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Introduction.

Let R be a commutative noetherian ring and let M be a finitely generated R -module. In this paper we shall study the homology of the complex $X \otimes_R M$ where X is a differential, graded skew-commutative R -algebra of the form

$$X = R\langle V_1, \dots, V_m; dV_i = v_i \rangle,$$

that is an R -algebra obtained from R by successive adjunction of finitely many variables V_i killing cycles v_i in the sense of Tate [7]. If $H_0(X \otimes M)$ has finite length over R , then so has $H_i(X \otimes M)$ for all i and we can consider the Hilbert series of $H(X \otimes M)$, which we denote by $|H(X \otimes M)|$. We have reasons to conjecture that this Hilbert series is a rational function. The purpose of this paper is to prove the conjecture in two important cases. The fact that $|H(X \otimes M)|$ is rational in these cases has applications to the Poincaré series of certain local rings. Recall that the Poincaré series of a local ring R is the Hilbert series of the graded vectorspace $\text{Tor}^R(k, k)$, where k is the residue field of R . It is well known that it is possible to construct a minimal R -algebra resolution of k of the form

$$Y = R\langle V_1, V_2, \dots, dV_i = v_i \rangle \qquad \text{cf. [1]}$$

For some classes of rings there exists a sub- R -algebra X of Y satisfying

- (i) X is obtained from R by the adjunction of finitely many of the variables V_i ,
- (ii) X satisfies the assumptions of the cases 1 or 2 below,
- (iii) the Poincaré series of R can be written

$$P^R(t) = |X \otimes k|(t)[1 - t(|H(X)|(t) - 1)]^{-1}.$$

Now $|X \otimes k|(t)$ is easily seen to be a rational function, hence the rationality of $P^R(t)$ follows from the rationality of $|H(X)|(t)$. Besides Golod rings, examples of such rings can be found in [3] and [5, corollary 2.7]. Recently Löfwall has

found an example where the sub- R -algebra X is generated by variables of degree 1, 2 and 3.

Our method of proof is to define on $H(X \otimes M)$ a structure of a graded artinian module over a graded ring generated by certain derivations on X . The two cases considered are the following:

Case 1: R has prime characteristic. In this case $H(X \otimes M)$ is an artinian module over a negatively graded commutative noetherian ring; the rationality of the Hilbert series follows by a well known argument.

Case 2: No assumption is made on the characteristic, but the derivation associated with the adjunction of the variables are assumed to be extendable to derivations on X . In this case $H(X \otimes M)$ is considered as a module over a ring which is not necessarily commutative. To obtain the rationality of the Hilbert series in this case, we introduce the notion of π -rational modules.

Another application of the module structure introduced in the present paper has recently been obtained by Levin [4] in his study of the Yoneda Ext-algebra of a local ring.

Notation and basic definitions.

R always denotes a commutative noetherian ring with unit. The term "graded" means \mathbb{Z} -graded. If H is a graded object, H_i denotes the homogeneous component of degree i . Graded modules will be left modules which are concentrated in non-negative degrees, that is the homogeneous components of negative degrees are all zero. The term "submodule" of a graded module means graded submodule. Maps are assumed to be homogeneous, but not necessarily of degree zero. $\deg j$ denotes the degree of the map j .

Graded rings are not assumed to be commutative, but they do have identity element.

Let H be a graded module over the graded ring T . If each H_i is a T_0 -module of finite length $l(H_i)$, we say that the Hilbert series of H is defined. The Hilbert series of H is the power series

$$\sum_i l(H_i)t^i$$

It will be denoted by $|H|(t)$ or just $|H|$.

The term " R -algebra" will be used in the sense of Tate [7], i.e. a differential graded strictly skew-commutative algebra X over R , such that each X_i is a finitely generated R -module, $X_0 = R \cdot 1$ and $X_i = 0$ for $i < 0$. d denotes the differential on X . The R -algebra $R\langle V_1, \dots, V_m; dV_i = v_i \rangle$ will briefly be denoted by $R\langle V_1, \dots, V_m \rangle$ if the values of dV_i are irrelevant. We will always assume that

the variables have been adjoined according to increasing degrees, that is $\deg V_{i+1} \geq \deg V_i$ for all i . This assumption causes no loss of generality.

A derivation j on an R -algebra X is an R -linear map $j: X \rightarrow X$ commuting with the differential and satisfying

$$j(xy) = (-1)^{wq}j(x)y + xj(y)$$

where $w = \deg j$ and $y \in X_q$.

All tensor products are taken over R .

1. π -rational modules.

DEFINITION. Let $\pi = \pi(t)$ be a polynomial in $\mathbb{Z}[t]$ such that $\pi(0) = \pm 1$. A power series $q(t)$ in $\mathbb{Z}[[t]]$ will be called π -rational if there exists a polynomial $r(t)$ such that $q(t) = r(t)\pi(t)^{-1}$.

A graded module H will be called π -rational if the Hilbert series $|H|$ is defined, and for all graded submodules $N \subseteq H$, $|N|$ is π -rational. Observe that since $H_i = 0$ for $i < 0$, then for any integer n $t^n|H|(t)$ is π -rational if and only if $|H|(t)$ is π -rational.

LEMMA 1. Let $0 \rightarrow H' \rightarrow H \rightarrow H'' \rightarrow 0$ be an exact sequence of graded modules. Let $\pi = \pi(t)$ be a polynomial in $\mathbb{Z}[t]$. Then H is π -rational if and only if H' and H'' are π -rational.

PROOF. Assume that H' and H'' are π -rational. Let N be a submodule of H , then there exists an exact sequence

$$0 \rightarrow N' \xrightarrow{I} N \xrightarrow{J} N'' \rightarrow 0$$

where N' and N'' are submodules of H' and H'' respectively. According to the degrees of I and J there exists non-negative integers n', n and n'' such that

$$t^{n'}|N'| + t^{n''}|N''| = t^n|N|.$$

It follows that $t^n|N|$ is π -rational, hence so is $|N|$.

Conversely, assume that H is π -rational. It is easily seen that H' and H'' are π -rational.

LEMMA 2. Let $H \xrightarrow{I} H' \xrightarrow{J} H'$ be an exact sequence of graded modules over the graded ring G . Assume that I has degree zero and that J denotes the multiplication map $J(x) = Jx$ where J is an element in G of negative degree $-w$. Let $\pi = \pi(t)$ be a polynomial and assume that H is artinian and π -rational. Then H' is artinian and π' -rational where $\pi'(t) = (1 - t^w)\pi(t)$.

PROOF. Clearly $H/\text{Ker } I$ is artinian. By lemma 1 it is also π -rational. Hence we may assume without loss of generality that I is injective. It follows from lemma 1.2 in [2] that H' is artinian. It remains to show that it is π' -rational.

Let N be a submodule of H' . We have exact sequences

$$(1) \quad 0 \rightarrow K_i \hookrightarrow J^{i-1}N \rightarrow J^iN \rightarrow 0 \quad \text{for } i > 0.$$

Since H' is artinian, there exists an integer s such that

$$J^sN = J^{s+1}N.$$

So (1) yields and exact sequence

$$(2) \quad 0 \rightarrow K_{s+1} \hookrightarrow J^sN \xrightarrow{J} J^sN \rightarrow 0.$$

As a submodule of H , K_{s+1} is π -rational. It follows from (2) that $|J^sN|$ is π' -rational. Using (1) it follows by descending induction on i that $|N|$ is π' -rational.

2. The case of prime characteristic.

In this section p denotes a fixed prime number and the ring R is assumed to have characteristic p , that is $pR = 0$.

LEMMA 3. *Let X be an R -algebra and let j be a derivation on X of even degree. Then so is j^p .*

PROOF. Let x and y be homogeneous elements in X . Then we have

$$j^p(xy) = \sum \binom{p}{k} j^k(x)j^{p-k}(y) = j^p(x)y + xj^p(y).$$

THEOREM 1. *Let X be an R -algebra obtained from R by the adjunction of a finite set of variables. Let M be a finitely generated R -module such that $H_0(X \otimes M)$ has finite length over R . Then $H(X \otimes M)$ is a graded artinian module over a commutative noetherian ring. Moreover, the Hilbert series $|H(X \otimes M)|$ is a rational function.*

PROOF. Let T_1, \dots, T_n be the adjoined variables of degree 1, and S_1, \dots, S_r the variables of degree > 1 .

Put

$$X^0 = R\langle T_1, \dots, T_n \rangle,$$

and inductively

$$X^{q+1} = X^q\langle S^{q+1} \rangle \quad \text{for } q \geq 0.$$

We are going to construct commuting derivations f_1, \dots, f_m of even negative degree on X such that $H(X \otimes M)$ becomes a graded artinian module over the commutative, noetherian (negatively) graded ring $R[f_1, \dots, f_m]$. This suffices since the rationality of $|H(X \otimes M)|$ then follows by a well known argument. Cf. [2, lemma 1.3].

Let $J_q: X^q \rightarrow X^q$ be the canonical derivation associated with the adjunction of the variable S_q , $q = 1, \dots, r$. Cf. [1]. Let S_{q_1}, \dots, S_{q_m} be the variables of even degree. Let t be an integer so large that

$$2p^t > \deg dS_{q_i} \quad \text{for } i = 1, \dots, m.$$

Put

$$f_i = J_{q_i}^{p^t} \quad \text{for } i = 1, \dots, m.$$

By lemma 3 each f_i is a derivation on X^{q_i} . Clearly f_i has degree less than or equal to $-2p^t$. Hence f_i vanish on all elements of degree less than $2p^t$. Since all the killed cycles dS_q ($q = 1, \dots, r$) have degree less than $2p^t$, it is possible to extend f_i successively to a derivation f_i on X in such a way that

$$f_i(S_l^{(k)}) = 0 \quad \text{for all } l \neq q_i \text{ and all } k$$

(if $\deg S_l$ is odd $S_l^{(k)}$ shall mean S_l for $k = 1$ and 0 for $k > 1$). See the proof of Lemma in [1].

To see that f_i and f_j commute for all i and j , put

$$f_{ij} = f_i f_j - f_j f_i.$$

f_{ij} is itself a derivation on X . It is easily seen that f_{ij} vanish on the R -algebra generators $S_l^{(k)}$, hence f_{ij} vanish on all of X .

The maps f_1, \dots, f_r restrict to operators on X^q for all q . Since they are R -linear maps commuting with the differential on X^q , they operate on $H(X^q \otimes M)$ in the natural way. By induction on q we shall show that $H(X^q \otimes M)$ is artinian as a module over the ring $G := R[f_1, \dots, f_m]$. For $q = 0$ this is trivial since X^0 is a finite complex, in fact $X_t^0 = 0$ for $t > n$. Observe also that

$$l(H_0(X^q \otimes M)) < \infty \text{ implies } l(H_t(X^q \otimes M)) < \infty \text{ for all } t.$$

Before we do the induction step, let us recall some basic facts about adjunction of variables. To simplify the notation, put

$$X^{q*} := X^q \quad \text{if } \deg S_q \text{ is even}$$

$$X^{q*} := X^{q-1} \quad \text{if } \deg S_q \text{ is odd.}$$

Put $J := J_q$ and observe that $\text{Im } J = X^{q*}$, thus we have an exact sequence of R -free complexes

$$(3) \quad 0 \rightarrow X^{q-1} \hookrightarrow X^q \xrightarrow{J} X^{q*} \rightarrow 0.$$

For each i , the map f_i gives rise to a commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & X^{q-1} & \hookrightarrow & X^q & \xrightarrow{J} & X^{q^*} \rightarrow 0 \\ & & \downarrow f_i & & \downarrow f_i & & f_i \downarrow \\ 0 & \rightarrow & X^{q-1} & \hookrightarrow & X^q & \xrightarrow{J} & X^{q^*} \rightarrow 0 \end{array}$$

which shows that the maps in the sequence

$$(4) \quad H(X^{q-1} \otimes M) \rightarrow H(X^q \otimes M) \rightarrow H(X^{q^*} \otimes M)$$

induced from (3) are G -linear.

Now let $q > 0$, and assume that $H(X^{q-1} \otimes M)$ is artinian as a G -module. Let us first assume that $\deg S_q$ is odd. In this case we have $X^{q^*} = X^{q-1}$, so by (4), $H(X^q \otimes M)$ is artinian. Let us now assume that $\deg S_q$ is even. For all $s \geq 1$ we have an exact sequence of R -free complexes and G -linear maps

$$0 \rightarrow X^{q-1} \hookrightarrow J^{-s}(0) \rightarrow J^{-(s-1)}(0) \rightarrow 0$$

which gives rise to an exact sequence of G -modules

$$H(X^{q-1} \otimes M) \rightarrow H(J^{-s}(0) \otimes M) \rightarrow H(J^{-(s-1)}(0) \otimes M) .$$

By induction on s it follows that $H(J^{-s}(0) \otimes M)$ is artinian for all s . From the split-exact sequence

$$0 \rightarrow \text{Ker } f_i \hookrightarrow X^q \xrightarrow{f_i} X^q \rightarrow 0$$

we obtain an exact sequence of G -modules

$$H(\text{Ker } f_i \otimes M) \rightarrow H(X^q \otimes M) \xrightarrow{f_i} H(X^q \otimes M) .$$

Since $\text{Ker } f_i = J^{-p}(0)$, $H(\text{Ker } f_i \otimes M)$ is artinian. Since $f_i \in G$ it follows from lemma 2 that $H(X^q \otimes M)$ is artinian.

3. The case of extendable derivations.

We now return to arbitrary characteristic. As before let X be an R -algebra obtained from R by the adjunction of a finite set of variables. Let the sub- R -algebras X^q and X^{q^*} and the canonical derivation $J^q: X^q \rightarrow X^q$ be as in the proof of theorem 1. We will say that a derivation j on X respects divided powers if for each adjoined variable S of even degree and each $k \geq 1$ we have

$$j(S^{(k)}) = S^{(k-1)}j(S) .$$

Observe that if R contains the rationals, then any derivation respects divided powers.

THEOREM 2. *Assume that the canonical derivations J_{q_1}, \dots, J_{q_m} associated with the adjunction of variables of even degree can all be extended to derivations on X respecting divided powers. Let M be a finitely generated R -module such that $H_0(X \otimes M)$ has finite length over R . Then $H(X \otimes M)$ is an artinian module over the tensoralgebra $T = T_R(J_{q_1}, \dots, J_{q_m})$. Moreover, there exists a polynomial $f(t)$ with integral coefficients such that*

$$|H(X \otimes M)|(t) = f(t) \left[\prod_{i=1}^m (1 - t^{w_i}) \right]^{-1}$$

where $w_i = -\deg J_{q_i}$.

PROOF. For each $i = 1, \dots, m$ we fix an extension of J_{q_i} to a derivation on X which respects divided powers. By abuse of notation the extended derivation will also be denoted by J_{q_i} . Since it respects divided powers and since we have assumed that the variables have been adjoined according to increasing degrees, it is easily seen that J_{q_i} operates on every subalgebra X^q . Hence $H(X^q \otimes M)$ becomes a module over the tensor algebra T in the obvious way.

Fix i . Put $J = J_{q_i}$. It is straight forward to check that we have a commutative diagram for all q

$$\begin{array}{ccccccc} 0 & \rightarrow & X^{q-1} & \hookrightarrow & X^q & \xrightarrow{J_q} & X^{q*} \rightarrow 0 \\ & & \downarrow J & & \downarrow J & & \downarrow J \\ 0 & \rightarrow & X^{q-1} & \hookrightarrow & X^q & \xrightarrow{J_q} & X^{q*} \rightarrow 0 \end{array}$$

Since the rows are split-exact we obtain an exact sequence of graded T -modules

$$(5) \quad H(X^{q-1} \otimes M) \rightarrow H(X^q \otimes M) \rightarrow H(X^{q*} \otimes M).$$

By induction on q we will prove that $H(X^q \otimes M)$ is an artinian T -module which is π_q -rational, where

$$\pi_q(t) = \prod_J (1 - t^{-\deg J}),$$

J running through the set $\{J_{q_i} \mid q_i \leq q \text{ and } 1 \leq i \leq m\}$. If this set is empty, we take the product to be equal to 1.

We first consider the case $q=0$. We have $\pi_0=1$. X^0 is a finite complex with $l(H_i(X^0 \otimes M)) < \infty$ for all i . Hence $|H(X^0 \otimes M)|$ is polynomial. Clearly $H(X^0 \otimes M)$ is an artinian T -module which is 1-rational.

Now let $q \geq 1$ and assume that $H(X^{q-1} \otimes M)$ is artinian and π_{q-1} -rational. Let us first assume that $\deg S_q$ is odd. In this case (5) reads

$$H(X^{q-1} \otimes M) \rightarrow H(X^q \otimes M) \rightarrow H(X^{q-1} \otimes M).$$

Then clearly $H(X^q \otimes M)$ is artinian as a T -module. From lemma 1 it follows that it is also π_{q-1} -rational. Since no new variable of even degree has been adjoined, we have $\pi_q = \pi_{q-1}$.

We will now assume that $\deg S_q$ is even. In this case we have $q = q_i$ for suitable i . Now (5) reads

$$H(X^{q-1} \otimes M) \rightarrow H(X^q \otimes M) \rightarrow H(X^q \otimes M),$$

where the right hand map is left multiplication by the element J_{q_i} in T . Put $w = -\deg J_{q_i}$. It follows from lemma 2 that $H(X^q \otimes M)$ is artinian over T and $(1-t^w)\pi_{q-1}(t)$ -rational. Since

$$\pi_q(t) = (1-t^w)\pi_{q-1}(t)$$

it follows that $H(X^q \otimes M)$ is π_q -rational.

COROLLARY. *Let R, \mathfrak{m} be a local noetherian ring, and let $Y = R\langle V_1, V_2, \dots, V_m, \dots \rangle$ be a minimal R -algebra resolution of R/\mathfrak{m} as constructed in [1]. Put*

$$Y^m = R\langle V_1, \dots, V_m; dV_i = v_i \rangle$$

and let M be a finitely generated R -module. Then if the Hilbert series $|H(Y^m \otimes M)|$ is defined, it is a rational function.

PROOF. In [1] it is shown how the canonical derivations associated with the adjunction of the variables can be extended to derivations on Y^m . It is immediate from the construction of the extended derivations that they respect divided powers. Hence the corollary follows from theorem 2.

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