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A II_1 FACTOR ANTI-ISOMORPHIC TO ITSELF BUT WITHOUT INVOLUTORY ANTIAUTOMORPHISMS

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Abstract.

We construct a type II_1 factor $\mathscr{A}$ which is anti-isomorphic to itself but has no involutory antiautomorphisms. The proof uses an invariant $\varkappa(\theta)$ for elements θ in Connes' group $\chi(\mathscr{A})$. We use $\varkappa(\theta)$ to show that if $\mathscr{M}$ is a II_1 factor without non-trivial hypercentral sequences and θ is an element of $\chi(\mathscr{M})$, then $\gamma(\theta) = \pm 1$, where γ is Connes' invariant for elements of $\operatorname{Out}\mathscr{M}$. We give an example which shows that $\gamma(\theta)$ can be -1 for θ in $\chi(\mathscr{M})$.

1. Introduction.

An antiisomorphism of a von Neumann algebra is a vector space isomorphism $\Phi\colon \mathscr{M} \to \mathscr{M}$ with $\Phi(a^*) = \Phi(a)^*$ and $\Phi(ab) = \Phi(b)\Phi(a)$.

If $\mathscr{M}$ is a factor, there are many mathematical objects associated with $\mathscr{M}$, say $\mathcal{O}_{\mathscr{M}}$, satisfying the following type of theorem: "If $\psi\colon \mathcal{O}_{\mathscr{M}} \to \mathcal{O}_{\mathscr{N}}$ is an isomorphism preserving the structure in $\mathcal{O}_{\mathscr{M}}$, there is an isomorphism *or* antiisomorphism $\varphi\colon \mathscr{M} \to \mathscr{N}$ inducing ψ". Examples for $\mathcal{O}$ are: the projection lattice [7], the unitary group [7], the Lie algebra [12], the Jordan algebra [11] and the automorphism group [9]. It was important to know whether all von Neumann algebras were antiisomorphic to themselves or not. This question was answered in the negative by Connes in [2] and [3] where he gave first a type III factor and then a type II_1 factor which were not antiisomorphic to themselves.

If $\mathcal{O}$ is a functor as above, there will be an exact sequence $1 \to \operatorname{Aut}\mathscr{M} \to \operatorname{Aut}\mathcal{O}_{\mathscr{M}} \to H \to 1$, where H is $\mathbf{Z}_2$ when $\mathscr{M}$ is antiisomorphic to itself and 1 otherwise. It is natural to ask the question whether this exact sequence splits when $H = \mathbf{Z}_2$. This is equivalent to the question "Does $\mathscr{M}$ possess an involutory antiautomorphism?". In this paper we answer this question in the negative by exhibiting a II_1 factor $\mathscr{A}$ which is antiisomorphic to itself but has no involutory antiautomorphisms. The method we use is a development of [3] so we begin by fixing notation and explaining some terms used in [3].

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If Aut $\mathcal{M}$ is the automorphism group of a II_1 factor $\mathcal{M}$ (always supposed to have separable predual, trace τ and trace-norm $\|-\|_2$), give it the topology of pointwise $\|-\|_2$ convergence on $\mathcal{M}$. The subgroup Int $\mathcal{M}$ of inner automorphisms is not necessarily closed so let $\overline{\mathrm{Int}}\,\mathcal{M}$ be its closure. An automorphism $\psi \in \mathrm{Aut}\,\mathcal{M}$ is said to be *centrally trivial* if, given a $\|-\|$-bounded sequence (x_n) in $\mathcal{M}$ with $\|[x_n, y]\|_2 \to 0$ as $n \to \infty$ for every $y \in \mathcal{M}$ (i.e. a central sequence), we have $\lim_{n\to\infty} \|\psi(x_n) - x_n\|_2 = 0$. Centrally trivial automorphisms form a normal subgroup Ct $\mathcal{M}$ of Aut $\mathcal{M}$ and if ε: Aut $\mathcal{M}$ $\to$ Out $\mathcal{M}$ is the quotient map onto Aut $\mathcal{M}$/Int $\mathcal{M}$, we define $\chi(\mathcal{M}) = \varepsilon(\mathrm{Ct}\,\mathcal{M} \cap \overline{\mathrm{Int}}\,\mathcal{M})$. In [4] Connes shows that $\chi(\mathcal{M})$ is abelian. It is a powerful isomorphism invariant for $\mathcal{M}$ and can be just about any abelian group.

For the most familiar II_1 factors (e.g. the hyperfinite one and those associated with free groups), $\chi(\mathcal{M})$ is trivial. The same is true for tensor products (finite or infinite) of these factors. The easiest way to construct factors $\mathcal{M}$ with non-trivial $\chi(\mathcal{M})$ is using crossed products by actions of finite groups. This is the method we shall employ.

In section 2 we shall define a conjugacy (in Out $\mathcal{M}$) invariant $\varkappa(\theta)$ for elements θ of $\chi(\mathcal{M})$, which is a complex number of modulus 1 when $\mathcal{M}$ has no non-trivial hypercentral sequences (a central sequence (y_n) is called hypercentral when $\lim_{n\to\infty} \|[y_n, x_n]\|_2 = 0$ for every central sequence (x_n)). This invariant $\varkappa$ is sensitive to conjugation by antiautomorphisms and will be used to prove that $\mathcal{A}$ has no involutory antiautomorphisms. It is defined using the action (or rather the lack of action) of θ on central sequences. We could have defined it for arbitrary factors, but we prefer to keep technical simplicity by supposing that $\mathcal{M}$ is a II_1 factor without non-trivial hypercentral sequences.

The group $\chi(\mathcal{M})$, being a subgroup of Out $\mathcal{M}$, gives an example of a Q-kernel in the sense of [8] or [16]. Thus one can associate an element of $H^3(\chi(\mathcal{M}), \mathsf{T})$ with $\chi(\mathcal{M})$, where T is the circle group and the action of $\chi(\mathcal{M})$ on T is trivial. This is done as follows. For each element θ of $\chi(\mathcal{M})$ choose an $\alpha_\theta \in \mathrm{Aut}\,\mathcal{M}$ with $\varepsilon(\alpha_\theta) = \theta$. Once this choise is made, we have

$$\alpha_\theta \alpha_\nu = \mathrm{Ad}\, u(\theta, \nu) \alpha_{\theta\nu} \quad \text{for } \theta, \nu \in \chi(\mathcal{M})$$

and unitaries $u(\theta, \nu)$ *in* $\mathcal{M}$. Associativity implies

$$\mathrm{Ad}\,(u(\theta, \nu) u(\theta\nu, \eta)) = \mathrm{Ad}\,(\alpha_\theta(u(\nu, \eta)) u(\theta, \nu\eta)) .$$

Thus

$$u(\theta, \nu) u(\theta\nu, \eta) = \gamma(\theta, \nu, \eta) \alpha_\theta(u(\nu, \eta)) u(\theta, \nu\eta)$$

for some $\gamma(\theta, \nu, \eta) \in \mathsf{T}$. The function γ: $\chi(\mathcal{M}) \times \chi(\mathcal{M}) \times \chi(\mathcal{M}) \to \mathsf{T}$ is seen to be a 3-cocycle and its cohomology class $\Omega_{\mathcal{M}}$ in $H^3(\chi(\mathcal{M}), \mathsf{T})$ does not depend on the choices made. Hence there is a further invariant $\Omega_{\mathcal{M}}$ for $\mathcal{M}$.

It is not yet known which elements of $H^3(A,\mathrm{T})$ can occur as $\Omega_{\mathcal{M}}$ for $A = \chi(\mathcal{M})$ and some fixed abelian group A, but there are severe restrictions. For example, when $\chi(\mathcal{M})$ is cyclic, $\Omega_{\mathcal{M}}$ is just the invariant $\gamma(\theta)$ of θ for a generator θ of $\chi(\mathcal{M})$. The interplay between $\varkappa(\theta)$ and $\gamma(\theta)$ allows us to show that $\gamma(\theta) = \pm 1$, whereas a priori one might have thought that $\gamma(\theta)$ could be any nth root of unity. So in this case $\Omega_{\mathcal{M}}$ can take at most two values. It also follows that $\gamma(\theta)$ is insensitive to conjugation by antiautomorphisms and is useless for distinguishing between an algebra and its opposite algebra, at least for $\theta \in \chi(\mathcal{M})$.

In section 6 we give an example showing that $\gamma(\theta)$ can be -1 for $\theta \in \chi(\mathcal{M})$, and hence that $\Omega_{\mathcal{M}}$ can be non-zero.

A further word about central sequences and ultrafilters is in order. Let $\mathcal{I}_\infty$ be the ideal of $l^\infty(\mathrm{N}, \mathcal{M})$ consisting of all bounded sequences (x_n) with $\lim_{n\to\infty} \|x_n\|_2 = 0$. The quotient C*-algebra $l^\infty(\mathrm{N},\mathcal{M})/\mathcal{I}_\infty$ is written $\mathcal{M}^\infty$. Note that $\mathcal{M}$ is embedded in $\mathcal{M}^\infty$ as constant sequences. The commutant $\mathcal{M}' \cap \mathcal{M}^\infty$ is denoted $\mathcal{M}_\infty$ and is the algebra of central sequences. One may repeat this same process for a free ultrafilter ω on N instead of ∞ (i.e. the Frechet filter). One obtains algebras $\mathcal{M}^\omega$ with $\mathcal{M} \subset \mathcal{M}^\omega$ and $\mathcal{M}_\omega$ as $\mathcal{M}' \cap \mathcal{M}^\omega$. The advantage of an ultrafilter is that $\mathcal{M}^\omega$ and hence $\mathcal{M}_\omega$ are von Neumann algebras. Automorphisms α of $\mathcal{M}$ induce automorphisms of $\mathcal{M}^\infty$, $\mathcal{M}^\omega$, $\mathcal{M}_\infty$, and $\mathcal{M}_\omega$, which will all also be written α. If (x_n) is a bounded sequence in $\mathcal{M}$, $[x_n]$ will denote its image in $\mathcal{M}^\infty$ or $\mathcal{M}^\omega$. It is important to note that if $\alpha = \lim_{n\to\infty} \mathrm{Ad}\, u_n$ for $\alpha \in \mathrm{Aut}\,\mathcal{M}$ and unitaries u_n in $\mathcal{M}$, and if $U = [u_n]$, then $\alpha = \mathrm{Ad}\, U$ on $\mathcal{M} \subset \mathcal{M}^\infty$, i.e. if $X \in \mathcal{M} \subset \mathcal{M}^\infty$, $\alpha(X) = UXU^*$. The same for ω in the place of ∞.

If $\mathcal{M}$ is an algebra, $Z(\mathcal{M})$ will denote its centre. To say that a sequence (x_n) is hypercentral is just to say that $[x_n] \in Z(\mathcal{M}_\infty)$. To say that $\mathcal{M}$ has no non-trivial hypercentral sequences means that if $[x_n] \in Z(\mathcal{M}_\infty)$, then $[x_n] = [\lambda_n 1]$ for a bounded sequence of scalars $\lambda_n \in \mathrm{C}$. If ω is an ultrafilter, such a sequence has a unique limit an $n \to \omega$, so that "$\mathcal{M}$ has no non-trivial hypercentral sequences" is the same as saying "$\mathcal{M}_\omega$ is a factor" for some (and hence all) free ultrafilter ω on N.

To say that an automorphism α is centrally trivial is the same as saying $\alpha = \mathrm{id}$ on $\mathcal{M}_\infty$ (or $\mathcal{M}_\omega$).

2. An invariant for elements of $\chi(\mathcal{M})$.

In this section we shall define an invariant $\varkappa(\theta)$ for elements $\theta \in \chi(\mathcal{M})$ and examine its relationship with the invariant $\gamma(\theta)$ of [6]. We also show how $\varkappa(\theta)$ behaves under conjugation by antiautomorphisms. We develop the definition of $\varkappa$ in three lemmas.

LEMMA 2.1. *Let* $\alpha \in \overline{\mathrm{Int}}\,\mathcal{M} \cap \mathrm{Ct}\,\mathcal{M}$ *and let* $\alpha = \mathrm{Ad}\,U$ *for* $U \in \mathcal{M}^\infty$ *on* $\mathcal{M} \subset \mathcal{M}^\infty$. *Then* $U^*\alpha(U) \in Z(\mathcal{M}_\infty)$.

PROOF. Let $X \in \mathcal{M} \subset \mathcal{M}^\infty$. Then $X = \alpha(U^*XU) = \alpha(U^*)\alpha(X)\alpha(U) = \alpha(U^*)UXU^*\alpha(U)$. Thus $U^*\alpha(U) \in \mathcal{M}_\infty = \mathcal{M}' \cap \mathcal{M}^\infty$. Moreover if $Y \in \mathcal{M}_\infty$,

$$YU^*\alpha(U) \;=\; U^*(UYU^*)\alpha(U) \;=\; U^*\alpha(UYU^*U) \;=\; U^*\alpha(U)Y,$$

since $UYU^* \in \mathcal{M}_\infty$ and $\alpha \in \mathrm{Ct}\,\mathcal{M}$. Thus $YU^*\alpha(U) = U^*\alpha(U)Y$ for all $Y \in \mathcal{M}_\infty$, and $U^*\alpha(U) \in Z(\mathcal{M}_\infty)$.

LEMMA 2.2. *Let* α *and* U *be as in lemma* 2.1 *and suppose* $\mathcal{M}$ *has no non-trivial hypercentral sequences. Then there is a* $\lambda_\alpha \in \mathbf{T}$ *such that* $\alpha(U) = \lambda_\alpha U$. *This* λ_α *does not depend on the choice of* U *with* $\alpha = \mathrm{Ad}\,U$ *on* $\mathcal{M} \subset \mathcal{M}^\infty$.

PROOF. Since there are no hypercentral sequences, we know that there is a bounded sequence λ_n with $\lim_{n\to\infty} \|\lambda_n 1 - u_n^*\alpha(u_n)\|_2 = 0$, where (u_n) is a representing sequence for U. We shall show that λ_n converges by showing that it has at most one accumulation point.

Suppose λ and μ are two accumulation points for the sequence (λ_n). Then let (n_i) and (m_i) be infinite sequences with $\lim_{i\to\infty} \lambda_{n_i} = \lambda$ and $\lim_{i\to\infty} \lambda_{m_i} = \mu$. Let V and $W \in \mathcal{M}^\infty$ be defined by $V = [u_{n_i}]$ and $W = [u_{m_i}]$. Clearly $\alpha(V) = \lambda V$ and $\alpha(W) = \mu W$ so that $\alpha(VW^*) = \lambda\bar{\mu} VW^*$. But $\mathrm{Ad}\,V = \mathrm{Ad}\,W = \mathrm{Ad}\,U$ on $\mathcal{M} \subset \mathcal{M}^\infty$ so that $VW^* \in \mathcal{M}_\infty$. Since $\alpha = \mathrm{id}$ on $\mathcal{M}_\infty$, $\lambda\bar{\mu} = 1$ so $\lambda = \mu$.

The same argument shows that λ_α does not depend on U with $\alpha = \mathrm{Ad}\,U$ on $\mathcal{M} \subset \mathcal{M}_\infty$.

LEMMA 2.3. *If* α, U *and* $\mathcal{M}$ *are as in lemma* 2.2, *and* v *is a unitary in* $\mathcal{M}$, $\lambda_\alpha = \lambda_{\mathrm{Ad}\,v\alpha}$.

PROOF. We have $\mathrm{Ad}\,v\alpha = \mathrm{Ad}\,(vU)$ on $\mathcal{M} \subset \mathcal{M}^\infty$. By definition,

$$\lambda_{\mathrm{Ad}\,v\alpha} vU \;=\; \mathrm{Ad}\,v\alpha(vU) \;=\; v\alpha(vU)v^* \;=\; vUvU^*\alpha(U)v^* \;=\; \lambda_\alpha vU\,.$$

Hence $\lambda_\alpha = \lambda_{\mathrm{Ad}\,v\alpha}$.

DEFINITION 2.4. Let $\mathcal{M}$ be a II_1 factor without non-trivial hypercentral sequences and $\theta \in \chi(\mathcal{M})$. Then $\varkappa(\theta) = \lambda_\alpha$ for any α with $\varepsilon(\alpha) = \theta$. This means that if u_n are unitaries with $\alpha = \lim_{n\to\infty} \mathrm{Ad}\,u_n$, $\lim_{n\to\infty} \|\alpha(u_n) - \lambda_\alpha u_n\|_2 = 0$. (Such a sequence of unitaries exists since $\alpha \in \overline{\mathrm{Int}}\,\mathcal{M}$.)

Each automorphism $\beta \in \mathrm{Aut}\,\mathcal{M}$ determines an automorphism $\bar{\beta}$ of $\chi(\mathcal{M})$ by conjugation, i.e. $\bar{\beta}(\theta) = \varepsilon(\beta)\theta\varepsilon(\beta^{-1})$. The next lemma shows that $\varkappa$ is a conjugacy invariant.

LEMMA 2.5. *If $\beta \in \operatorname{Aut}\mathcal{M}$ and $\theta \in \chi(\mathcal{M})$, then $\varkappa(\theta)=\varkappa(\bar{\beta}(\theta))$.*

PROOF. If α is such that $\varepsilon(\alpha)=\theta$ and $\alpha=\operatorname{Ad} U$ for $U \in \mathcal{M}^\infty$ on $\mathcal{M} \subset \mathcal{M}^\infty$, then $\beta\alpha\beta^{-1}=\operatorname{Ad}\beta(U)$ on $\mathcal{M} \subset \mathcal{M}^\infty$, and $\beta\alpha\beta^{-1}(\beta(U))=\lambda_\alpha\beta(U)$.

Each antiautomorphism Φ of $\mathcal{M}$ determines an automorphism $\tilde{\Phi}$ of $\chi(\mathcal{M})$ by conjugation, i.e. $\tilde{\Phi}(\theta)=\varepsilon(\Phi\alpha\Phi^{-1})$ where $\varepsilon(\alpha)=\theta$.

LEMMA 2.6. *If Φ is an antiautomorphism of $\mathcal{M}$ and $\theta \in \chi(\mathcal{M})$, then $\overline{\varkappa(\theta)} =\varkappa(\tilde{\Phi}(\theta))$.*

PROOF. Exactly as for 2.5 exept that $\Phi\alpha\Phi^{-1}=\operatorname{Ad}(\Phi(U^*))$.

We will also be interested in $\varkappa(\theta^{-1})$.

LEMMA 2.7. *If $\theta \in \chi(\mathcal{M})$, $\varkappa(\theta^{-1})=\varkappa(\theta)$.*

PROOF. If $\varepsilon(\alpha)=\theta$ and $\alpha=\operatorname{Ad} U$ on $\mathcal{M} \subset \mathcal{M}^\infty$, $\alpha^{-1}=\operatorname{Ad} U^*$ and $\alpha^{-1}(U^*) =\lambda_\alpha U^*$.

Remember that for $\theta \in \operatorname{Out}\mathcal{M}$, $\gamma(\theta)$ is defined by $\alpha(v)=\gamma(\theta)v$ where $\varepsilon(\alpha)=\theta$ and $\alpha^n=\operatorname{Ad} v$ with $n=$period of θ.

LEMMA 2.8. *Let $\theta \in \chi(\mathcal{M})$ have period n and suppose $\mathcal{M}$ has no hypercentral non-trivial sequences. Then $\gamma(\theta)=\varkappa(\theta)^n$.*

PROOF. If $\varepsilon(\alpha)=\theta$, $\alpha=\operatorname{Ad} U$ on $\mathcal{M} \subset \mathcal{M}^\infty$ and $\alpha^n=\operatorname{Ad} v$ on $\mathcal{M}$, then $v^*U^n \in \mathcal{M}_\infty$ so $\alpha(v^*U^n)=v^*U^n$. But also $\alpha(v^*U^n)=\alpha(v^*)\varkappa(\theta)^n U^n$ $=\overline{\gamma(\theta)}\varkappa(\theta)^n v^*U^n$. Hence $\gamma(\theta)=\varkappa(\theta)^n$.

LEMMA 2.9. *With θ and $\mathcal{M}$ as in 2.8, $\gamma(\theta)=\varkappa(\theta)^{-n}$.*

PROOF. If $\varepsilon(\alpha)=\theta$, $\alpha=\operatorname{Ad} U$ on $\mathcal{M} \subset \mathcal{M}^\infty$ and $\alpha^n=\operatorname{Ad} v$,

$$\gamma(\theta)v = \alpha(v) = UvU^* = UvU^*v^*v = U\alpha^n(U^*)v = \varkappa(\theta)^{-n}v .$$

Thus $\gamma(\theta)=\varkappa(\theta)^{-n}$.

COROLLARY 2.10. *If θ and $\mathcal{M}$ are as in 2.8, $\gamma(\theta)=\pm 1$.*

We shall see in section 6 that $\gamma(\theta)$ can be -1.

REMARK 2.11. If ω is a free ultrafilter on $\mathbf{N}$, and $\operatorname{Ad} U=\alpha$ on $\mathcal{M}\subset\mathcal{M}^{\omega}$, then as above $U^*\alpha(U)\in Z(\mathcal{M}_{\omega})$. By hypothesis $Z(\mathcal{M}_{\omega})$ is just the scalars so that $\lambda_{\alpha}=U^*\alpha(U)$. This shows that if λ_{α} had been defined with an ultrafilter, the result does not depend on the ultrafilter.

3. Definition of $\mathcal{A}$.

Let F_{24} be the free group on the 24 generators g_i, $i=1,2,\ldots 24$. Let $\lambda(g_i)$ be the unitaries of the left regular representation of F_{24}, and let UF_{24} be the von Neumann algebra generated by the $\lambda(g_i)$. UF_{24} is a II_1 factor. For each 25th root μ of unity define the automorphisms $\zeta_{\mu}\colon UF_{24}\to UF_{24}$ by $\zeta_{\mu}(\lambda(g_i))=\mu^i\lambda(g_i)$. By [13], ζ_{μ} is outer if $\mu\neq 1$ and the period of ζ_{μ} is the order of μ as a root of unity. Also $\zeta_{\mu}\zeta_{\nu}=\zeta_{\mu\nu}$.

Let $\sigma=e^{2\pi i/25}$ and $\gamma=\sigma^5$. Let $\mathcal{P}$ be the crossed product $W^*(UF_{24},\mathbf{Z}_5)$, where $\mathbf{Z}_5$ acts on UF_{24} by $n(x)=\zeta_{\gamma^n}(x)$ for $n\in\mathbf{Z}_5$. Write the elements of $\mathcal{P}$ as sums of the form $\sum_{i=0}^{4}a_iu^i$ where $a_i\in UF_{24}$ and u is a unitary, $u^5=1$ and $\operatorname{Ad} u=\zeta_{\gamma}$ on UF_{24}. Define $r_5^{\bar\gamma}\in\operatorname{Aut}\mathcal{P}$ by

$$r_5^{\bar\gamma}\left(\sum_{i=0}^{4}a_iu^i\right)=\sum_{i=0}^{4}\gamma^{-i}\zeta_{\sigma}(a_i)u^i\,.$$

Then $(r_5^{\bar\gamma})^5=\operatorname{Ad} u$ and $r_5^{\bar\gamma}(u)=\bar\gamma u$. (This is just the construction of [8, theorem 2.1] in the cyclic case.)

Now choose an automorphism s_5^{γ} of the hyperfinite II_1 factor R with $(s_5^{\gamma})^5=\operatorname{Ad} w$ and $s_5^{\gamma}(w)=\gamma w$. (See [6, proposition 1.6].) Let $\mathcal{B}$ be the tensor product $\mathcal{P}\otimes R$ and define $r_5^{\bar\gamma}\otimes s_5^{\gamma}$ on $\mathcal{B}$. Then $(r_5^{\bar\gamma}\otimes s_5^{\gamma})^5=\operatorname{Ad}(u\otimes w)$ and $r_5^{\bar\gamma}\otimes s_5^{\gamma}(u\otimes w)=u\otimes w$ is in the centre of the fixed point algebra $\mathcal{L}$ of $r_5^{\bar\gamma}\otimes s_5^{\gamma}$. Choose a 5th root t of $(u\otimes w)^*$ in $Z(\mathcal{L})$ and let $\psi=\operatorname{Ad} t\ (r_5^{\bar\gamma}\otimes s_5^{\gamma})$. Then $\psi^5=\mathrm{id}$.

The von Neumann algebra $\mathcal{A}$ is the crossed product $W^*(\mathcal{B},\mathbf{Z}_5)$ where ψ determines the action of $\mathbf{Z}_5$ on $\mathcal{B}$. Since ψ^i is outer for $0<i<5$, $\mathcal{A}$ is a type II_1 factor.

Before going on to prove that $\mathcal{A}$ is antiisomorphic to itself, we prove a fact about $\mathcal{P}$ which allows us to control central sequences in $\mathcal{P}\otimes R$.

LEMMA 3.1. *There is a* $K>0$ *such that if* $x\in\mathcal{P}$ *satisfies* $\|[x,\lambda(g_k)]\|_2<\varepsilon$ *for* $k=1,2,\ldots 24$, *then* $\|x-\tau(x)\|_2<K\varepsilon$

Here τ denotes the trace both on UF_{24} and $\mathcal{P}$. Thus $\tau(\sum_{i=0}^{4}x_iu^i)=\tau(x_0)$.

PROOF. Write $x\in\mathcal{P}$ in the form $x=\sum_{i=0}^{4}x_iu^i$ with $x_i\in UF_{24}$. For $k=5$ and 10, $[\lambda(g_k),u]=0$ so that for these values of k,

$$\|[\lambda(g_k), x]\|_2^2 = \sum_{i=0}^{4} \|[x_i, \lambda(g_k)]\|_2^2 < \varepsilon^2 .$$

Thus for each i, $\|[x_i, \lambda(g_k)]\|_2 < \varepsilon$. Hence by [14, lemma 4.3.3], $\|x_i - \tau(x_i)\|_2 < 14\varepsilon$. But if $y = \sum_{i=0}^{4} \tau(x_i)u^i$, then $\|x-y\|_2 < \sqrt{5}\,14\varepsilon$, so that

$$\|[y, \lambda(g_k)]\|_2 < \sqrt{5}\,28\varepsilon + \varepsilon \quad \text{for } k = 1,2,\ldots 24 .$$

In particular

$$\|[y, \lambda(g_1)]\|_2^2 = \sum_{i=0}^{4} |\tau(x_i)|^2 |1-\gamma^i|^2 < (28\sqrt{5}+1)^2 \varepsilon^2$$

(since we know how u commutes with $\lambda(g_1)$: $u\lambda(g_1)u^* = \gamma\lambda(g_1)$). Thus for $i \neq 0$,

$$|\tau(x_i)| < ((28\sqrt{5}+1)/|1-\gamma|)\varepsilon$$

and

$$\begin{aligned}\|x-\tau(x)\|_2 &\leqq \|x-y\|_2 + \|y-\tau(x)\|_2 \\ &< 14\sqrt{5}\varepsilon + 2((28\sqrt{5}+1)/|1-\gamma|)\varepsilon .\end{aligned}$$

REMARK 3.2. Lemma 3.1 implies (by [1, lemma 2.11 and corollary 3.6]), that $\mathscr{P}$ is *full*, i.e. $\overline{\mathrm{Int}\,\mathscr{P}} = \mathrm{Int}\,\mathscr{P}$.

4. $\mathscr{A}$ is antiisomorphic to itself.

To prove this assertion we use the following special case of a result which must be known to many authors.

LEMMA 4.1. *Let $\mathscr{M}$ be a II_1 factor and G a (discrete) group of automorphisms of $\mathscr{M}$. If Ψ is an antiautomorphism of $\mathscr{M}$ such that $\Psi g \Psi^{-1} = \varrho(g)$ for some automorphism ϱ of G, then the formula*

$$\Phi\Big(\sum_{g\in G} a_g u_g\Big) = \sum_{g\in G} \varrho(g)^{-1}\Psi(a_g)u_{\varrho(g)^{-1}} = \sum_{g\in G} \Psi(g^{-1}(a_g))u_{\varrho(g)^{-1}}$$

defines an antiautomorphisms of $W^*(\mathscr{M}, G)$.

Here $\{u_g\}$ is the usual unitary representation in the crossed product implementing G on $\mathscr{M}$.

PROOF. Since Φ preserves the trace there is no problem extending it from finite sums to all of $W^*(\mathscr{M}, G)$. Thus we only need to verify the relations $\Phi(xy) = \Phi(y)\Phi(x)$ and $\Phi(x^*) = \Phi(x)^*$. By linearity it suffices to do this for a pair au_g and bu_h. Now

$$\Phi(au_g) = \varrho(g)^{-1}\Psi(a)u_{\varrho(g)^{-1}}$$
$$\Phi(bu_h) = \varrho(h)^{-1}\Psi(b)u_{\varrho(h)^{-1}}$$

so that

$$\begin{aligned}\Phi(bu_h)\Phi(au_g) &= \varrho(h)^{-1}\Psi(b)\varrho(h)^{-1}\varrho(g)^{-1}\Psi(a)u_{\varrho(gh)^{-1}}\\ &= \Psi(h^{-1}b)\Psi((gh)^{-1}a)u_{\varrho(gh)^{-1}}\\ &= \Psi((gh)^{-1}(ag(b)))u_{\varrho(gh)^{-1}}\\ &= \Phi(ag(b)u_{gh})\\ &= \Phi(au_g bu_h)\,.\end{aligned}$$

Moreover

$$\Phi(au_g)^* = u^*_{\varrho(g)^{-1}}\varrho(g)^{-1}\Psi(a^*) = \Psi(a^*)u_{\varrho(g)}$$

and

$$\Phi((au_g)^*) = \Phi(g^{-1}(a^*)u_g-1) = \Psi(a^*)u_{\varrho(g)}\,.$$

THEOREM 4.2. *$\mathscr{A}$ is antiisomorphic to itself.*

PROOF. We want to use lemma 4.1 so we begin by constructing appropriate antiautomorphisms of $\mathscr{P}$ and R.

Define the involutory antiautomorphism Δ of UF_{24} by $\Delta(\lambda(g))=\lambda(g)^{-1}$ for all $g\in F_{24}$; For a 25th root of unity μ, $\Delta\zeta_\mu\Delta^{-1}=\zeta_{\mu^{-1}}$. Now define the automorphism $\pi\colon UF_{24}\to UF_{24}$ by the permutation of the generators $\pi(\lambda(g_i))=\lambda(g_{-2i \bmod 25})$. One checks that $\pi^{-1}\zeta_\mu\pi=\zeta_{\mu^{-2}}$ so that $\Psi=\pi^{-1}\Delta$ is an antiautomorphism and $\Psi\zeta_\mu\Psi^{-1}=\zeta_{\mu^2}$. Thus by lemma 4.1, the formula

$$\Phi\left(\sum_{i=0}^{4} a_i u^i\right) = \sum_{i=0}^{4}\zeta_{\gamma^{-2i}}\Psi(a_i)u^{-2i}$$

defines an antiautomorphism Φ of $\mathscr{P}=W^*(UF_{24},\mathbf{Z}_5)$. Also

$$\begin{aligned}\Phi r_5^{\bar\gamma}\Phi^{-1}(au^i) &= \Phi r_5^{\bar\gamma}(\Psi^{-1}\zeta_{\gamma^{-i}}(a)u^{2i}) \quad \text{(see below)}\\ &= \Phi(\gamma^{-2i}\zeta_\sigma\Psi^{-1}\zeta_{\gamma^{-i}}(a)u^{2i})\\ &= \gamma^{-2i}\zeta_{\gamma^i}\Psi\zeta_\sigma\Psi^{-1}\zeta_{\gamma^{-i}}(a)u^i\\ &= \gamma^{-2i}\zeta_{\sigma^2}(a)u^i = (r_5^{\bar\gamma})^2(au^i)\,.\end{aligned}$$

Thus $\Phi r_5^{\bar\gamma}\Phi^{-1}=(r_5^{\bar\gamma})^2$.

(To calculate $\Phi^{-1}(au^i)$, note that $\Phi(au^i)=\zeta_{\gamma^{-2i}}\Psi(a)u^{-2i}$ so that

$$\Phi^{-1}(\zeta_{\gamma^{-2i}}\Psi(a)u^{-2i}) = au^i .$$

Put $a'=\zeta_{\gamma^{-2i}}\Psi(a)$ and $j=-2i$. Then $a=\Psi^{-1}\zeta_{\gamma^{2i}}(a')$ and $i=2j \bmod 5$. Thus $\Phi^{-1}(a'u^j)=\Psi^{-1}\zeta_{\gamma^{-j}}(a')u^{2j}$.)

Now by [6, theorem 1.11] there is an antiautomorphism Γ of R such that $\Gamma s_5^{\gamma}\Gamma^{-1}=(s_5^{\gamma})^2$.

Remember that $\mathscr{A}$ is the crossed product of $\mathscr{P}\otimes R$ by $\psi=\mathrm{Ad}\,t\ (r_5^{\bar\gamma}\otimes s_5^{\gamma})$. If we define the antiautomorphism $\Phi\otimes\Gamma$ of $\mathscr{P}\otimes R$, then

$$\Phi\otimes\Gamma(\mathrm{Ad}\,t(r_5^{\bar\gamma}\otimes s_5^{\bar\gamma}))\Phi^{-1}\otimes\Gamma^{-1} = \mathrm{Ad}(\Phi\otimes\Gamma(t^*))(r_5^{\bar\gamma}\otimes s_5^{\gamma})^2.$$

But this means that the automorphisms $(\Phi\otimes\Gamma)\psi(\Phi\otimes\Gamma)^{-1}$ and ψ^2 differ by an inner automorphism. By [6, corollary 2.6], there is an automorphism β of $\mathscr{P}\otimes R$ such that

$$\beta(\Phi\otimes\Gamma)\psi(\Phi\otimes\Gamma)^{-1}\beta^{-1} = \psi^2 .$$

Thus by lemma 4.1, the formula

$$\sum_{i=0}^{4} a_i z^i \mapsto \sum_{i=0}^{4} \psi^{-2i}\beta(\Phi\otimes\Gamma)(a_i)z^{-2i}$$

defines an antiautomorphism of $\mathscr{A}$ (here $a_i \in \mathscr{B}=\mathscr{P}\otimes R$ and z is a unitary with $z^5=1$ and $\mathrm{Ad}\,z=\psi$ on $\mathscr{B}$).

5. $\mathscr{A}$ possesses no involutory antiautomorphisms.

To prove the assertion of this section we shall calculate $\chi(\mathscr{A})$ and $\varkappa(\theta)$ for generator θ of $\chi(\mathscr{A})$ using the methods of [3]. We give a brief sketch of these methods.

Given a finite subgroup G of $\mathrm{Aut}\,\mathscr{M}$ ($\mathscr{M}$ a II_1 factor without non-trivial hypercentral sequences) with $G\cap\overline{\mathrm{Int}}\,\mathscr{M}=\{\mathrm{id}\}$, Connes defines $K=G\cap\mathrm{Ct}\,\mathscr{M}$ and $K^{\perp}$, the group of homomorphisms from G to T which vanish on K. Let $\mathscr{M}^G$ be the fixed point algebra of the group G and let Fint be the subgroup of $\mathrm{Aut}\,\mathscr{M}$ consisting of all those automorphisms of the form $\mathrm{Ad}\,x$ with $x \in \mathscr{M}^G$. Let $\overline{\mathrm{Fint}}$ be its closure in $\mathrm{Aut}\,\mathscr{M}$. Connes defines $L=\varepsilon(G\,\mathrm{Ct}\,\mathscr{M}\cap\overline{\mathrm{Fint}})$, a subgroup of $\mathrm{Out}\,\mathscr{M}$. He shows that there is an exact sequence

$$0 \to K^{\perp} \xrightarrow{\partial} \chi(W^*(\mathscr{M},G)) \xrightarrow{\Pi} L \to 0 .$$

The map ∂ comes from the dual action: write elements in the crossed product in the form $\sum_{g\in G} a_g u_g$ with $a_g \in \mathscr{M}$ and $\mathrm{Ad}\,u_g=g$ on $\mathscr{M}$. Then for each $\eta \in K^{\perp}$, $\eta\colon G \to \mathrm{T}$, define

$$\delta(\eta)\Bigl(\sum_{g\in G} a_g u_g\Bigr) = \sum_{g\in G} \eta(g)a_g u_g .$$

One checks that $\delta(\eta) \in \operatorname{Ct} \mathcal{N} \cap \overline{\operatorname{Int}} \mathcal{N}$ ($\mathcal{N}$ is $W^*(\mathcal{M}, G)$) and ∂ is defined by taking the quotient $\varepsilon \circ \delta$. The injectivity of ∂ comes from the fact that the dual action is outer.

The map $\Pi\colon \chi(\mathcal{N}) \to L$ is defined as follows. Given $\alpha \in \operatorname{Ct} \mathcal{N} \cap \overline{\operatorname{Int}} \mathcal{N}$, one can show, using the detailed knowledge of central sequences in $\mathcal{N}$ coming from the hypothesis $G \cap \overline{\operatorname{Int}} \mathcal{M} = \{\mathrm{id}\}$, that there is a unitary $v \in \mathcal{N}$ and a sequence u_n of unitaries in $\mathcal{M}^G$ such that $\alpha = \operatorname{Ad} v \lim_{n\to\infty} \operatorname{Ad} u_n$. Let $\psi_\alpha = (\operatorname{Ad} v^* \alpha)|_{\mathcal{M}}$. By construction $\psi_\alpha \in \overline{\operatorname{Fint}}$ and one checks that $\psi_\alpha \in G \operatorname{Ct} \mathcal{M}$ (use Galois theory on $\mathcal{M}_\omega$). The image of ψ_α in $\operatorname{Out} \mathcal{M}$ depends only on $\varepsilon(\alpha)$, so we may define a map $\Pi\colon \chi(\mathcal{N}) \to L$ by $\Pi(\theta) = \varepsilon(\psi_\alpha)$ with $\varepsilon(\alpha) = \theta$.

The surjectivity of Π involves constructing a set-theoretic section for Π. This will be the most important construction for this paper. It is done as follows: given $\mu \in L$, let $\alpha_\mu \in \overline{\operatorname{Fint}} \cap G \operatorname{Ct} \mathcal{M}$ be such that $\varepsilon(\alpha_\mu) = \mu$. This α_μ commutes with G since it is the limit of such automorphisms. Thus we may define an automorphism β_μ of $W^*(\mathcal{M}, G)$ by

$$\beta_\mu\left(\sum_{g \in G} a_g u_g\right) = \sum_{g \in G} \alpha_\mu(a_g) u_g .$$

One checks that $\beta_\mu \in \operatorname{Ct} \mathcal{N} \cap \overline{\operatorname{Int}} \mathcal{N}$ and it is clear that $\Pi(\varepsilon(\beta_\mu) = \mu$. Thus $\mu \mapsto \varepsilon(\beta_\mu)$ provides the required section.

Remark 5.1. A bonus of this description is that we can easily calculate $\varkappa(\varepsilon(\beta_\mu))$. For if u_n^μ are unitaries in $\mathcal{M}^G$ with $\lim_{n\to\infty} \operatorname{Ad} u_n^\mu = \alpha_\mu$ in $\operatorname{Aut} \mathcal{M}$, then $\beta_\mu = \lim_{n\to\infty} \operatorname{Ad} u_n^\mu$ in $\operatorname{Aut} \mathcal{M}$ so that we only need to calculate

$$\lim_{n\to\infty} (u_n^\mu)^* \beta_\mu(u_n^\mu) = \lim_{n\to\infty} (u_n^\mu)^* \alpha_\mu(u_n^\mu)$$

and this is $\varkappa(\varepsilon(\beta_\mu))$.

We want to show that if $\mathcal{A}$ and $\mathcal{B}$ are as in section 3, $\chi(\mathcal{A}) = \mathbf{Z}_{25}$ and if $\sigma = e^{2\pi i/25}$, a generator θ of $\chi(\mathcal{A})$ satisfies $\varkappa(\theta) = \bar{\sigma}$. To do this we whall show that $K = \mathbf{Z}_5$, $L = \mathbf{Z}_5$ and that a lifting to $\chi(\mathcal{A})$ of a generator of L satisfies $\theta^5 \neq 1$ and $\varkappa(\theta) = \bar{\sigma}$.

To apply [3, theorem 4] we need to know that $\mathcal{B}$ has no non-trival hypercentral sequences. Together with our lemma 3.1, [1, lemma 2.11] shows that all central sequences in $\mathcal{P} \otimes R$ come from R and it it is well known that R has no non-trivial hypercentral sequences. We also need to know that $G \cap \overline{\operatorname{Int}} \mathcal{B} = \{\mathrm{id}\}$. Here G is the group generated by $\psi = \operatorname{Ad} t\ (r_5^{\bar{1}} \otimes s_5^{\bar{1}})$. It suffices to show that $\psi \notin \overline{\operatorname{Int}} \mathcal{B}$. This follows immediately from [5, 3.3] and our remark 3.2. Thus we may apply [3, theorem 4] with impunity.

We need to know $K = G \cap \operatorname{Ct} \mathcal{B}$.

LEMMA 5.2. *The group* $K = G \cap \operatorname{Ct}\mathcal{B}$ is just the identity.

PROOF. It suffices to show that $\psi \notin \operatorname{Ct}\mathcal{B}$. Bu s_5^γ is not in $\operatorname{Ct} R$ (see [6]) and if (x_n) is central in R, $(1 \otimes x_n)$ is central in $\mathcal{P} \otimes R$ so that $r_5^{\bar\gamma} \otimes s_5^\gamma \notin \operatorname{Ct}\mathcal{B}$.

Next we determine L and a lifting of a generator.

LEMMA 5.3. *With notation as above,* $\varepsilon(G \operatorname{Ct}\mathcal{B} \cap \overline{\operatorname{Fint}}) \cong \mathbf{Z}_5$, *and a generator is*

$$\mu = \varepsilon(\mathrm{id} \otimes (\operatorname{Ad} v s_5^\gamma))$$

where v *is a unitary in* R *such that* $s_5^\gamma(v) = \sigma v$.

If $\beta_\mu \in \operatorname{Ct}\mathcal{A} \cap \overline{\operatorname{Int}}\,\mathcal{A}$ *is a described above, then* $\varkappa(\varepsilon(\beta_\mu)) = \bar\sigma$.

PROOF. Note first that such a v exists by [6, corollary 2.6].

We now want to show that $\mathrm{id} \otimes (\operatorname{Ad} v s_5^\gamma) \in G \operatorname{Ct}\mathcal{B} \cap \overline{\operatorname{Fint}}$. Consider first of all $G \operatorname{Ct}\mathcal{B}$: multiplication of $\mathrm{id} \otimes (\operatorname{Ad} v s_5^\gamma)$ by $\psi^{-1} = (\operatorname{Ad} t(r_5^{\bar\gamma} \otimes s_5^\gamma))^{-1}$ yields

$$\operatorname{Ad} t^*(1 \otimes v)((r_5^{\bar\gamma})^{-1} \otimes \mathrm{id})$$

which is certainly in $\operatorname{Ct}\mathcal{B}$. Thus $\mathrm{id} \otimes (\operatorname{Ad} v s_5^\gamma) \in G \operatorname{Ct}\mathcal{B}$. To show that

$$\mathrm{id} \otimes (\operatorname{Ad} v s_5^\gamma) \in \overline{\operatorname{Fint}}$$

we must exhibit a sequence $\{u_n\}$ of unitaries in $\mathcal{B}$ with $\psi(u_n) = u_n$ and

$$\mathrm{id} \otimes (\operatorname{Ad} v s_5^\gamma) = \lim_{n \to \infty} \operatorname{Ad} u_n .$$

We begin by showing the existence of a sequence x_n of unitaries in R with $s_5^\gamma = \lim_{n \to \infty} \operatorname{Ad} x_n$ and $s_5^\gamma(x_n) = \bar\sigma x_n$. Since $\overline{\operatorname{Int}}\, R = \operatorname{Aut} R$, there is a sequence y_n of unitaries of R such that $s_5^\gamma = \lim_{n \to \infty} \operatorname{Ad} y_n$. Choose a free ultrafilter ω on $\mathbf{N}$ and let $Y = [y_n] \in R^\omega$. By the same calculation as in lemma 2.1, $W = Y^* s_5^\gamma(Y) \in R_\omega$. Moreover

$$W s_5^\gamma(W) \ldots (s_5^\gamma)^4(W) = Y^*(s_5^\gamma)^5(Y) ,$$

and since $(s_5^\gamma)^5 = \operatorname{Ad} w$ with $s_5^\gamma(w) = Y w Y^* = \gamma$, we obtain

$$(\sigma W) s_5^\gamma(\sigma W) \ldots (s_5^\gamma)^4(\sigma W) = 1 ,$$

and by [6, corollary 2.6], $\sigma W = Z^* s_5^\gamma(Z)$ for $Z \in R_\omega$. If (z_n) is a representing sequence of unitaries for Z, then

$$\lim_{n \to \omega} \operatorname{Ad} y_n z_n^* = s_5^\gamma \qquad \text{(since (z_n) is ω-central)} ,$$

and

$$\lim_{n \to \omega} \| s_5^\gamma(y_n z_n^*) - \bar\sigma y_n z_n^* \|_2 = 0 .$$

By standard approximation arguments (e.g. [10, 3.5], we may assume $s_5^{\gamma}(y_n z_n^*) = \bar{\sigma} y_n z_n^*$. Putting $x_n = y_n z_n^*$ gives the required result.

Now put $u_n = v x_n$. Then $\operatorname{Ad} v s_5^{\gamma} = \lim_{n\to\omega} \operatorname{Ad} u_n$ with $s_s^{\gamma}(u_n) = u_n$, so that

$$r_5^{\bar{\gamma}} \otimes s_5^{\gamma}(1 \otimes u_n) = 1 \otimes u_n ,$$

and since t was chosen in the centre of the fixed point algebra for $r_5^{\bar{\gamma}} \otimes s_5^{\gamma}$,

$$\psi(1 \otimes u_n) = \operatorname{Ad} t(r_5^{\bar{\gamma}} \otimes s_5^{\gamma})(1 \otimes u_n) = 1 \otimes u_n .$$

Thus $1 \otimes u_n$ is the desired sequence and $\operatorname{id} \otimes (\operatorname{Ad} v s_5^{\gamma}) \in \overline{\operatorname{Fint}}$.

Let us now calculate $\varkappa(\varepsilon(\beta_\mu))$. Choosing a subsequence we may assume

$$\operatorname{id} \otimes (Ad\, v s_5^{\gamma}) = \lim_{n\to\infty} \operatorname{Ad} 1 \otimes u_n .$$

By remark 5.1, to calculate $\varkappa(\varepsilon(\beta_\mu))$ we need only calculate $(\operatorname{id} \otimes (\operatorname{Ad} v s_5^{\gamma}))(1 \otimes u_n)$. But

$$\operatorname{id} \otimes s_5^{\gamma}(1 \otimes u_n) = 1 \otimes u_n$$

so

$$\operatorname{id} \otimes (\operatorname{Ad} v s_5^{\gamma})(1 \otimes u_n) = 1 \otimes v u_n v^* .$$

Now $\lim_{n\to\infty} \operatorname{Ad} u_n = s_5^{\gamma}$ and $s_5^{\gamma}(v) = \sigma v$. Thus $\lim_{n\to\infty} u_n v^* u_n^* = \bar{\sigma} v^*$, and

$$\lim_{n\to\infty} \|\operatorname{id} \otimes (\operatorname{Ad} v s_5^{\gamma})(1 \otimes u_n) - \bar{\sigma} 1 \otimes u_n\|_2 = 0 .$$

By definition $\varkappa(\varepsilon(\beta_\mu)) = \bar{\sigma}$.

Thus far we know that $\varepsilon(\operatorname{id} \otimes (\operatorname{Ad} v s_5^{\gamma}))$ is an element (of period 5) of $L = (G \operatorname{Ct} \mathscr{B} \cap \overline{\operatorname{Fint}})$. We want to show that it generates L. This means that if $\varphi \in G \operatorname{Ct} \mathscr{B} \cap \overline{\operatorname{Fint}}$ then it differs by an inner automorphsism from a power of $\operatorname{id} \otimes (\operatorname{Ad} v s_5^{\gamma})$. By the same argument as in [2, theorem 3.2 a)], if $\alpha \in \operatorname{Ct} \mathscr{B}$, then $\alpha = \operatorname{Ad} z(\nu \otimes \operatorname{id})$ for some unitary $z \in \mathscr{B}$ and an automorphism ν of $\mathscr{P}$ (basically because otherwise α would act on central sequences coming from R). Thus since $\varphi \in G \operatorname{Ct} \mathscr{B}$, there is an $n \in \{0,1,2,3,4\}$ such that $\varphi \operatorname{Ad} t^n (r_5^{\bar{\gamma}} \otimes s_5^{\gamma})^n$ is of the form $\operatorname{Ad} z\ (\nu \otimes \operatorname{id})$. Solving for φ gives

$$\varphi = \operatorname{Ad} x(\nu (r_5^{\bar{\gamma}})^{-n} \otimes (s_5^{\gamma})^{-n})$$

for some $x \in \mathscr{B}$. Since $\varphi \in \overline{\operatorname{Int}} \mathscr{B}$ and $\mathscr{B}$ is full we may apply [5, corollary 3.3] to conclude that

$$\varphi = \operatorname{Ad} x' (\operatorname{id} \otimes (s_5^{\gamma})^{-n})$$

(or argue as in [2, 2.1]). Thus $\varepsilon(\varphi) = \mu^{-n}$, and μ generates L.

LEMMA 5.4. *We have* $\chi(\mathscr{A}) \cong \mathbf{Z}_{25}$ *and* $\Omega_{\mathscr{A}} = 0$.

PROOF. By 5.2, 5.3 and [3, theorem 4], we have an exact sequence

$$0 \to \mathbf{Z}_5 \to \chi(\mathscr{A}) \to \mathbf{Z}_5 \to 0 .$$

A lifting to $\chi(\mathscr{A})$ of a generator of $\mathbf{Z}_5$ is defined on $\mathscr{A} = W^*(\mathscr{B}, \mathbf{Z}_5)$ by $\varepsilon(\alpha)$ where

$$\alpha\left(\sum_{i=0}^{4} a_i z_i^i\right) = \sum_{i=0}^{4} \mathrm{id} \otimes (\mathrm{Ad}\, v s_5^{\gamma})(a_i) z^i .$$

Since 5 is a prime number, to prove that $\chi(\mathscr{A}) \cong \mathbf{Z}_{25}$, it suffices to show that $\varepsilon(\alpha)^5 \neq 1$, i.e. $\alpha^5 \notin \mathrm{Int}\,\mathscr{A}$.

By definition,

$$\alpha^5\left(\sum_{i=0}^{4} a_i z^i\right) = \sum_{i=0}^{4} \mathrm{Ad}\,(1 \otimes v^5 w)(a_i) z^i = \mathrm{Ad}\,(1 \otimes v^5 w)\left(\sum_{i=0}^{4} \gamma^{2i} a_i z^i\right).$$

To see this last step, remember that $r_5^{\bar{\gamma}} \otimes s_5^{\gamma}(1 \otimes v^5 w) = \sigma^5 \gamma (1 \otimes v^5 w)$, so that

$$\mathrm{Ad}\, t(1 \otimes v^5 w) = 1 \otimes v^5 w$$

and

$$\mathrm{Ad}\, t(r_5^{\bar{\gamma}} \otimes s_5^{\gamma})(1 \otimes v^5 w) = \gamma^2 (1 \otimes v^5 w) .$$

Also $\mathrm{Ad}\, z = \mathrm{Ad}\, t(r_5^{\bar{\gamma}} \otimes s_5^{\gamma})$ on $\mathscr{B}$. Thus α^5 is a dual action times an inner automorphism, which is outer. Hence $\chi(\mathscr{A}) \cong \mathbf{Z}_{25}$.

The H^3 obstruction $\Omega_{\mathscr{A}}$ is represented by $\gamma(\theta)$ for generator of $\chi(\mathscr{A})$. But from the above calculation, the period of θ is 25, and if $\theta = \varepsilon(\beta_m)$, we may suppose, by lemma 5.3, that $\varkappa(\theta) = \bar{\sigma}$. By lemma 2.7, $\gamma(\theta) = 1$, which means $\Omega_{\mathscr{A}} = 0$.

THEOREM 5.5. *$\mathscr{A}$ possesses no involutory antiautomorphism.*

PROOF. If Φ were such an antiautomorphism, conjugation by Φ would induce an automorphism of period 1 or 2 of $\chi(\mathscr{A})$. Since $\chi(\mathscr{A}) = \mathbf{Z}_{25}$, the only such automorphisms are the identity and the map $\theta \mapsto \theta^{-1}$. By lemmas 2.6 and 2.7, neither of these is possible since $\varkappa(\theta) = \bar{\sigma}$, and $\sigma \neq \bar{\sigma}$.

REMARK 5.6. Since everything happened in $\mathrm{Out}\,\mathscr{A}$, it follows from the above that $\mathscr{A}$ has no antiautomorphism whose square is an inner automorphism. This may also be deduced from [15, theorem 5.5 and theorem 4.4].

REMARK 5.7. It is clear that $\mathscr{A}$ is not the von Neumann algebra generated by the left regular representation of a discrete group. Thus a modernised version of [14, problem 4.4.30] would be: "If $\mathscr{M}$ is a II_1 factor with an involutory antiautomorphism, is there a discrete group G such that $\mathscr{M} = U(G)$?"

6. Example of a II_1 factor $\mathscr{M}$ with $\Omega_{\mathscr{M}} \neq 0$.

We have seen that for an element $\theta \in \chi(\mathscr{M})$, $\gamma(\theta) = \pm 1$ (if $\mathscr{M}$ has no non-trivial hypercentral sequences). We shall give an example of such a factor with $\chi(\mathscr{M}) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ and an element $\theta \in \chi(\mathscr{M})$ with $\gamma(\theta) = -1$. This also implies $\Omega_{\mathscr{M}} \neq 0$.

The example is obtained by replacing 5 by 2 and 24 by 3 in the above construction of $\mathscr{A}$ and $\mathscr{B}$. (If one doesn't believe lemma 3.1 any more, just add a few dummy generators to F_3.) Now γ will be -1 and σ will be i. All the calculations work in the same way up to 5.4. Thus we obtain a $\theta \in \chi(\mathscr{A})$ lifting a generator of $\varepsilon(\mathrm{Ct}\,\mathscr{B} \cap \overline{\mathrm{Fint}}) \cong \mathbf{Z}_2$ with $\varkappa(\theta) = i$. But in the calculation of 5.4, we notice that

$$\alpha^2\left(\sum_{i=0}^{2} a_i z^i\right) = \mathrm{Ad}\,(1 \otimes v^2 w)\left(\sum_{i=0}^{2} \gamma^{2i} a_i z^i\right).$$

And now $\gamma^{2i} = 1$, so that $\varepsilon(\alpha)^2 = 1$. Thus the sequence $0 \to \mathbf{Z}_2 \to \chi(\mathscr{A}) \to \mathbf{Z}_2 \to 0$ is split, and since $\theta^2 = 1$ and $\varkappa(\theta) = i$, by 2.8. $\gamma(\theta) = -1$.

That $\mathscr{A}$ has no non-trivial hypercentral sequences follows from the fact that $\mathscr{B}$ has none and the control over central sequences given by the hypothesis $G \cap \overline{\mathrm{Int}\,\mathscr{B}} = \mathrm{id}$ (see [5, theorem 3.1]).

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