MATH. SCAND. 46 (1980), 95–102

A DUAL PROOF OF THE UPPER BOUND CONJECTURE FOR CONVEX POLYTOPES

AAGE BONDESEN and ARNE BRØNDSTED

1. Introduction.

In the paper [2], P. McMullen proved the following celebrated result:

THEOREM (P. McMullen). *There exist numbers* $f_k(v,d)$ *such that*

$$f_k(P) \leqq f_k(v,d), \qquad k=1,\ldots,d-1\ ,$$

for any simplicial d-polytope P with v vertices, and

$$f_k(P) = f_k(v,d), \qquad k=1,\ldots,d-1\ ,$$

for any simplicial neighbourly d-polytope P with v vertices.

(We use standard terminology, cf. [1], [3]. In particular, $f_k(P)$ denotes the number of k-faces of P.)

In this note we shall present a dual proof of the theorem. It was already pointed out by P. McMullen in [2] that his proof may be given a dual formulation. Our proof, however, is not just a straightforward dualization of the proof in [2], it also contains substantial simplifications. When comparing the two proofs one should note that at several points details have been omitted from the proof in [2].

In section 2 we shall review some basic facts about duality of convex polytopes. In particular, we shall give a dual formulation of the theorem to be proved. The proof follows in section 3.

2. Duality of convex polytopes.

Two d-polytopes P and Q are said to be *dual*, if there exists a one-to-one inclusion-reversing correspondence between the faces of P and the faces of Q. Then

$$\dim F + \dim G = d-1$$

Received May 16, 1979.

for corresponding faces F of P and G of Q; in particular, vertices of Q correspond to facets of P. Also,

$$f_k(P) = f_{d-k-1}(Q)$$

when P and Q are dual d-polytopes.

A d-polytope P is called *simplicial*, if all its proper faces are simplices; in fact, it suffices that all facets are simplices. Now, k-simplices can be characterized as k-polytopes with $k+1$ vertices, and therefore the duals of simplicial d-polytopes are characterized by the property that each $(d-k-1)$-face is contained in (and hence is the intersection of) exactly $k+1$ facets, $k=0,\ldots,d-1$; in fact, this holds if it holds for $k=d-1$. Such polytopes are called *simple*.

From what has been said above it is clear that the theorem of P. McMullen may be given the following equivalent formulation:

THEOREM (P. McMullen). *There exist numbers $f_k(v,d)$ such that*

$$f_k(Q) \leqq f_{d-k-1}(v,d), \qquad k=0,\ldots,d-2\,,$$

for any simple d-polytope Q with v facets, and

$$f_k(Q) = f_{d-k-1}(v,d), \qquad k=0,\ldots,d-2\,,$$

for any simple d-polytope Q with v facets which is the dual of a neighbourly polytope.

In the proof of the theorem we shall need some elementary facts about simple polytopes which we shall formulate here as propositions 1–4. It is understood that P and Q are dual d-polytopes, P simplicial and Q simple.

If F is a $(d-k-1)$-face of a $(d-1)$-simplex S, then F is contained in exactly k $(d-2)$-faces of S. Dually:

PROPOSITION 1. *Let G be a k-face of Q, $k=0,\ldots,d$, and let x be a vertex of G. Then there are exactly k edges in G containing x.*

If $F_1,\ldots,F_k$ are $(d-2)$-faces of a $(d-1)$-simplex S, then $F_1 \cap \ldots \cap F_k$ is a $(d-k-1)$-face of S, and $F_1,\ldots,F_k$ are the only $(d-2)$-faces of S containing $F_1 \cap \ldots \cap F_k$. Dually:

PROPOSITION 2. *Let $E_1,\ldots,E_k$ be edges in Q, $k=0,\ldots,d$, containing a common vertex x. Then the smallest face G of Q containing $E_1,\ldots,E_k$ has dimension k, and $E_1,\ldots,E_k$ are the only edges in G which contain x.*

If x is a vertex of a $(d-1)$-simplex S, then the number of edges in S containing x is $d-1$. Dually:

PROPOSITION 3. *Let G be a facet of Q. Then G is also a simple polytope.*

Consider the statement: Any k vertices of P are contained in a proper face of P. The dual of this is the following: Any k facets of Q have a non-empty intersection. Since P is simplicial, the statement about P is equivalent to the following: Any k vertices of P are the vertices of a proper face of P. By definition, this holds for all $k \leqq [d/2]$, if and only if P is neighbourly. Therefore, we have:

PROPOSITION 4. *The polytope P is neighbourly, if and only if any $[d/2]$ or fewer facets of Q have a non-empty intersection.*

3. Proof of the theorem.

We shall divide the proof into three sections. In section A we shall introduce certain numbers γ_k, $k=0,\ldots,d$, associated with a simple d-polytope Q, and we shall express the number of k-faces of Q by the numbers γ_j, $j=0,\ldots,[d/2]$,—cf. (7). In section B we shall obtain relations between the numbers γ_k for Q, and the corresponding numbers γ_k^X for facets X of Q,—cf. (9) and (10). Finally, in section C we shall combine the results of section A and B to obtain the desired conclusion.

A. Let Q be a simple d-polytope in $\mathbf{R}^d$. Let w be a non-zero vector in $\mathbf{R}^d$ such that

(1) no hyperplane in $\mathbf{R}^d$ with w as a normal contains more than one of the vertices of Q.

(The existence of w is clear.) The vector w induces an orientation of each edge of Q according to the following rule: An edge with endpoints x and y is oriented towards x (and away from y) when

$$\langle x,w\rangle > \langle y,w\rangle ,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product. Calling the direction of w the "down direction", this amounts to orienting the edges "downwards". The condition (1) ensures that all edges will be oriented.

For a vertex x of Q we next define the *in-degree* of x as the number of edges in Q which have x as an endpoint and are oriented towards x. Similarly, the *out-degree* of x is the number of edges in Q which have x as an endpoint and are oriented away from x. Taking $G=Q$ in proposition 1 we see that for each vertex x of Q,

(2) the sum of the in-degree of x and the out-degree of x equals d.

We shall also need the following definitions. A *k-star*, $k=0,\ldots,d$, is a vertex of Q together with a set of k edges in Q having the vertex x in common; this common vertex is called the *centre* of the k-star. A k-star whose edges are all oriented towards the centre is called a *k-in-star*, and a k-star whose edges are all oriented away from the centre is called a *k-out-star*.

We want to establish a one-to-one correspondence between the k-faces of Q and the k-in-stars in Q. Let G be a k-face of Q. Then each vertex of G is the centre of a unique k-star contained in G; this follows from proposition 1. The particular k-star in G whose centre is the "lowest" vertex of G is clearly a k-in-star. To reach the desired conclusion we shall show that conversely each k-in-star in Q is contained in a unique k-face G and the centre of the k-in-star is the "lowest" vertex of G. The first statement of proposition 2 tells that the smallest face G of Q containing the given k-in-star has dimension k; any other face containing the k-in-star must therefore have dimension $>k$. To see that the centre x of the k-in-star is the "lowest" vertex of G, let $[x,y_1],\ldots,[x,y_k]$ be the k edges forming the k-in-star. Then the points $y_1,\ldots,y_k$ are all "above" x, and therefore there exists a hyperplane H with w as normal which separates x from $y_1,\ldots,y_k$. Since $[x,y_1],\ldots,[x,y_k]$ are the only edges in G containing x, cf. the second statement of proposition 2, then H must in fact separate x from any other vertex of G, and therefore x is the "lowest" vertex of G.—From this we conclude that

(3) the number of k-faces of Q equals the number of k-in-stars in Q, $k=0,\ldots,d$.

For $k=0,\ldots,d$, let γ_k denote the number of vertices of Q whose in-degree equals k. Then clearly the total number of k-in-stars in Q is

$$\sum_{j=0}^{d}\binom{j}{k}\gamma_j .$$

Denoting the number of k-faces of Q by f_k, it then follows from (3) that we have

$$f_k = \sum_{j=0}^{d}\binom{j}{k}\gamma_j, \quad k=0,\ldots,d . \tag{4}$$

An important consequence of this is the following. The matrix of coefficients $\binom{j}{k}$, $j,k=0,\ldots,d$, in the "equations" (4) is a triangular matrix with 1's in the diagonal, and therefore one can "solve" the "equations", and express the γ_j's by the f_k's. This shows that although the definition of the numbers γ_j apparently depends on the particular choice of the vector w (satisfying (1)), then actually

(5) the γ_j's are independent of w .

Now, if the condition (1) holds for w, it also holds for $-w$. When one replaces w by $-w$, all orientations of the edges in Q are reversed; vertices having in-degree k with respect to w will have out-degree k, and hence in-degree $d-k$, with respect to $-w$, cf. (2). The number of vertices having in-degree k with respect to w therefore equals the number of vertices having in-degree $d-k$ with respect to $-w$. Bearing in mind (5), we see that

$$\gamma_k = \gamma_{d-k}, \quad k=0,\ldots,d . \tag{6}$$

Combining now (4) and (6) we obtain

$$f_k = \sum_{j=0}^{[d/2]} \left[\binom{j}{k} + (1-\delta(d,2j))\binom{d-j}{k}\right]\gamma_j, \quad k=0,\ldots,d , \tag{7}$$

where $\delta(\cdot,\cdot)$ denotes the Kronecker-symbol. Note that the coefficient of γ_j is non-negative.

B. Let X be a facet of a simple d-polytope Q in $\mathbf{R}^d$. Then X is a simple $(d-1)$-polytope by proposition 3, and therefore we have numbers γ_k^X, $k=0,\ldots,d-1$, associated with X in the same way as we have numbers γ_k associated with Q, cf. section A. In other words, γ_k^X is the number of vertices of X having in-degree k (with respect to any vector w satisfying (1) for X).

Let w be a vector in $\mathbf{R}^d$ such that (1) holds for Q, and consider the orientation of the edges of Q induced by w. For a vertex x of X, let the *relative in-degree* of x be the number of edges in X which contain x and are oriented towards x. Now, when (1) holds for Q, it also holds for X. Therefore, for any vertex x of X, the in-degree of x in X (with respect to the orientation of the edges in X induced by w) equals the relative in-degree of x (with respect to the orientation of the edges in Q induced by w). Hence,

(8) γ_k^X is the number of vertices of X having relative in-degree k.

Suppose that we choose w such that (1) holds and each vertex of Q not in X is "below" each vertex of X (which is clearly possible). Then the relative in-degree of a vertex x of X is simply the in-degree of x. By (8), this implies that

$$\gamma_k^X \leqq \gamma_k, \quad k=0,\ldots,d-1 . \tag{9}$$

Suppose that we have strict inequality in (9) for some k. Then there is a vertex x of Q not in X such that the in-degree of x is k. Therefore, the out-degree of x is $d-k$, cf. (2), and hence x is a vertex of a (unique) $(d-k)$-face F of

Q whose remaining vertices are all "below" x; this follows from proposition 2 as in the argument leading to (3). Since Q is simple, F is the intersection of k facets $X_1,\ldots,X_k$ of Q. Since F is disjoint from X, we see that the $k+1$ facets $X, X_1,\ldots,X_k$ have an empty intersection. Therefore, proposition 4 shows that

(10) $\gamma_k^X = \gamma_k$, $k=0,\ldots,[d/2]-1$, when Q is the dual of a neighbourly d-polytope.

C. Let Q be a simple d-polytope in $\mathbf{R}^d$, and let w be a vector in $\mathbf{R}^d$ such that (1) holds. By a *k-incidence* we shall mean a pair (X,x), where X is a facet of Q, and x is a vertex of X whose relative in-degree is k. We denote the total number of k-incidences in Q by I_k.

It follows from (8) that

(11) $$I_k = \sum \gamma_k^X, \quad k=0,\ldots,d-1 ,$$

where we sum over all facets X of Q. From (11) and (9) we obtain

(12) $$I_k \leqq v\cdot\gamma_k, \quad k=0,\ldots,d-1 ,$$

where v denotes the number of facets of Q. From (11) and (10) we obtain

(13) $I_k = v\cdot\gamma_k$, $k=0,\ldots,[d/2]-1$, when Q is the dual of a neighbourly polytope.

We shall next determine I_k by summing over the vertices of Q (rather than over the facets as we did above). Let x be a vertex of Q. By proposition 1 there are exactly d edges in Q containing x, and since Q is simple, there are also exactly d facets of Q containing x. Again by proposition 1, each of the d facets containing x contains $d-1$ of the d edges containing x. Hence, for each of the facets exactly one of the edges is not in the facet; we shall call this edge the *exterior edge* of the facet. Note that conversely each of the d edges containing x is the exterior edge of some facet containing x. Now, for a facet X containing the vertex x, the pair (X,x) is a k-incidence, if and only if one of the following two conditions holds:

(i) x has in-degree k in Q, and the exterior edge of X is oriented away from x;

(ii) x has in-degree $k+1$ in Q, and the exterior edge of X is oriented towards x.

If x has in-degree k, there are $d-k$ facets X such that (i) holds. If x has in-degree $k+1$, there are $k+1$ facets X such that (ii) holds. Therefore,

(14) $$I_k = (d-k)\gamma_k + (k+1)\gamma_{k+1}, \quad k=0,\ldots,d-1 .$$

Combining (12) and (14) we obtain

$$(15)\qquad \gamma_{k+1} \leqq \frac{v-d+k}{k+1}\gamma_k, \quad k=0,\ldots,d-1\ .$$

Since $\gamma_0=1$, cf. the argument leading to (3), it follows from (15) that

$$(16)\qquad \gamma_k \leqq \binom{v-d+k-1}{k}, \quad k=0,\ldots,d\ .$$

Similarly, combining (13) and (14) we obtain

$$(17)\quad \gamma_k = \binom{v-d+k-1}{k}, \quad k=0,\ldots,[d/2], \text{ when } Q \text{ is the dual of a}$$

neighbourly polytope .

Finally, it follows immediately from (7), (16) and (17) that the theorem holds with

$$f_{d-k-1}(v,d) = \sum_{j=0}^{[d/2]}\left[\binom{j}{k}+(1-\delta(d,2j))\binom{d-j}{k}\right]\binom{v-d+j-1}{j},$$

$$k=0,\ldots,d-2\ .$$

4. Concluding remarks.

By inspection of the proof in section 3, and in particular noting that the coefficient of γ_j in (7) is strictly positive for $k=0,\ldots,[d/2]$, one easily deduces the following:

If Q is a simple d-polytope with v facets which is not the dual of a neighbourly polytope, then $f_k(Q)<f_{d-k-1}(v,d)$ for $k=0,\ldots,[d/2]$.

This is actually the dual formulation of a supplementary statement of the theorem in [2]. The theorem in [2] contains two more supplements. In the dual formulation, they are as follows:

The inequality $f_k(Q)\leqq f_{d-k-1}(v,d)$, $k=0,\ldots,d-2$, holds for any d-polytope Q with v facets.

If Q is a non-simple d-polytope with v facets, then $f_k(Q)<f_{d-k-1}(v,d)$ for $k=0,\ldots,[d/2]$.

If desired, these two statements can of course also be proved in the dual setting.

REFERENCES

1. B. Grünbaum, *Convex polytopes*, John Wiley and Sons, London - New York - Sydney, 1967.
2. P. McMullen, *The maximum numbers of faces of a convex polytope*, Mathematika 17 (1970), 179–184.
3. P. McMullen and G. Shephard, *Convex polytopes and the upper bound conjecture*, Cambridge University Press, London - New York, 1971.

MATEMATISK INSTITUT
DANMARKS LÆRERHØJSKOLE, KØBENHAVN
DANMARK

AND

MATEMATISK INSTITUT
KØBENHAVNS UNIVERSITET
DANMARK