MATH. SCAND. 46 (1980), 154–156

A NOTE ON POWER BOUNDED RESTRICTIONS OF FOURIER–STIELTJES TRANSFORMS

ERIK SVENSSON*

In this note we extend the case $\mathbf{R}^n$ of the main result, Theorem 5.1, in Andersson [1] to some non-compact situations.

Definition. If E is a subset of $\mathbf{R}^n$, then the supremum of all radii of closed balls contained in E is called the width of E and is written $w(E)$. (Possibly $w(E) = \infty$.)

Undefined notations below are as in Andersson [1].
We shall prove the following

Theorem. *Let E be a subset of $\mathbf{R}^n$ such that E is the closure of its interior and suppose that for each $c \in \mathbf{R}_+$ there are only finitely many components of E of width less than c. Suppose $\varphi \in \mathscr{B}(E)$, i.e. suppose $\varphi \in B(E)$, $|\varphi| = 1$ on E, and $\sup_{k \in \mathbf{Z}_+} \|\varphi^k\| < \infty$. Then on all of E except possibly on finitely many components of finite width, φ equals one and the same product $ae^{ib \cdot x}$, where $a \in \mathbf{C}$, $|a| = 1$, and $b \in \mathbf{R}^n$.*

We note that the theorem in particular implies that if each component of E has infinite width, then on all of E the function φ equals one and the same product $ae^{ib \cdot x}$.

Proof of the theorem. It is sufficient to prove the theorem in the case when E is the union of infinitely many pairwise disjoint balls whose radii tend to infinity. To see this, let $E'_1, E'_2, \ldots$ be an enumeration of E's component of finite width, and let $E''_1, E''_2, \ldots$ be an enumeration of those of infinite width. For each E'_i choose a ball A'_i contained in E'_i and having radius equal to $\frac{1}{2}w(E'_i)$, and for each E''_i choose an infinite sequence $A''_{i1}, A''_{i2}, \ldots$ of pairwise disjoint balls, each contained in E''_i and such that radius $(A''_{ij}) = i + j$. These choices are

* Partially supported by the Swedish Natural Science Research Council, contract no. F2220-100.

Received August 14, 1979.

clearly possible. Then let $B_1, B_2, \ldots$ be an enumeration of all the balls $A'_1, A'_2, \ldots, A''_{11}, A''_{12}, \ldots, A''_{21}, A''_{22}, \ldots, \ldots$. The radii of $B_1, B_2, \ldots$ obviously tend to infinity. Trivially $\varphi \in \mathscr{B}(E) \Rightarrow \varphi \in \mathscr{B}(\bigcup_1^\infty B_k)$. Now suppose we could prove that on all but finitely many of the balls $B_1, B_2, \ldots, \varphi$ equals one and the same product $ae^{ib\cdot x}$. Since we know that on the whole of each of E's components, φ is a product $ae^{ib\cdot x}$ (see [1, Theorem 5.1]), and since each of E's components of infinite width contains infinitely many of the balls $B_1, B_2, \ldots$, the theorem would then follow. From now on we suppose that E is a union of an infinite sequence $B_1, B_2, \ldots$ of pairwise disjoint balls whose radii tend to infinity, and we want to show that on all but finitely many of the balls $B_1, B_2, \ldots, \varphi$ equals one and the same product $ae^{ib\cdot x}$.

LEMMA. *Let $E_1, E_2, \ldots$ be subsets of R^n having the property that there are points $x_1, x_2, \ldots$ in R^n such that each compact subset of R^n is contained in every $E_k + x_k$ if k is large enough. Suppose $\hat{\mu} \in B(\mathsf{R}^n)$ coincides with a constant λ_k on each E_k. Then $\lim_{k\to\infty} \lambda_k$ exists and is equal to $\mu(\{0\})$, the mass of μ at the origin in R^n.*

PROOF OF THE LEMMA. Fix $\varepsilon > 0$. By Fubini's theorem

$$\int_{\mathsf{R}^n} f(x)\hat{\mu}(x - x_k)\,dx = \int_{\mathsf{R}^n} \hat{f}(\gamma)e^{i2\pi x_k\cdot\gamma}\,d\mu(\gamma) \tag{1}$$

for each $f \in L^1(\mathsf{R}^n)$. Now fix (see for example Rudin [2, p. 48, Theorem 2.6.1]) an $f \in L^1(\mathsf{R}^n)$ with $0 \leqq \hat{f} \leqq 1$ so that $\hat{f}(0) = 1$ and so that the support of $\hat{f}$ is contained in so small a neighborhood of the origin in R^n that

$$\int_{\mathsf{R}^n} |\hat{f}(\gamma)|\,d|\mu_1(\gamma)| < \varepsilon\,,$$

where $\mu_1 = \mu - \mu(\{0\})\delta_0$ and δ_0 is the Dirac measure at the origin in R^n. Then also

$$\left|\int_{\mathsf{R}^n} \hat{f}(\gamma)e^{i2\pi x_k\cdot\gamma}\,d\mu_1(\gamma)\right| < \varepsilon$$

for each x_k. I.e.

$$\left|\int_{\mathsf{R}^n} \hat{f}(\gamma)e^{i2\pi x_k\cdot\gamma}\,d\mu(\gamma) - \mu(\{0\})\right| < \varepsilon \tag{2}$$

for each x_k. Now choose a compact set C in R^n such that

$$\int_{\mathsf{R}^n\setminus C} |f(x)| < \varepsilon\,.$$

If k is so large that $E_k + x_k \supset C$, then

$$\begin{aligned}
(3) \qquad & \left| \int_{\mathbf{R}^n} f(x)\hat{\mu}(x - x_k)\,dx - \lambda_k \right| \\
& \leqq \left| \int_C f(x)\hat{\mu}(x - x_k)\,dx - \lambda_k \right| + \|\mu\|\varepsilon \\
& = \left| \int_C f(x)\lambda_k\,dx - \lambda_k \int_{\mathbf{R}^n} f(x)\,dx \right| + \|\mu\|\varepsilon \\
& \leqq 2\|\mu\|\varepsilon\,.
\end{aligned}$$

Combining (1), (2), and (3) the lemma follows.

We shall now see how this lemma implies the theorem.

On each B_k the function φ is of the form $a_k e^{ib_k \cdot x}$, where $a_k \in \mathbf{C}$, $|a_k| = 1$, and $b_k \in \mathbf{R}^n$. In [1, Theorem 4.1] it is shown that the b_k's can take only a finite number of values. We claim that in our case b_k is constant if k is large enough. If not, then there are two different values b' and b'' which occur infinitely many times among the b_k's. Choose $x_0 \in \mathbf{R}^n$ so that $e^{ib' \cdot x_0} = 1$ but $e^{ib'' \cdot x_0} \neq 1$. Set $\psi(x) = \varphi(x + x_0) - \varphi(x)$. Also for all but those finitely many B_k for which radius $(B_k) < |x_0|$, let B_k' be the ball B_k with its radius decreased by $|x_0|$. Then $|\psi|^2 \in B(\mathbf{R}^n)$, $|\psi|^2 = |e^{ib' \cdot x_0} - 1|^2 = 0$ on all those infinitely many balls B_k' for which $b_k = b'$, and $|\psi|^2 = |e^{ib'' \cdot x_0} - 1|^2 > 0$ on all those infinitely many balls B_k' for which $b_k = b''$. But this contradicts the lemma, and thus b_k is constant if k is large enough. It remains to show that a_k is constant if k is large enough. Suppose this is not the case. Then, exactly as in the proof of Lemma 5.2 in [1], it is possible to construct a Fourier–Stieltjes transform on $\mathbf{R}^n$ taking constant values λ_k on each E_k, where $E_1, E_2, \ldots$ is an infinite subsequence of $B_1, B_2, \ldots$, and where the λ_k's are such that $\lim_{k \to \infty} \lambda_k$ does not exist. Since this contradicts the lemma, we conclude that a_k is constant if k is large enough. The theorem is proved.

ACKNOWLEDGEMENT. I wish to think Professor Yngve Domar for suggestions making it possible to simplify my original proof.

REFERENCES

1. R. Andersson, *Power bounded restrictions of Fourier–Stieltjes transforms*, Math. Scand. 46 (1980), 129–153.
2. W. Rudin, *Fourier analysis on groups* (Interscience Tracts in Pure and Applied Mathematics 12), Interscience Publ., New York, 1967.

ERIK SVENSSON
UPPSALA UNIVERSITY
DEPARTMENT OF MATHEMATICS
THUNBERGSVÄGEN 3
S-752 38 UPPSALA
SWEDEN