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A SPLITTING LEMMA FOR NON-REFLEXIVE BANACH SPACES

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The original splitting lemma is a result of Gromoll and Meyer [3], about germs of smooth functions on a Hilbert space H. It asserts that if $f\colon H \to \mathbf{R}$ is a smooth function, and if $f'(0)=0$, and $f''(0)$ is a Fredholm operator T from H into H^*, then in a neighbourhood of 0 there is a smooth change of coordinates in H which transforms f into a function on $\ker T$ plus the quadratic form $1/2\langle Tx, x\rangle$, which is non-degenerate on $(\ker T)^\perp$. This has been generalized by adding parameters and replacing the Hilbert space by a reflexive Banach space [4]. The lemma and its generalisations are useful for extending the theory of generic/versal unfoldings to a Banach space (see [1], [2], [4], [5], [6]). However, a reflexive Banach space other than a Hilbert space occurs in few cases, mainly because there are no natural non-Hilbert Banach spaces which are isomorphic to their duals. (An example is $L^p(\Omega)\times L^q(\Omega)$ for a measure space Ω where $(1/p)+(1/q)=1$ and $p \neq q$; not a convenient example for applications). We shall present a splitting lemma which can be used in non-reflexive spaces, and a particular benefit will be the possibility of using spaces of continuous or bounded measurable functions, instead of the integrable functions.

We let X be a Banach space and suppose we have another Banach space Y and a continuous linear operator $\sigma\colon Y \to X^*$. We define M^σ to be the set of germs at 0 of C^∞ mappings $f\colon X \to \mathbf{R}$ such that $f'\colon X \to X^*$ is a composition

$$f' = \sigma h ,$$

where $h\colon X \to Y$ is a C^∞ mapping and $h(0)=0$. If A is another Banach space we define U_A^σ to be the set of germs at $(0,0)$ of C^∞ mappings $f\colon X\times A \to \mathbf{R}$ such that $D_1 f$ is a composition

$$D_1 f = \sigma h$$

where $h\colon X\times A \to Y$ is a C^∞ mapping and $h(0,0)=0$. These are the unfoldings which occur in the theory.

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The main theorem is as follows; note that the same symbol is used for a germ as for a representative mapping.

THEOREM 1. *Let* $f \in U_A^\sigma$, *let* $D_1 f = \sigma h$, *let* $D_1 h(0,0) = T$ *and let* $V = \ker T$. *Suppose that* TX *is closed in* Y *and that there exist closed subspaces* $Z \subset X$, $W \subset Y$ *such that* $X = V \oplus Z$, $Y = TX \oplus W$ *and such that* σW *annihilates* Z. *Then* f *is equivalent to an unfolding* $(x, a) \mapsto 1/2\langle \sigma Tz, z\rangle + g(v, a)$ *where* $x = v + z$, $v \in V$, $z \in Z$, g *is a real-valued* C^∞ *mapping defined on a neighbourhood of* $(0,0) \in V \times A$, *and* $\langle .,. \rangle$ *is the bilinear pairing of* X^* *and* X. *The equivalence is realised by a diffeomorphism*

$$\varphi_{(v,a)} : Z, 0 \to Z, 0$$

where $(z, v, a) \mapsto \varphi_{(v,a)}(z)$ *is* C^∞ *and*

$$f(\varphi_{(v,a)}(z) + v, a) = 1/2\langle \sigma Tz, z\rangle + g(v, a) .$$

If, for some $a \in A$, $D_1 f(v, a)$ *annihilates* Z *for every* v *in a neighbourhood of* $0 \in V$, *then* $g(\cdot, a)$ *is the restriction of* $f(\cdot, a)$ *to* V.

COROLLARY 1. (Morse lemma). *Let* $f \in M^\sigma$, *let* $f' = \sigma h$ *and let* $h'(0) = T$. *If* T *is an isomorphism of* X *onto a subspace of* Y, *and* TX *has a complement* W *such that* σW *annihilates* X, *then* f *is equivalent to the germ at* 0 *of* $x \mapsto 1/2\langle \sigma Tx, x\rangle + f(0)$.

The proof of Theorem 1 is really just the same as the proof of Theorem 1 of [4], except that everything is suitably "factorized" through the space Y. This factorisation will be explained in some detail, while the rest of the proof, based on the "flow" technique, will be more or less sketched.

LEMMA 1. *Let* Z, U, B, *be Banach spaces and* $d: Z \times B \to U$ *a* C^∞ *mapping such that* $D_1 d(0,0)$ *is an isomorphism of* Z *onto* U. *Let* χ *be a continuous linear mapping of* U *into* Z^* *and let* $\{\cdot,\cdot\}$ *be the pairing of* Z^* *and* Z. *Suppose that* $\chi D_1 d(z, b)$ *is symmetric for each* $z \in Z$ *and* $b \in B$, *that is,*

$$\{\chi D_1 d(z,b) z_1, z_2\} = \{\chi D_1 d(z,b) z_2, z_1\}$$

for all z_1, z_2 *in* Z. *Then given a smooth* $f: Z \times B \to \mathbf{R}$ *there exists a smooth* $k: Z \times B \to Z$ *and a smooth* $r: B \to \mathbf{R}$ *(defined in neighbourhoods of the origins) such that*

$$f(z,b) = \{\chi d(z,b), k(z,b)\} + r(b)$$

whenever $D_1 f$ *can be factorized through* χ, *that is, whenever* $D_1 f = \chi h$ *for some* $h: Z \times B \to U$.

PROOF. It is useful to compare this lemma with the comparatively simpler but similar Lemma 2 of [4]. The extra assumption of symmetry is used here to get around the lack of reflexiveness. Both lemmas are "division" results, in which a function, $f(z,b)$, is "divided" by a "one-form" $d(z,b)$, to get rid of z-dependence.

The mapping $(z,b) \mapsto (d(z,b),b)$ is a diffeomorphism of a neighbourhood of $(0,0)$ in $Z \times B$ onto a neighbourhood of $(0,0)$ in $U \times B$. Let its inverse be

$$(u,b) \mapsto (\gamma(u,b),b)$$

Now

$$f(\gamma(u,b),b) = f(\gamma(0,b),b) + \int_0^1 \frac{d}{dt} f(\gamma(tu,b),b)\,dt$$

$$= f(\gamma(0,b),b) + \int_0^1 \{\chi h(\gamma(tu,b),b), [D_1 d(\gamma(tu,b),b)]^{-1} u\}\,dt\,.$$

To simplify things let us put

$$h(\gamma(tu,b),b) = u_t \quad \text{and} \quad \gamma(tu,b) = z_t\,.$$

The integrand is then

$$\begin{aligned}&\{\chi u_t, [D_1 d(z_t,b)]^{-1} u\}\\ &\quad = \{\chi D_1 d(z_t,b)[D_1 d(z_t,b)]^{-1} u_t, [D_1 d(z_t,b)]^{-1} u\}\\ &\quad = \{\chi u, [D_1 d(z_t,b)]^{-1} u_t\}\end{aligned}$$

by the symmetry property. If we now set $\gamma(u,b) = z$, which implies $u = d(z,b)$, the integral assumes the form

$$\{\chi d(z,b), k(z,b)\}$$

where

$$k(z,b) = \int_0^1 [D_1 d(z_t,b)]^{-1} u_t\,dt$$

in which the substitution $u = d(z,b)$ must be made. This proves the lemma.

PROOF OF THEOREM 1. A comparison with [4] is again useful, since we have to establish the following equation, which is just equation (1) of [4]:

$$\begin{aligned}(2)\qquad & f(v+z,a) - \tfrac{1}{2}\langle \sigma T z, z\rangle\\ &\quad = \langle t D_1 f(v+z,a) + (1-t)\sigma T z, k(z,v,a,t)\rangle + \psi(v,a,t)\,.\end{aligned}$$

The problem here is the existence of k and ψ so that (2) is satisfied on the following domain: v, z and a are in neighbourhoods of the origin in V, Z and A respectively, whilst t is in an interval $(-\varepsilon, 1+\varepsilon) \subset \mathbf{R}$. The mapping k has range in Z, ψ is real-valued, both are to be C^∞. We use Lemma 1 for t in a neighbourhood of each $t_0 \in [0,1]$. A finite number of determinations of "k" and "ψ" can be assembled by means of a partition of unity over $[0,1]$. (It might be possible to avoid this by adjusting Lemma 1 in order to allow the variable "b" to occupy a "macroscopic" set).

We let Z be the space Z of Lemma 1 and $V \times A \times \mathbf{R}$ the space B. Let τ: $X^* \to Z^*$ be the linear operator which restricts functionals on X to Z. Let P: $Y \to TX$ be the projection with kernel W. For all $\bar{z} \in Z$

$$\begin{aligned} &\langle tD_1 f(v+z,a) + (1-t)\sigma Tz, \bar{z}\rangle \\ &= \langle t\sigma Ph(v+z,a) + (1-t)\sigma Tz, \bar{z}\rangle \\ &= \{t\tau\sigma Ph(v+z,a) + (1-t)\tau\sigma Tz, \bar{z}\} \end{aligned}$$

where $\{\cdot, \cdot\}$ is the pairing of Z^* and Z. Hence we can let TX be the space U of Lemma 1, with

$$\chi = \tau\sigma \,|\, TX$$

and if $b = (v, a, t)$

$$d(z,b) = (t_0+t)Ph(v+z,a) + (1-t_0-t)Tz\ .$$

Then

$$\begin{aligned} &\{\chi D_1 d(z,b) z_1, z_2\} \\ &= \{(t_0+t)\chi PD_1 h(v+z,a)z_1 + (1-t_0-t)\chi Tz_1, z_2\} \\ &= \langle (t_0+t)\sigma PD_1 h(v+z,a)z_1 + (1-t_0-t)\sigma Tz_1, z_2\rangle \\ &= \langle (t_0+t)D_1^2 f(v+z,a)z_1 + (1-t_0-t)D_1^2 f(0,0)z_1, z_2\rangle \end{aligned}$$

whence the symmetry property follows because of the symmetry of second derivatives. Furthermore $D_1 d(0,0) = T \,|\, Z$ which is an isomorphism of Z onto U. Lastly we check that the z-derivative of $f(v+z,a) - \frac{1}{2}\langle \sigma Tz, z\rangle$ can be factorized through χ. We have for all $\bar{z} \in Z$

$$\begin{aligned} &\langle D_1 f(v+z,a), \bar{z}\rangle - \langle \sigma Tz, \bar{z}\rangle \\ &= \{\tau\sigma Ph(v+z,a) - \tau\sigma Tz, \bar{z}\} = \{\chi[Ph(v+z,a) - Tz], \bar{z}\} \end{aligned}$$

which is what we need.

The rest of the proof is based on equation (2) and will be summarised. Consider the differential equation in Z

$$\zeta' = -k(\zeta, v, a, t)$$

where $\zeta' = d\zeta/dt$, and v and a are parameters. Let $t \mapsto \Phi(z, v, a, t)$ be the solution satisfying the initial condition $\zeta(0) = z$. If

$$M(z, v, a, t) = tf(v+z, a) + (1-t)\tfrac{1}{2}\langle \sigma Tz, z\rangle$$

then (2) implies

$$\frac{d}{dt} M(\Phi(z, v, a, t), v, a, t) = \psi(v, a, t) .$$

Hence, integrating from $t=0$ to $t=1$ gives

$$M(\Phi(z, v, a, 1), v, a, 1) = \tfrac{1}{2}\langle \sigma Tz, z\rangle + g(v, a)$$

where $g(v, a) = \int_0^1 \psi(v, a, t)\, dt$. Then $\varphi_{(v,a)} = \Phi(\cdot, v, a, 1)$.

To obtain the last part of the theorem, suppose $\langle D_1 f(v, a), z\rangle = 0$ for all $z \in Z$, and all v in a neighbourhood of $0 \in V$. Then setting $z=0$ in equation (2) gives $f(v, a) = \psi(v, a, t)$. Hence $g(v, a) = \int_0^1 f(v, a)\, dt = f(v, a)$. This concludes the proof of Theorem 1.

The pair of spaces X and Y form what is called a dual pair [8], provided we add the rather natural conditions that:

(i) if $\langle \sigma y, x\rangle = 0$ for all $x \in X$ then $y = 0$.
(ii) if $\langle \sigma y, x\rangle = 0$ for all $y \in Y$ then $x = 0$.

These conditions lead to some elementary results which are useful in applying Theorem 1. The most familiar example of a dual pair is a space and its dual, in which case these results are so familiar that it would be unnecessary to prove them. If asked to prove them, the Hahn–Banach theorem comes to mind at once, which casts doubt on them in the case of an arbitrary dual pair. For this reason the extremely elementary proofs will be given. As σ plays little role we write $\{y, x\}$ for $\langle \sigma y, x\rangle$.

LEMMA 2. *Let X and Y form a dual pair and let T: $X \to Y$ be linear and symmetric in the sense that* $\{Tx_1, x_2\} = \{Tx_2, x_1\}$ *for all x_1, x_2 in X. Let* $V = \ker T$ *and suppose that* $Y = \overline{TX} \oplus W$ *where* $\dim W < \infty$. *Let* $Z = \{x \in X : \{w, x\} = 0$ *for all* $w \in W\}$, *that is, Z is the annihilator of W. Then*

(a) $\dim V < \infty$ *and* $\dim V \leq \dim W$.
(b) $V \cap Z = \{0\}$.
(c) $V \oplus Z = X$ *iff* $\dim V = \dim W$.

PROOF. (a) Each v induces a linear functional $\tilde{v}$ on W by the formula $\tilde{v}(w) = \{w, v\}$. If $\tilde{v}=0$ then $\{w, v\}=0$ for all $w \in W$. But $\{Tx, v\}=\{Tv, x\}=0$ for all $x \in X$, and so $\{y, v\}=0$ for all $y \in \overline{TX}$. Therefore $\{y, v\}=0$ for all $y \in Y$ which implies $v=0$. Hence linearly independent members of V induce linearly independent functionals on W, so $\dim V \leqq \dim W$.

(b) If $x \in V \cap Z$ then $\tilde{x}=0$ and so $x=0$ as in the proof of (a).

(c) Suppose $\dim V = \dim W$. Then V induces all linear functionals on W. If $x \in X$ the functional $w \mapsto \{w, x\}$ on W *has the form* $w \mapsto \tilde{v}(w)=\{w, v\}$ for some $v \in V$. Hence $\{w, v-x\}=0$ for all $w \in W$ and so $v-x \in Z$. Hence $x \in V \oplus Z$, and so $V \oplus Z = X$. Conversely let $V \oplus Z = X$. Now we let W induce linear functionals on V by the formula $\tilde{w}(v)=\{w, v\}$. If $\tilde{w}=0$ we have $\{w, v\}=0$ for all $v \in V$. But $\{w, z\}=0$ for all $z \in Z$ and so $\{w, x\}=0$ for all $x \in X$. Hence $w=0$. This shows that linearly independent members of W induce linearly independent functionals on V and so $\dim W = \dim V$. This concludes the proof.

In applications of Theorem 1 it is important to have an effective knowledge of g (I hesitate to say "calculate g"; a Taylor expansion is the best one can hope for, but this is usually enough, combined with the theory of determinacy and universal unfoldings). The point is that the critical points of g and their bifurcation geometry correspond to those of f. If $D_1 f(v, a)$ annihilates Z for all v in a neighbourhood of $0 \in V$ then $g(v, a) = f(v, a)$. This is an exceptional situation, but one that can be exploited to find an unfolding in V *strongly equivalent* [5] to g. This results in a "reduction procedure" as in [5]. Suppose $\dim V < \infty$ and let $\eta(x) = f(x, 0)$. There is a germ γ: $V, 0 \to Z, 0$ such that $\eta'(v+\gamma(v))$ annihilates Z for each v in a neighbourhood of $0 \in V$. Just solve for $\gamma \in Z$ the equation

$$Ph(v+\gamma, 0) = 0$$

which defines γ by the implicit function theorem. (This is connected with the Liapunov–Schmidt reduction, but is slightly simpler because the bifurcation parameters are suppressed). Then f is strongly equivalent to the unfolding

$$(x, a) \mapsto f(v+\gamma(v)+z, a)$$

which is in turn strongly-equivalent to

$$(x, a) \mapsto \tfrac{1}{2}\langle \sigma Tz, z\rangle + \tilde{g}(v, a)$$

where $\tilde{g}(v, 0) = \eta(v+\gamma(v))$, by Theorem 1, and note how the last clause of the theorem is used. It follows that f is versal *within the class* U_A^q if and only if $\tilde{g}$ is versal. To decide on the latter we need to know $\tilde{g}(v, 0) = \eta(v+\gamma(v))$ to sufficiently high order only, and also the linearised part $D_2\tilde{g}(v, 0)\cdot a$. But as shown in the proof of Theorem 2 of [5]

$$D_2\tilde{g}(v,0)\cdot a \;=\; D_2\bar{g}(v,0)\cdot a$$

where $\bar{g}(v,a) = f(v+\gamma(v),a)$.

Thus only a knowledge of $f(v+\gamma(v),a)$ to *first order* in a and to *sufficiently high order* in v is needed. For more details see [5] and [6], (these deal with a reflexive Banach space only because the splitting lemma used is valid only in this case; with any splitting lemma there is a reduction procedure provided the term "versality" is understood in connection only with those unfoldings which can be split).

We now consider some examples.

EXAMPLE 1. Let $E\colon \mathbf{R}^2 \to \mathbf{R}$ be a C^∞ mapping. We wish to define a function

$$\eta(x) \;=\; \int_0^1 E(t,x(t))\,dt\;. \tag{3}$$

Without growth restrictions on E this does not in general define a function on $L^2[0,1]$, and even with growth restrictions such a function cannot, with one kind of exception, be in C^2, (it can be in C^1 if $D_2E(t,u)$ has at most linear growth in u for large $|u|$). The exception is

$$E(t,u) \;=\; a(t)+b(t)u+c(t)u^2\;.$$

This is the only kind of integrand for which (3) defines a C^∞, or even a C^2, function on $L^2[0,1]$.

However (3) does define a C^∞ function on $C[0,1]$ and on $L^\infty[0,1]$ and then the results of the present paper can be used. Let us consider $X = C[0,1] = Y$, and define $\sigma\colon Y \to X^*$ by

$$\langle \sigma x_1, x_2\rangle \;=\; \int_0^1 x_1(t)x_2(t)\,dt\;.$$

In this way X forms a dual pair with itself. Since

$$\langle \eta'(x),\bar{x}\rangle \;=\; \int_0^1 D_2E(t,x(t))\bar{x}(t)\,dt$$

we set $h(x) = D_2E(\cdot,x(\cdot))$. Then $h'(x)$ is just multiplication of functions in $C[0,1]$ by $D_2^2E(t,x(t))$. So $h'(x)$ is an isomorphism iff $D_2^2E(t,x(t)) \neq 0$ for all $t \in [0,1]$. If $h(x_0) = 0$ and $h'(x_0)$ is an isomorphism we can apply the Morse Lemma (Corollary 1) and find that the germ at 0 of

$$x \mapsto \int_0^1 E(t,x_0(t)+x(t))\,dt$$

is equivalent to the germ at 0 of

$$x \mapsto \tfrac{1}{2} \int_0^1 D_2^2 E(t, x_0(t))(x(t))^2 \, dt + \text{constant} .$$

It would of course be quite easy to prove this directly.

If $D_2^2 E(t, x(t)) = 0$ for some $t \in [0, 1]$ there is no application of Theorem 1 since the range of $h'(x_0)$ is then not closed. It is still, however, possible to discuss the question of universally unfolding such a singularity, and to describe the singularities which occur generically in a p-parameter family, if only the integrand E is allowed to vary. On this matter see [7]. If we have a Morse point x_0, that is, $h(x_0) = 0$ and $h'(x_0) = T$ is an isomorphism, Theorem 1 implies that any unfolding of η, whose x-derivative may be factorized through σ is equivalent to the "constant" unfolding $x \mapsto \frac{1}{2}\langle \sigma T x, x \rangle$. This class of unfoldings of η includes, and is strictly larger than, the class obtained by incorporating parameters in E.

EXAMPLE 2. Let E: $\mathbf{R}^3 \to \mathbf{R}$ be a C^∞ mapping. We wish to define a function

$$\eta(x) = \int_0^1 E(t, x(t), x'(t)) \, dt . \tag{4}$$

This is the classical case of the calculus of variations. If x is to belong to a Hilbert space the only reasonable candidate is $H^1[0, 1]$, and then we need E to be quadratic in x' in order to get a C^∞ function. Instead we choose $x = C^1[0, 1]$. Then η is C^∞ and

$$\begin{aligned} \langle \eta'(x), \bar{x} \rangle &= \int_0^1 [D_2 E(t, x(t), x'(t))\bar{x}(t) + D_3 E(t, x(t), x'(t))\bar{x}'(t)] \, dt \\ &= \int_0^1 \bar{x}'(t)\beta_x(t) \, dt + \bar{x}(1) \int_0^1 D_2 E(t, x(t), x'(t)) \, dt \end{aligned}$$

where

$$\beta_x(t) = D_3 E(t, x(t), x'(t)) - \int_0^t D_2 E(s, x(s), x'(s)) \, ds .$$

So we take $Y = C[0, 1] \times \mathbf{R}$ and define σ: $Y \to X^*$ by

$$\langle \sigma(f, \lambda), \bar{x} \rangle = \int_0^1 \bar{x}'(t) f(t) \, dt + \lambda \bar{x}(1) .$$

It is rather easy to see that this makes X and Y form a dual pair. Thus

$$h(x) = \left(\beta_x, \int_0^1 D_2 E(t, x(t), x'(t)) \, dt \right)$$

and $T = h'(0)$ is the linear map $x \mapsto (f, \lambda)$ where

$$f(t) = D_2D_3E(t,0,0)x(t) + D_3^2E(t,0,0)x'(t) - \int_0^t [D_2^2E(s,0,0)x(s) + D_3D_2E(s,0,0)x'(s)]\,ds$$

and

$$\lambda = \int_0^1 [D_2^2E(t,0,0)x(t) + D_3D_2E(t,0,0)x'(t)]\,dt\,.$$

T is a Fredholm operator of index 0 whenever $D_3^2E(t,0,0) \neq 0$ for all $t \in [0,1]$. For the mapping $x \mapsto f$ is Fredholm of index 1, being a differential operator without singular point plus an integral operator which is compact. Assuming that this condition holds, T is an isomorphism if the boundary value problem

$$\begin{aligned} -(a(t)x')' + b(t)x &= 0 \\ px(1) + qx'(1) = rx(0) + sx'(0) &= 0 \end{aligned} \tag{5}$$

has only the solution $x = 0$, where

$$\begin{aligned} a(t) &= D_3^2E(t,0,0) \\ b(t) &= D_2^2E(t,0,0) - D_1D_2D_3E(t,0,0) \\ p &= D_2D_3E(1,0,0) \\ q &= D_3^2E(1,0,0) \\ r &= D_2D_3E(0,0,0) \\ s &= D_3^2E(0,0,0)\,. \end{aligned}$$

The condition $h(x) = 0$ for x to be a critical point of η is equivalent to the Euler–Lagrange equations

$$\frac{d}{dt}D_3E(t,x,x') - D_2E(t,x,x') = 0$$

$$D_3E(0,x(0),x'(0)) = D_3E(1,x(1),x'(1)) = 0$$

of which (5) constitute the linearisation at $x = 0$. If $x = 0$ is a critical point and $T = h'(0)$ is an isomorphism then η is equivalent to the germ at 0 of the quadratic form $\langle \sigma Tx, x \rangle$ which is

$$\tfrac{1}{2}\int_0^1 [c(t)(x(t))^2 + 2d(t)x(t)x'(t) + a(t)(x'(t))^2]\,dt$$

where $a(t)$ is defined above and

$$c(t) = D_2^2E(t,0,0)$$
$$d(t) = D_2D_3E(t,0,0)\,.$$

T fails to be an isomorphism when (5) has non-trivial solutions. The kernel of T then has dimension 1 or 2, the latter only occurring if $p=q=r=s=0$. Consider the case $\dim\ker T=1$. If $(f,\lambda)\in Y$ the equation

$$Tx = (f,\lambda)$$

has a solution if and only if $\langle\sigma(f,\lambda),v_0\rangle=0$ where v_0 spans $\ker T$. (This is clearly necessary, and so sufficient, since the range of T has codimension 1). Hence we can take W to be spanned by $(w_0,0)$ where

$$w_0(t) = \int_0^t v_0(s)\,ds\Big/\int_0^1 (v_0(s))^2\,ds\,,$$

since then $\langle\sigma(w_0,0),v_0\rangle=1$. The projection $P\colon Y\to TX$ with kernel W is then given by

$$(f,\lambda)\mapsto(f-\langle\sigma(f,\lambda),v_0\rangle w_0,\lambda)$$

and Z is the annihilator of W so that $x\in Z$ if and only if

$$\int_0^1 x(t)v_0(t)\,dt = 0\,.$$

The equations determining γ are complicated and will not be given here. We conclude by giving the condition for a fold singularity at $x=0$ (assuming T is Fredholm, $\dim\ker T=1$, $h(0)=0$ so that $x=0$ is critical). This is that the third derivative of $v\mapsto\eta(v+\gamma(v))$ is not 0 at $v=0$. In view of the properties $\gamma(0)=0$, $\gamma'(0)=0$ of γ this third derivative is just the *number* $\eta^{(3)}(0)v_0^3$, which is independent of γ, (in fact, to calculate $\eta(v+\gamma(v))$ to nth order we need to know γ to $(n-2)$-th order). The condition is therefore

$$\int_0^1 [A(t)(v_0(t))^3+3B(t)(v_0(t))^2v_0'(t)+3C(t)v_0(t)(v_0'(t))^2$$
$$+D(t)(v_0'(t))^3]\,dt \neq 0$$

where

$$A(t) = D_2^3E(t,0,0)$$
$$B(t) = D_2^2D_3E(t,0,0)$$
$$C(t) = D_2D_3^2E(t,0,0)$$
$$D(t) = D_3^3E(t,0,0)\,.$$

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