (ii).

Suppose now that (ii) is valid, and choose an arbitrary $\mu \in \check{P}(C')$ of type 1. Then for $\varphi = L\mu$ we have

$$\varphi(0) - \varphi(x) \leqq \int_{\mathbf{R}_+} t \, d\mu_x(t) \leqq k\|x\|$$

for all $x \in C$. This implies that φ is continuous. By assumption therefore μ is a Radon measure on C'. This means that the identity on E' transforms every cylindrical measure of type 1 into a Radon measure. Now theorem 3.5 yields that the identity is cone absolutely summing, and therefore E is an (AM)-space by corollary 3.4.

3.7. THEOREM. *Let E denote a locally convex vector lattice with positive cone C. Then the following assertions are equivalent:*

(i) *E is a C^b-lattice.*

(ii) *For every neighborhood U of 0 in E, such that p_U is a lattice seminorm, there exists a neighborhood $V \subset U$ of the same type, such that the canonical imbedding*

$$\chi_{U,V} \colon E'_{U^0} \to E'_{V^0}$$

is cone absolutely summing.

(iii) *Every equicontinuous family of positive definite functions on C is the set of the Laplace transforms of a family of measures on C', which is uniformly tight with respect to the system of all weakly compact and equicontinuous subsets of C'.*

PROOF. The proof of the implication (i) $\Rightarrow$ (iii) is contained in the proofs of theorem 2.5 and theorem 2.10. So we have still to show the implications (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i).

(iii) $\Rightarrow$ (ii): Let S_U denote the set of all finite sequences $(a'_k)_{1 \leqq k \leqq n}$ $(n \in \mathbf{N})$ in $C' \setminus \{0\}$ with

$$\sup_{x \in U \cap C} \sum_{k=1}^{n} \langle x, a'_k \rangle \leqq 1 .$$

We have to show that there is a neighborhood V such that $\sum_{k=1}^{n} p_{V^0}(a'_k) \leqq 1$ for all $(a'_k) \in S_U$. To every $a' = (a'_k)_{1 \leqq k \leqq n} \in S_U$ we define the measures

$$\mu_{a',p} := \sum_{k=1}^{n} p_k \varepsilon_{p_k^{-1} a'_k}$$

for all $p = (p_1, \ldots, p_n) \in \mathbf{R}_+^n$ with $p_k > 0$ $(k = 1, \ldots, n)$ and $\sum_{k=1}^{n} p_k = 1$. Let us denote this family by $M_{U,P}$. For $\mu \in M_{U,P}$ one computes $L\mu(0) - L\mu(x) \leqq p_U(x)$, i.e. the family $L(M_{U,P})$ of Laplace transform is equicontinuous. By assumption therefore there is a $V \subset U$ (neighborhood of 0, p_V lattice seminorm) such that for all $\mu \in M_{U,P}$ we have $\mu(V^0) > 0$. For a fixed $a' = (a'_k) \in S_U$ we define

$$p_k := p_{V^0}(a'_k) \Big/ \left(\sum_{j=1}^{n} p_{V^0}(a'_j) \right).$$

Then for the associated measure $\mu_{a',p}$ the condition $\mu_{a',p}(V^0) > 0$ implies that there is a k such that $p_k^{-1} a'_k \in V^0$. This gives

$$\sum_{k=1}^{n} p_{V^0}(a'_k) \leqq 1 .$$

Therefore $\chi_{U,V}$ is cone absolutely summing.

(ii) $\Rightarrow$ (i): From [9; theorem 3.8 of ch. IV] it follows that the adjoint map $\pi_{U,V}\colon \tilde{E}_V \to \tilde{E}_U$ of $\chi_{U,V}$ is majorizing (see [9] for the definition). Therefore by [9; IV, th. 3.4] $\pi_{U,V}$ possesses a factoring

$$\tilde{E}_V \xrightarrow{\psi_V} M \xrightarrow{\psi_U} \tilde{E}_U ,$$

where M is an (AM)-space, $\psi_V \in \mathscr{L}(\tilde{E}_V, M)$, $\psi_U \in \mathscr{L}(M, \tilde{E}_U)$ and ψ_U positive. Since U was arbitrary, this shows that E is a C^b-lattice.

The above theorem can be generalized as follows:

3.8. THEOREM. *Let E be a locally convex vector lattice and let F be an arbitrary locally convex vector space. Let further $\mathscr{U}$ ($\mathscr{V}$) denote a neighborhood base in E (F) of 0 consisting of closed convex circled neighborhoods, such that in addition for all $U \in \mathscr{U}$ the p_U are lattice seminorms. For $T \in \mathscr{L}(F, E)$ the following assertions are equivalent:*

(i) *T admits a factoring $F \xrightarrow{T_1} G \xrightarrow{T_2} E$, where $T_1 \in \mathscr{L}(F, G)$, $0 \leqq T_2 \in \mathscr{L}(G, E)$ and G is a C^b-lattice.*

(ii) *$S := {}^tT$ is cone absolutely summing in the following sense: To every $U \in \mathscr{U}$ there is a $V \in \mathscr{V}$ such that the following diagram commutes*

$$\begin{array}{ccc} E' & \xrightarrow{S} & F' \\ {\scriptstyle \chi_U}\uparrow & & \uparrow{\scriptstyle \chi_V} \\ E'_{U^0} & \xrightarrow[S_{U,V}]{} & F'_{V^0} \end{array},$$

where χ_U, χ_V are the canonical injections and $S_{U,V}$ is cone absolutely summing.

(iii) *For any equicontinuous set Φ of positive definite functions on E_+ the set $\Phi \circ T := \{\varphi \circ T : \varphi \in \Phi\}$ is the family of Laplace transforms of a uniformly tight* *family of Radon measures on F'_σ (tightness with respect to the equicontinuous subsets of F').*

The proof of (3.8) is similar to the proof of (3.7) and we omit it.

4. An application to the weak convergence of probability measures on a C^b-lattice.

In the following let E always denote a C^b-lattice with weak dual F. On F we consider the system $\mathscr{K}_s$ of all equicontinuous, closed, convex and circled sets K such that the Minkowski functional p_K is an (AL)-norm on E'_K. Then clearly the C^b-topology on E is exactly the topology of uniform convergence on $\mathscr{K}_s$, and from theorem 2.10 we get the following tightness criterion.

4.1. THEOREM. *Let $M \subset M^b_+(F_+)$ be a norm uniformly bounded family. Then the following assertions are equivalent:*

(i) *M is uniformly tight.*
(ii) *$L(M) := \{L\mu : \mu \in M\}$ is equicontinuous for a topology $\mathscr{S}$ with $\lambda^1(E,F) \leqq \mathscr{S} \leqq c^b(E,F)$.*
(iii) *For all $\varepsilon > 0$ there is a $K \in \mathscr{K}_s$ such that*

$$\sup_{\mu \in M} \int \inf(p_K, 1_F)\, d\mu < \varepsilon .$$

4.2. COROLLARY. *Let $(\mu_i)_{i\in I}$ be a net on $M^b_+(F_+)$ such that the net $(L\mu_i)_{i\in I}$ of Laplace transforms converges pointwise to a function φ, which is continuous at zero for a topology between the L^1- and the C^b-topology. Then there is a $\mu \in M^b_+(F_+)$ such that $L\mu = \varphi$ and $\lim \mu_i = \mu$ weakly.*

PROOF. The pointwise limit φ is again positive definite. Therefore the continuity of φ implies by (2.10) that there is a $\mu \in M^b_+(F_+)$ with $L\mu = \varphi$. The convergence now follows from the compactness of $\{\mu \in C^b(F_+)' : \|\mu\| \leqq c\}$ in the weak dual of $C^b(F_+)$.

4.3. LEMMA. *For a family $M \subset P(F_+)$ of Poisson measures (i.e. of probability measures μ of the form*

$$\mu = e(G) := e^{-\|G\|}\left(\sum_{k \geqq 0} \frac{1}{k!} G^k\right)$$

with $G \in M^b_+(F_+)$) the following assertions are equivalent:

(i) *M is uniformly tight.*

(ii) *For every* $\varepsilon > 0$ *there is a* $K \in \mathscr{K}_s$ *such that*

$$\sup_{\mu = e(G) \in M} \int \inf(p_K, 1_F)\, dG < \varepsilon .$$

PROOF. (i) $\Rightarrow$ (ii): For all $K \in \mathscr{K}_s$ we set $L_K(\mu) := \int e^{-p_K} d\mu$. For a Poisson measure $\mu = e(G)$ we then have

$$L_K(e(G)) = \exp\left(-\left(\|G\| - \int e^{-p_K} dG\right)\right).$$

From this we get for all $n \in \mathbf{N}$:

$$-\log L_K(e(G)^{1/n}) = 1/n\left(\|G\| - \int e^{-p_K} dG\right).$$

For $\|G\| \leqq n$ this implies (because of the inequality $t/2 \leqq 1 - e^{-t}$ valid for $0 \leqq t \leqq 1$)

$$1/n\left(\|G\| - \int e^{-p_K} dG\right) \leqq 2(1 - \exp(\log L_K(e(G)^{1/n})$$

or

$$\|G\| - \int e^{-p_K} dG \leqq 2n(1 - L_K(e(G))^{1/n} .$$

Now $1 - a \leqq \varepsilon$ $(0 \leqq a \leqq 1)$ always implies $n(1 - a^{1/n}) \leqq 2\varepsilon$. Therefore

$$\int \inf(p_K, 1_F)\, dG \leqq 2 \int (1 - e^{-p_K})\, dG \leqq \varepsilon$$

if $1 - L_K(e(G)) \leqq \varepsilon/8$, and this inequality shows that (i) $\Rightarrow$ (ii) is valid.

The implication (ii) $\Rightarrow$ (i) follows from the estimation

$$1 - L_K(e(G)) \leqq \|G\| - \int e^{-p_K} dG \leqq \int \inf(p_K, 1_F)\, dG .$$

For the following application let us recall that a family $(\alpha_{nj})_{n \in \mathbf{N}, 1 \leqq j \leqq k_n}$ of probability measures on a topological vector space is called an *infinitesimal system*, if for every (Borel) neighborhood U of 0 one has

$$\lim_{n \to \infty} \inf_{1 \leqq j \leqq k_n} \alpha_{nj}(U) = 1 .$$

4.4. THEOREM. *Let* (α_{nj}) *be an infinitesimal system of probability measures on* F_+. *Then the following assertion are valid*:

(i) *The sequence*

$$\left(\mathop{*}_{j=1}^{k_n} \alpha_{nj}\right)_{n\in\mathbf{N}}$$

is uniformly tight if and only if the sequence

$$\left(e\left(\sum_{j=1}^{k_n} \alpha_{nj}\right)\right)_{n\in\mathbf{N}}$$

is uniformly tight, and then both sequences possess the same limit points.

(ii) *The sequence*

$$\left(\mathop{*}_{j=1}^{k_n} \alpha_{nj}\right)_{n\in\mathbf{N}}$$

is uniformly tight and converges towards some (infinitely divisible) measure if and only if the following two conditions hold:

(1) *there is a* $K \in \mathscr{K}_s$ *such that*

$$\sup_{n\in\mathbf{N}} \sum_{j=1}^{k_n} \int \inf(p_K, 1_F)\, d\alpha_{nj} < \infty, \text{ and}$$

(2) *for all* $x \in E_+$ *the limit*

$$\lim_{n\to\infty} \sum_{j=1}^{k_n} (1 - L\alpha_{nj}(x)) \text{ exists .}$$

PROOF. (i) follows from the inequalities

$$1 - L\left(e\left(\sum_{j=1}^{k_n} \alpha_{nj}\right)\right) \leqq 1 - L\left(\mathop{*}_{j=1}^{k_n} \alpha_{nj}\right) \leqq \sum_{j=1}^{k_n} (1 - L\alpha_{nj}) .$$

(ii): If the sequence (μ_n) with $\mu_n := *_{j=1}^{k_n} \alpha_{nj}$ is uniformly tight and convergent, then by (i) the sequence (λ_n) with $\lambda_n := e(\sum_{j=1}^{k_n} \alpha_{nj})$ is also uniformly tight and convergent, and the conditions (1) and (2) follow with the aid of theorem 4.1. If conversely the conditions (1) and (2) are valid, then again by theorem 4.1. the sequence (λ_n) must be uniformly tight and convergent, and therefore by (i) also the sequence (μ_n).

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MATHEMATISCHES INSTITUT
DER UNIVERSITÄT TÜBINGEN
AUF DER MORGENSTELLE 10
D-74 TÜBINGEN
GERMANY