MATH. SCAND. 46 (1980), 157–172

WEAKLY ALMOST PERIODIC FUNCTIONS AND THE IRREGULARITY OF MULTIPLICATION IN SEMIGROUP ALGEBRAS

HENERI A. M. DZINOTYIWEYI

1. Introduction.

Let G denote a locally compact topological group throughout this introduction. Burckel [4] proved that G is compact if and only if all the bounded continuous functions on G are weakly almost periodic. Granirer [10] improved the result to: G is compact if and only if all the bounded uniformly continuous functions on G are weakly almost periodic. Their proofs are rather deep — in particular they both rely on the fact that the weakly almost periodic functions on G admit an invariant mean. In this paper we prove such results for a very large class of topological semigroups from which the results of Burckel and Granirer follow as "very" special cases. Moreover we do not invoke the theory of invariant means in our proof.

Civin and Yood [5] proved that if G is infinite and abelian, then the two Arens products on the second dual of $L^1(G)$ do not coincide — that is, the *multiplication in* $L^1(G)$ *is irregular.* Young [21] improved the result to include the case where G is not necessarily abelian. Pym [17] studied the same problem for certain convolution measure algebras supported on semigroups. Our second aim in this paper is to investigate this phenomenon further. In particular the result of Young [21] and Theorem 5 of Pym [17] turn out to be "very" special cases of our results — see section 4.

In section 2 we clarify part of our notation and include some definitions needed in this paper. In section 3 we study the subject of weakly almost periodic functions on topological semigroups. Some related results on the Stone–Čech compactification of a topological semigroup are also proved in section 3. Section 4 is devoted to a study of the irregularity of multiplication in certain algebras of measures on topological semigroups.

2. Preliminaries.

Let A and B be any subsets of a semigroup S and x any element of S. We

Received July 5, 1978.

take AB, $A^{-1}B$, $x^{-1}B$ and $A^{-1}x$ to denote $\{ab : a \in A \text{ and } b \in B\}$, $\{y \in S : ay \in B \text{ for some } a \in A\}$, $\{x\}^{-1}B$ and $A^{-1}\{x\}$ (respectively). By symmetry the definitions of BA^{-1}, Bx^{-1} and xA^{-1} must be clear. By a *cancellation semigroup* we mean a semigroup S such that whenever $xy = xz$ or $yx = zx$ then $y = z$, for all $x, y, z \in S$.

Throughout this paper, a semigroup S is called a *topological semigroup* if S is endowed with a Hausdorff topology with respect to which the semigroup operation $(x, y) \to xy$ is a jointly continuous mapping of $S \times S$ into S.

Let S be a topological semigroup, $m(S)$ the space of all bounded complex-valued functions on S with the usual sup-norm $\| \ \|_S$,

$$C(S) := \{f \in m(S) : f \text{ is continuous}\}$$

and $M(S)$ the set of all bounded complex-valued Radon measures on S. For every $\mu \in M(S)$ we take $|\mu|$ to be the measure arising from the total variation of μ and, if E is a Borel subset of S, we take $\mu|_E$ to be the measure given by $\mu|_E(B) := \mu(B \cap E)$ for every Borel subset B of S. For every point x in S we take $\bar{x}$ to be the point mass at x. Let

$$M_a^l(S) := \{\mu \in M(S) : \text{ the map } x \to |\mu|(x^{-1}C) \text{ of } S \text{ into } \mathrm{R} \text{ is continuous, for every compact subset } C \text{ of } S\},$$

$$M_a^r(S) := \{\mu \in M(S) : \text{ the map } x \to |\mu|(Cx^{-1}) \text{ of } S \text{ into } \mathrm{R} \text{ is continuous, for every compact subset } C \text{ of } S\}$$

and $M_a(S) := M_a^l(S) \cap M_a^r(S)$. The set $M_a(S)$ has been studied in many publications—see e.g. [1], [2], [7], [8], [18] and [19]. In particular for a locally compact topological group G it is well known that $M_a(G)$ can be identified with $L^1(G)$—the usual group algebra of G (see e.g. [12]).

Now let $\mathscr{A}$ be any subset of $M(S)$. For each $\mu \in \mathscr{A}$ let

$$\operatorname{supp}(\mu) := \{x \in S : \text{ if } V \text{ is any open neighbourhood of } x \text{ then } |\mu|(V) > 0\}.$$

The closure of the set

$$\bigcup \{\operatorname{supp}(\mu) : \mu \in \mathscr{A}\}$$

is called *the foundation* of $\mathscr{A}$. A Borel set $E \subset S$ is said to be *$\mathscr{A}$-negligible* if $|\mu|(E) = 0$ for all $\mu \in \mathscr{A}$.

Following [7] we say a topological semigroup S is a *C-distinguished topological semigroup* if the real-valued functions in $C(S)$ separate the points of S. For such a topological semigroup $M(S)$ can be identified with the strictly continuous linear functionals on $C(S)$—see also [13].

Let S be any C-distinguished topological semigroup. We noted in [7] that the maps $x \to \int f(yx)\,d\mu(y)$ of S into $\mathbb{C}$ are continuous ($f \in C(S)$ and $\mu \in M(S)$). This enabled us to define a convolution operation on $M(S)$, given by

$$\nu * \mu(f) := \iint f(xy)\,d\nu(x)\,d\mu(y) \quad (\nu, \mu \in M(S) \text{ and } f \in C(S)),$$

with respect to which $M(S)$ is a normed algebra under the usual total variation norm $\|\ \|$. (We urge the reader to note the details from [7].) In particular for every Borel subset E of S we have (as noted in [7]) that, for all ν, μ in $M(S)$,

$$\nu * \mu(E) = \int \mu(x^{-1}E)\,d\nu(x) = \int \nu(Ex^{-1})\,d\mu(x).$$

Following [7] we say $\mathscr{A} \subseteq M(S)$ is *a neo-convolution measure algebra* if $\mathscr{A}$ is a norm closed subalgebra of $M(S)$ such that for all $\nu, \mu \in M(S)$ if $\nu \ll |\mu|$ and $\mu \in \mathscr{A}$ then $\nu \in \mathscr{A}$. In particular from [7, Theorem 4.1] we have

THEOREM 2.1. *For every* $\mathbb{C}$*-distinguished topological semigroup* S, *we have* $M_a(S)$ *a neo-convolution measure algebra.*

As remarked in [7], neo-convolution measure algebras are not necessarily complete. Also in the case of locally compact topological semigroups our definition of neo-convolution measure algebra coincides with that of convolution measure algebra in the sense of Taylor [20].

3. Weakly Almost Periodic Functions.

We start by giving a lemma which we also need in section 4. The idea in the construction of sequences of compact sets in the proof of Lemmas 3.1 and 4.2 is taken from Young [21].

LEMMA 3.1. *Let* S *be a topological semigroup.*

(i) *If* S *is not compact and such that* $C^{-1}D$ *and* DC^{-1} *are compact for all compact subsets* C *and* D *of* S, *then given any compact set* $K \subset S$ *we can find sequences* $\{x_n\}$, $\{y_m\}$ *of distinct points of* S *such that*

$$\left(\bigcup_{m=1}^{\infty} \bigcup_{n>m}^{\infty} \{Kx_ny_mK\}\right) \cap \left(\bigcup_{n=1}^{\infty} \bigcup_{n<m}^{\infty} \{Kx_ny_mK\}\right) = \varnothing.$$

(ii) *If* S *is infinite and such that* $x^{-1}\{y\}$ *and* $\{y\}x^{-1}$ *are finite sets for all* x *and* y *in* S, *we can find sequences* $\{x_n\}$, $\{y_m\}$ *of distinct points of* S *such that*

$$\left(\bigcup_{m=1}^{\infty} \bigcup_{n>m}^{\infty} \{x_ny_m\}\right) \cap \left(\bigcup_{n=1}^{\infty} \bigcup_{n<m}^{\infty} \{x_ny_m\}\right) = \varnothing.$$

PROOF. We start proving (i) by induction. Suppose, by the induction hypothesis, for some positive integer p we have finite sequences $\{x_1, x_2, \ldots, x_p\}$ and $\{y_1, y_2, \ldots, y_p\}$ such that

$$\left(\bigcup_{m=1}^{p-1} \bigcup_{p \geqq n > m} \{Kx_n y_m K\}\right) \cap \left(\bigcup_{n=1}^{p-1} \bigcup_{n < m \leqq p} \{Kx_n y_m K\}\right) = \emptyset .$$

For convenience we write $X_p := \{x_1, x_2, \ldots, x_p\}$, $Y_p := \{y_1, y_2, \ldots, y_p\}$

$$L_p := \bigcup_{m=1}^{p-1} \bigcup_{p \geqq n > m} \{Kx_n y_m K\} \quad \text{and} \quad R_p := \bigcup_{n=1}^{p-1} \bigcup_{n < m \leqq p} \{Kx_n y_m K\} .$$

Then with this notation we can write

$$L_p \cap R_p = \emptyset . \tag{1}$$

Since $(K^{-1}R_p)(Y_pK)^{-1}$ is a compact subset of S (by the condition stated in the lemma) we can choose $x_{p+1} \in S \setminus X_p$ such that

$$x_{p+1} \notin (K^{-1}R_p)(Y_pK)^{-1} .$$

Consequently

$$Kx_{p+1} Y_p K \cap R_p = \emptyset . \tag{2}$$

Next, noting that $((KX_p)^{-1}(L_p \cup Kx_{p+1} Y_p K))K^{-1}$ is compact, we can choose $y_{p+1} \in S \setminus Y_p$ such that

$$y_{p+1} \notin ((KX_p)^{-1}(L_p \cup Kx_{p+1} Y_p K))K^{-1} .$$

Consequently

$$KX_p y_{p+1} K \cap (L_p \cup Kx_{p+1} Y_p K) = \emptyset . \tag{3}$$

Recalling the above definition of L_p and R_p and similarly defining L_{p+1} and R_{p+1} we note that (1), (2) and (3) imply

$$\begin{aligned} L_{p+1} \cap R_{p+1} &= (L_p \cup Kx_{p+1} Y_p K) \cap (R_p \cup KX_p y_{p+1} K) \\ &= (L_p \cap R_p) \cup (Kx_{p+1} Y_p K \cap R_p) \cup ((L_p \cup Kx_{p+1} Y_p K) \cap KX_p y_{p+1} K) \\ &= \emptyset . \end{aligned}$$

By repeating the above argument countably many times we thus get item (i).

Next we prove item (ii). If S has an identity element e let $S' := S$, otherwise adjoin an isolated identity element e and let $S' := S \cup \{e\}$. Evidently the hypothesis of (ii) still holds with S' in place of S. Now if in the proof of item (i) we replace S by S', substitute the word "compact" by "finite", take $K := \{e\}$ and start the induction proof with $x_0 := e$ and $y_0 := e$ it is not difficult to see that we get the conclusion of item (ii). We leave the details for the reader to check.

Let S be any topological semigroup. For any function f defined on S and $x \in S$, we define the functions ${}_xf$ and f_x on S by ${}_xf(y) := f(xy)$ and $f_x(y) := f(yx)$ $(y \in S)$. In particular when f is in $C(S)$ we clearly have the functions ${}_xf$ and f_x in $C(S)$ for all x in S. Let

$$LUC(S) := \{f \in C(S) : \text{ the map } x \to {}_xf \text{ of } S \text{ into } C(S) \text{ is norm continuous}\}$$

and

$$RUC(S) := \{f \in C(S) : \text{ the map } x \to f_x \text{ of } S \text{ into } C(S) \text{ is norm continuous}\} .$$

We say a function f is *left* (or *right*) *uniformly continuous on* S if $f \in LUC(S)$ (or $f \in RUC(S)$, respectively). We then say a function in $UC(S)$: $= LUC(S) \cap RUC(S)$ is a *uniformly continuous function on* S. The set of all *weakly almost periodic functions* $WAP(S)$ on S is given by

$$WAP(S) := \{f \in C(S) : \text{ the set } \{{}_xf : x \in S\} \text{ is weakly relatively compact in } C(S)\} .$$

The function spaces $LUC(S)$, $RUC(S)$, $UC(S)$ and $WAP(S)$ have been studied in many publications — see e.g. [4], [6], [10] and [15].

Equipped with the preceding notation we are now in a position to prove our main result of this section.

Theorem 3.2. *Let S be a topological semigroup such that $M_a(S)$ contains a non-zero measure and if C and D are compact subsets of S then $C^{-1}D$ and DC^{-1} are compact sets. Then the following items are equivalent*:

(i) *S is compact*;
(ii) $C(S) = WAP(S)$;
(iii) $UC(S) = WAP(S)$.

Proof. First we note that the hypothesis of our theorem implies that S is locally compact. For if μ is any non-zero measure in $M_a(S)$ we can choose a compact subset C of S such that $|\mu|(C) > \frac{1}{2}\|\mu\|$. Now for any $x \in S$ we have that

$$O(x) := \{y \in S : |\mu|(y^{-1}(xC)) > \tfrac{1}{2}\|\mu\|\}$$

is an open neighbourhood of x, by the continuity of the map

$$y \to |\mu|(y^{-1}(xC))$$

of S into $\mathbf{R}$. But

$$O(x) \subseteqq \{y \in S : y^{-1}(xC) \cap C \neq \emptyset\} \subseteqq (xC)C^{-1} ,$$

and so $(xC)C^{-1}$ is a compact neighbourhood of x. It follows that S is locally compact. (The preceding argument is similar to that used in [18, Theorem 6.16].)

That (i) implies (ii) and (iii) is well known—see e.g. [4, Theorem 1.7].

We now show that (iii) implies (i) by an argument which also shows that (ii) implies (i), and so our proof will be complete.

We can choose a positive measure μ in $M_a(S)$ such that supp (μ) is compact (recall Theorem 2.1) and $\|\mu\| = 1$. Now since S is locally compact, the maps

$$x \to \bar{x} * \mu * \mu \quad \text{and} \quad x \to \mu * \mu * \bar{x}$$

of S into $M(S)$ are known to be norm continuous—see e.g. [18, Proposition 5.5] or [7, Theorem 4.9]. Let $\nu := \mu * \mu$ and $K := \operatorname{supp}(\nu)$. Then K is compact and $\|\nu\| = 1$. For any $h \in M(S)^*$ we define the function $\nu \circ h \circ \nu$ on S by

$$\nu \circ h \circ \nu(x) := h(\nu * \bar{x} * \nu) \qquad (x \in S)\,.$$

Since $|\nu \circ h \circ \nu(x) - \nu \circ h \circ \nu(y)| < \|h\| \; \|\bar{x} * \nu - \bar{y} * \nu\|$ $(x, y \in S)$ we thus have $\nu \circ h \circ \nu$ in $C(S)$. In fact we have $\nu \circ h \circ \nu \in UC(S)$ (for related remarks see also [16] and [18]). For if $x, y \in S$, then

$$\begin{aligned} \|{}_x(\nu \circ h \circ \nu) - {}_y(\nu \circ h \circ \nu)\|_S &= \sup\{|h(\nu * \overline{xs} * \nu) - h(\nu * \overline{ys} * \nu)| \;:\; s \in S\} \\ &< \|h\| \; \|\nu * \bar{x} - \nu * \bar{y}\| \end{aligned}$$

and so $\nu \circ h \circ \nu \in LUC(S)$. Similarly $\nu \circ h \circ \nu \in RUC(S)$ and hence $\nu \circ h \circ \nu \in UC(S)$.

Now suppose that (iii) holds but S is not compact. Then with $K := \operatorname{supp}(\nu)$ we can choose sequences $\{x_n\}$, $\{y_m\}$ in S such that if

$$H_1 := \bigcup_{m=1}^{\infty} \bigcup_{n>m}^{\infty} \{Kx_ny_mK\} \quad \text{and} \quad H_2 := \bigcup_{n=1}^{\infty} \bigcup_{n<m}^{\infty} \{Kx_ny_mK\}$$

then

$$H_1 \cap H_2 = \varnothing \tag{1}$$

by Lemma 3.1 (i). Noting that H_1 is σ-compact (and hence a Borel set) we can define $h \in M(S)^*$ by

$$h(\eta) := \eta(H_1) \qquad (\eta \in M(S))\,.$$

So $\nu \circ h \circ \nu \in UC(S)$, by the conclusion of the preceding paragraph. We claim that $\nu \circ h \circ \nu \notin WAP(S)$. By [11, Theorem 6] our claim follows if we can show that

$$(*) \qquad \begin{aligned} \lim_m \lim_n \nu \circ h \circ \nu(x_ny_m) &= 1 \\ \lim_n \lim_m \nu \circ h \circ \nu(x_ny_m) &= 0 \end{aligned}$$

Now since $\operatorname{supp}(\nu * \overline{x_n y_m} * \nu) = Kx_n y_m K$ and $Kx_n y_m K \subseteqq H_1$ if $n > m$ while $Kx_n y_m K \subseteqq H_2$ if $n < m$, it follows from (1) that

$$\nu \circ h \circ \nu(x_n y_m) = \nu * \overline{x_n y_m} * \nu(H_1) = \begin{cases} 1 & \text{if } n > m \\ 0 & \text{if } n < m . \end{cases}$$

The preceding equation clearly implies (*) and as remarked above, this completes the proof of our theorem.

The equivalence of (i) and (ii) in the following corollary is due to Burckel [4] and of (i) and (iii) due to Granirer [10].

COROLLARY 3.3. *If G denotes a locally compact topological group, then the* by taking any non-locally compact topological group.

(i) *G is compact*;
(ii) $C(G) = WAP(G)$;
(iii) $UC(G) = WAP(G)$.

EXAMPLES AND SOME REMARKS 3.4.

(i) If in Theorem 3.2 we drop the assumption that there is a non-zero measure in $M_a(S)$ then S is not necessarily locally compact as one can easily see by taking any non-locally compact topological group.

(ii) The assumption that $C^{-1}D$ and DC^{-1} are compact for all compact subsets C and D of S, is necessary for the conclusion of Theorem 3.2 to hold. For example let S be the set of all integers with discrete topology and multiplication given by $xy := 0$ $(x, y \in S)$. Then $M_a(S) = M(S)$ and $WAP(S) = UC(S) = C(S)$ but S is not compact.

(iii) In the first draft of this paper, we posed the following problem: Let S be a locally compact topological semigroup such that if C and D are compact subsets of S then $C^{-1}D$ and DC^{-1} are compact. Does $C(S) = WAP(S)$ or $UC(S) = WAP(S)$ imply that S is compact when $M_a(S)$ is zero? We thank Dr. G. L. G. Sleijpen for pointing out to us the following example which shows that S is not necessarily compact. In particular the example shows that a condition like $M_a(S)$ be non-zero is required for the conclusion of Theorem 3.2 to hold.

Recalling the usual well-ordering of the set of all ordinals and taking ω_1 to denote the first uncountable ordinal, let

$$S = \{\alpha : \alpha \text{ is an ordinal and } \alpha < \omega_1\}$$

with the interval topology and operation $xy = \max(x, y)$ $(x, y \in S)$. Thus S is a locally compact topological semigroup. The reader can easily check that $M_a(S)$

is zero and that $C^{-1}D$ and DC^{-1} are compact whenever C and D are compact subsets of S. Now the Stone–Čech Compactification βS of S is given by

$$\beta S = \{\alpha : \alpha \text{ is an ordinal and } \alpha \leqq \omega_1\}$$

—see e.g. [9, pages 74 and 88]. So βS with the operation $xy = \max(x, y)$ $(x, y \in S)$ is a compact topological semigroup. So

$$(*) \qquad C(\beta S) = UC(\beta S) = WAP(\beta S)$$

—see e.g. [4]. One can easily deduce from (*) that

$$C(S) = UC(S) = WAP(S)\,.$$

However S is not compact—thus completing our verification.

(iv) Let S be any topological semigroup and

$$AP(S) := \{f \in C(S) : \text{ the set } \{{}_xf : x \in S\} \text{ is relatively norm compact in } C(S)\}\,.$$

For a locally compact topological group G, Burckel [4] proved that $AP(G) = WAP(G)$ if and only if G is compact. We know that if S is compact then $AP(S) = C(S)$—see e.g. [4]. It is interesting to note that we may have $AP(S) = WAP(S)$ but S not compact, even when S is a cancellation semigroup satisfying the hypothesis of Theorem 3.2. (E.g. take S to be the additive semigroup of positive integers with discrete topology.)

(v) The referee has commented that there is a non-commutative version of Granirer's result: Theorem 12 in *Weakly almost periodic and uniformly continuous functionals on the Fourier algebra of a locally compact group*, by E. Granirer, Trans. Amer. Math. Soc. 189 (1974), 371–382, to which we refer the interested reader.

We conclude this section by shading some light on the structure of (completely regular) topological semigroups S for which the Stone–Čech compactification βS admits a separately continuous semigroup operation such that S is a subsemigroup of βS. Such a problem is studied in many papers—see e.g. [3], [14] and [22].

Let S be a discrete topological semigroup and recall that $\bar{x} \in m(S)^*$ where $\bar{x}(f) := f(x)$ for all $f \in m(S)$ and $x \in S$. The w^*-closure of $\{\bar{x} : x \in S\}$ in $m(S)^*$ is called *the Stone–Čech compactification of* S and denoted by βS.

THEOREM 3.5. *Let S be a discrete topological semigroup such that $x^{-1}\{y\}$ and $\{y\}x^{-1}$ are finite subsets for all $x, y \in S$. Then the following items are equivalent*:

(i) *βS admits a separately continuous multiplication with respect to which βS is a semigroup with S as a subsemigroup;*

(ii) *the characteristic function of every countable subset of S is weakly alomost periodic;*

(iii) *S is finite.*

PROOF. Evidently (iii) implies (i) and (ii). Now [22, Theorem 2] says that for any pair of sequences $\{x_n\}$, $\{y_m\}$ in S, if item (i) of our theorem holds then we cannot have

$$\left(\bigcup_{m=1}^{\infty} \bigcup_{n>m}^{\infty} \{x_n y_m\}\right) \cap \left(\bigcup_{n=1}^{\infty} \bigcup_{n<m}^{\infty} \{x_n y_m\}\right) = \varnothing \,. \tag{1}$$

So, by Lemma 3.1 (ii), (i) implies (iii).

Finally we show that (ii) implies (iii). If S is infinite, then Lemma 3.1 (ii) allows us to choose sequences $\{x_n\}$, $\{y_m\}$ in S such that equation (1) holds. Now if f is the characteristic function of the countable set

$$\bigcup_{m=1}^{\infty} \bigcup_{n>m}^{\infty} \{x_n y_m\}\,,$$

we have (by (1))

$$\begin{aligned} \lim_m \lim_n {}_{x_n}f(y_m) &= \lim_m \lim_n f(x_n y_m) = 1 \\ \lim_n \lim_m {}_{x_n}f(y_m) &= \lim_n \lim_m f(x_n y_m) = 0 \end{aligned} \tag{2}$$

Hence, by [11, Theorem 6] we have that (2) implies $f \notin WAP(S)$. Consequently (ii) implies (iii).

REMARK 3.6. We note that if S is a cancellation semigroup then $x^{-1}\{y\}$ and $\{y\}x^{-1}$ are finite $(x, y \in S)$. Of course there are many non cancellation semigroups S such that $x^{-1}\{y\}$ and $\{y\}x^{-1}$ are finite sets $(x, y \in S)$—e.g. take S to be the positive integers with maximum operation.

Macri [14] gave a long proof to show that if S is a cancellation semigroup with every characteristic function of a countable subset of S in $AP(S)$ then βS is a topological semigroup with S as a subsemigroup.

(Here S is assumed to be a discrete topological semigroup and $AP(S)$ is a defined in item 3.4 (iv).) Now noting that $AP(S) \subseteqq WAP(S)$ it follows from Theorem 3.5 that the conditions imposed on S (by Macri [14]) imply that S is finite and his conclusion follows trivially.

4. Irregularity of multiplication in semigroup algebras.

Let S be a C-distinguished topological semigroup and $\mathscr{A}$ a norm closed subalgebra of $M(S)$. We denote the first and second dual spaces of $\mathscr{A}$ by $\mathscr{A}^*$ and $\mathscr{A}^{**}$, respectively. For $\varphi \in \mathscr{A}^{**}$, $h \in \mathscr{A}^*$ and $\nu \in \mathscr{A}$ we define $h_\nu, h^\nu, h_\varphi, h^\varphi$ in $\mathscr{A}^*$ by

$$h_\nu(\mu) := h(\nu * \mu) \text{ and } h^\nu(\mu) := h(\mu * \nu) \qquad (\mu \in \mathscr{A})$$

$$h_\varphi(\mu) := \varphi(h^\mu) \text{ and } h^\varphi(\mu) := \varphi(h_\mu) \qquad (\mu \in \mathscr{A})$$

We then define the *Arens products* $\odot$ and $\odot'$ on $\mathscr{A}^{**}$ by

$$\varphi \odot \psi(h) := \psi(h_\varphi) \text{ and } \varphi \odot' \psi(h) := \varphi(h^\psi)$$

for all $\varphi, \psi \in \mathscr{A}^{**}$ and $h \in \mathscr{A}^*$.

With respect to either product $\odot$ or $\odot'$, $\mathscr{A}^{**}$ is a Banach algebra into which $\mathscr{A}$ is embedded isometrically by the canonical homomorphism $\nu \to \pi(\nu)$ $(\nu \in \mathscr{A})$ where

$$\pi(\nu)(h) := h(\nu) \qquad (h \in \mathscr{A}^*) .$$

If the two Arens products coincide (i.e. $\varphi \odot \psi = \varphi \odot' \psi$ for all $\varphi, \psi \in \mathscr{A}^{**}$) we say *multiplication in $\mathscr{A}$ is regular*. Multiplication in $\mathscr{A}$ is said to be *irregular* if it is not regular. It is immediate from the definitions that either Arens product on $\mathscr{A}^{**}$ is separately continuous for $\sigma(\mathscr{A}^{**}, \mathscr{A}^*)$. Any pair $\{\pi(\nu_n)\}, \{\pi(\mu_m)\}$ of bounded sequences in $\pi(\mathscr{A})$ will have cluster points φ, ψ respectively in $\mathscr{A}^{**}$ for $\sigma(\mathscr{A}^{**}, \mathscr{A}^*)$. If multiplication in $\mathscr{A}^*$ is regular, it follows that if $h \in \mathscr{A}^*$ is such that both repeated limits of the double sequence $\{h(\nu_n * \mu_m)\}$ exist, then these two limits are equal (being in fact $\varphi \odot \psi(h)$).

The preceding remarks are similar to those given in [21] and we summarise in

LEMMA 4.1. *If S denotes a C-distinguished topological semigroup and $\mathscr{A}$ a norm closed subalgebra of $M(S)$, then the following items are equivalent*:

(i) *$\mathscr{A}$ has regular multiplication*;

(ii) *if $\{\nu_n\}$, $\{\mu_m\}$ are any sequences in $\mathscr{A}$ and $h \in \mathscr{A}^*$ such that $b: = \lim_m \lim_n h(\nu_n * \mu_m)$ and $c := \lim_n \lim_m h(\nu_n * \mu_m)$ exist, then $b = c$.*

For a locally compact topological group G our next lemma is an improvement of that given by Young [21]. Recall that by a *non-$\mathscr{A}$-negligible set $E \subset S$* we mean a Borel set E such that $|\mu|(E) > 0$ for some $\mu \in \mathscr{A}$. We also take

$$d_{\mathscr{A}}(E) := \{x \in S : \text{ if } O \text{ is any open neighbourhood of } x \text{ then } O \cap E \text{ is not } \mathscr{A}\text{-negligible}\} .$$

LEMMA 4.2. *Let S be a C-distinguished topological semigroup with identity* *element e and $\mathscr{A}$ a neo-convolution measure algebra with foundation equal to S. Let W be a subset of S such that e is not isolated in W and $W \subseteqq d_{\mathscr{A}}(W)$. Then there exist sequences $\{C_n\}$, $\{D_m\}$ of non-$\mathscr{A}$-negligible compact subsets of W such that*

$$\left(\bigcup_{n=1}^{\infty} \bigcup_{n<m}^{\infty} \{C_n D_m\}\right) \cap \left(\bigcup_{m=1}^{\infty} \bigcup_{n>m}^{\infty} \{C_n D_m\}\right) = \varnothing .$$

PROOF. We prove the lemma by induction. Suppose, by the induction hypothesis, that for some positive integer p we have finite sequences $\{C_1, C_2, \ldots, C_p\}$, $\{D_1, D_2, \ldots, D_p\}$ of non-$\mathscr{A}$-negligible compact subsets of W such that if

$$X_p := \bigcup_{n=1}^{p} C_p, \; Y_p := \bigcup_{m=1}^{p} D_m, \; L_p := \bigcup_{n=1}^{p-1} \bigcup_{n<m\leqq p} \{C_n D_m\}$$

and

$$R_p := \bigcup_{m=1}^{p-1} \bigcup_{p\geqq n>m} \{C_n D_m\}$$

then

(α) $e \notin X_p \cup Y_p \cup L_p \cup R_p$

(β) $X_p \cap Y_p = \varnothing$

(γ_1) $L_p \cap Y_p = \varnothing$

(γ_2) $R_p \cap X_p = \varnothing$

(δ) $L_p \cap R_p = \varnothing$

We now prove the inductive step, that is to choose C_{p+1}, D_{p+1} and verify items (α) to (δ) with $p+1$ in place of p. (Note that all sets mentioned in items (α) to (δ) are compact.)

We can choose a non-$\mathscr{A}$-negligible compact set $C_{p+1} \subseteqq W$ such that

(1) $e \notin C_{p+1} \cup C_{p+1} Y_p$ by (α)

(2) $C_{p+1} \cap (Y_p \cup R_p) = \varnothing$ by (α)

(3) $C_{p+1} Y_p \cap (L_p \cup X_p \cup C_{p+1}) = \varnothing$ by (β), (γ_1) and since $e \notin Y_p$.

Next we can choose a non-$\mathscr{A}$-negligible compact set $D_{p+1} \subset W$ such that

(4) $e \notin D_{p+1} \cup X_p D_{p+1}$ by (α)

(5) $D_{p+1} \cap (X_p D_{p+1} \cup L_p \cup X_{p+1}) = \varnothing$ by (α) and since $e \notin C_{p+1}$ (note that $X_{p+1} = X_p \cup C_{p+1}$.)

(6) $X_pD_{p+1} \cap (C_{p+1}Y_p \cup R_p \cup Y_p \cup D_{p+1}) = \emptyset$ by (3), (γ_2), (β) and (α).

We now establish items (α) to (δ) with $p+1$ in place of p.

Noting that $L_{p+1} = L_p \cup X_pD_{p+1}$ and $R_{p+1} = R_p \cup C_{p+1}Y_p$, it follows from ($\alpha$), (1) and (4) that

$$e \notin X_{p+1} \cup Y_{p+1} \cup L_{p+1} \cup R_{p+1}.$$

From (β), (2) and (5) we get

$$X_{p+1} \cap Y_{p+1} = (X_{p+1} \cap Y_p) \cup (X_{p+1} \cap D_{p+1}) = \emptyset.$$

From (γ_1), (5) and (6) we get

$$\begin{aligned} L_{p+1} \cap Y_{p+1} &= (L_p \cup X_pD_{p+1}) \cap Y_{p+1} \\ &= (L_p \cap Y_{p+1}) \cup (X_pD_{p+1} \cap Y_{p+1}) = \emptyset. \end{aligned}$$

From (γ_2), (2) and (3) we get

$$\begin{aligned} R_{p+1} \cap X_{p+1} &= (R_p \cup C_{p+1}Y_p) \cap X_{p+1} \\ &= (R_p \cap X_{p+1}) \cup (C_{p+1}Y_p \cap X_{p+1}) = \emptyset. \end{aligned}$$

From (δ), (3) and (6) we have

$$\begin{aligned} L_{p+1} \cap R_{p+1} &= (L_p \cup X_pD_{p+1}) \cap (R_p \cup C_{p+1}Y_p) \\ &= (L_p \cap R_p) \cup (L_p \cap C_{p+1}Y_p) \cup (X_pD_{p+1} \cap (R_p \cup C_{p+1}Y_p) \\ &= \emptyset \end{aligned}$$

This completes our proof for the inductive step. By repeating the argument countably many times we note, in particular, that item (δ) leads to the conclusion of our lemma.

Our next Theorem contains the result of Young [21] as a special case.

THEOREM 4.3. *Let S denote a C-distinguished topological semigroup with an identity element e and $\mathscr{A}$ a neo-convolution measure algebra with foundation equal to S. Then*

(i) *If e is not an isolated element in S, then multiplication in $\mathscr{A}$ is irregular.*

(ii) *If the foundation of $M_a(S)$ coincides with S and the sets $x^{-1}\{y\}$ and $\{y\}x^{-1}$ are finite ($x, y \in S$), then $\mathscr{A}$ has regular multiplication if and only if S is finite.*

PROOF. To prove (i) we suppose that e is not isolated in S and choose sequences $\{C_n\}$, $\{D_m\}$ of compact subsets of S as stated in Lemma 4.2 (with W

$=S$). By our definition of a neo-convolution measure algebra we can choose positive measures ν_n, μ_m in $\mathscr{A}$ such that

$$\|\nu_n\| = \nu_n(C_n) = 1 \quad \text{and} \quad \|\mu_m\| = \mu_m(D_m) = 1$$

for all positive integers n, m. Since $H := \bigcup_{n=1}^{\infty} \bigcup_{n<m}^{\infty} \{C_n D_m\}$ is σ-compact, we can define $h \in \mathscr{A}^*$ by

$$h(\eta) := \eta(H) \qquad (\eta \in \mathscr{A}) .$$

By Lemma 4.2 and our definition of ν_n and μ_m we have that

$$h(\nu_n * \mu_m) := \nu_n * \mu_m(H) = \begin{cases} 1 & \text{if } n < m \\ 0 & \text{if } n > m \end{cases}$$

and so $\lim_n \lim_m h(\nu_n * \mu_m) = 1$ and $\lim_m \lim_n h(\nu_n * \mu_m) = 0$. Recalling Lemma 4.1, our proof for item (i) is complete.

To prove item (ii) we first note that if S is finite then it is trivial to see that $\mathscr{A}$ has regular multiplication. For the converse we first consider the case where S is discrete. Now if S is discrete and infinite, then recalling Lemma 3.1 (ii), an argument similar to that used in the proof of (i) (which we omit) easily shows that $\mathscr{A}$ has irregular multiplication. So S must be finite if $\mathscr{A}$ has regular multiplication. Finally we consider the case where S is not discrete. In view of item (i) our proof will be complete if we can show that e is not isolated in S. Since S is the foundation of $M_a(S)$, if e is isolated in S then $\bar{e} \in M_a(S)$. Now $\bar{e} \in M_a(S)$ implies that S is discrete—see e.g. [2, Proposition 2.8]. By this conflict if S is not discrete then e is not isolated in S.

One major difference between Lemma 3.1 and Lemma 4.2 is that in Lemma 3.1 the choice of the sequences $\{x_n\}, \{y_m\}$ is not restricted to a special portion of the semigroup S where as in Lemma 4.2 the sequences $\{C_n\}, \{D_m\}$ are chosen within a given "vicinity" of the identity element of S. Our next Theorem partly shows how such a choice in Lemma 4.2 can be exploited to give certain general results.

Let G be any topological group. (Of course the topology on G is assumed to be Hausdorff.) As such G is uniformizable, whence G is completely regular. In particular G is a C-distinguished topological semigroup—so we always have a convolution multiplication on $M(G)$, as defined in section 2. With these remarks in mind we prove the following generalization of Theorem 5 of Pym [17].

THEOREM 4.4. *Let $\mathscr{A}$ be a neo-convolution measure algebra whose foundation (semigroup) S is a Borel subsemigroup of a topological group G. Then the following items are equivalent:*

(i) *Multiplication in $\mathscr{A}$ is regular*;

(ii) *S is finite.*

PROOF. Of course (ii) implies (i). We now prove that (i) implies (ii). Suppose on the contrary (i) holds but S is infinite. Then S is not discrete, by an argument similar to the proof of Theorem 4.3 (ii). So G is not discrete. We can choose two distinct elements x, y in S such that every neighbourhood (in S) of x is infinite and every neighbourhood (in S) of y is infinite. Let U and V be disjoint open neighbourhoods (in S) of x and y, respectively. So if $W' := (x^{-1}U) \cap (Vy^{-1})$, where x^{-1} and y^{-1} denote the inverses of x and y (respectively) in G, then the identity element e of G is not isolated in W' when W' has the restriction topology (though the interior of W' in G may be empty).

With x and y as above, we consider the subsets

$$\overline{x^{-1}} * \mathscr{A} := \{\overline{x^{-1}} * \nu \;:\; \nu \in \mathscr{A}\}$$

and

$$\mathscr{A} * \overline{y^{-1}} := \{\nu * \overline{y^{-1}} \;:\; \nu \in \mathscr{A}\}$$

of $M(G)$. (Recall that $M(S) \subseteqq M(G)$ since S is a Borel subset of G.) Since U and V are open subsets of S and S is the foundation of $\mathscr{A}$, a standard argument easily shows that W' is contained in the foundation of $\overline{x^{-1}} * \mathscr{A}$ and $\mathscr{A} * \overline{y^{-1}}$. Now a similar argument to that used in the proof of Lemma 4.2 with W' replacing the W of Lemma 4.2 shows that we can choose sequences $\{C_n\}$, $\{D_m\}$ of compact subsets of W' such that each C_n is not negligible with respect to $\overline{x^{-1}} * \mathscr{A}$, each D_m is not negligible with respect to $\mathscr{A} * \overline{y^{-1}}$ and

$$\left(\bigcup_{n=1}^{\infty} \bigcup_{n<m}^{\infty} \{C_n D_m\}\right) \cap \left(\bigcup_{m=1}^{\infty} \bigcup_{n>m}^{\infty} \{C_n D_m\}\right) = \varnothing . \tag{1}$$

(We omit the details which the reader can easily supply by conveniently adjusting the proof of Lemma 4.2.)

Thus in particular we can choose sequences $\{\nu_n\}$, $\{\mu_m\}$ of positive measures in $\mathscr{A}$ such that

$$\begin{cases} \|\nu_n\| = \nu_n(xC_n) = \overline{x^{-1}} * \nu_n(C_n) = 1 \\ \|\mu_m\| = \mu_m(D_m y) = \mu_m * \overline{y^{-1}}(D_m) = 1 \end{cases} \tag{2}$$

$xC_n \subset x(x^{-1}U) \subset S$ and $D_m y \subset (Vy^{-1})y \subset S$, and (by cancellation) (1) implies that

$$\left(\bigcup_{n=1}^{\infty} \bigcup_{n<m}^{\infty} \{(xC_n)(D_m y)\}\right) \cap \left(\bigcup_{m=1}^{\infty} \bigcup_{n>m}^{\infty} \{(xC_n)(D_m y)\}\right) = \varnothing . \tag{3}$$

Using (2) and (3) we can define $h \in \mathscr{A}^*$ such that $\lim_n \lim_m h(\nu_n * \mu_m) = 1$ and $\lim_m \lim_n h(\nu_n * \mu_m) = 0$, in a way similar to the first paragraphy of the proof of Theorem 4.3. Recalling Lemma 4.1, we have a contradiction. By the conflict our proof is complete.

SOME REMARKS 4.5.

(i) The reader can easily supply a counter-example to show that the hypothesis of Theorem 4.4 does not necessarily require S to have non-empty interior in G.

(ii) From our Lemma 4.2 and Theorem 2 of Young [22] it follows that if S is a non-discrete C-distinguished topological semigroup with an identity element e such that e is not isolated in S, then if S_d denotes S with discrete topology we have that βS_d (the Stone–Čech compactification of S_d) does not admit a separately continuous multiplication with respect to which βS_d is a semigroup containing S_d as a subsemigroup.

ACKNOWLEDGEMENT. It is a pleasure to express my sincere gratitude to Professor A. C. M. van Rooij for many stimulating discussions and his usual characteristically intensive criticisms during the progress of this work. I am also grateful to the referee for helpful comments, and to Dr. G. L. G. Sleijpen for the neat example mentioned in 3.4 (iii).

For financial support I held a Faculty of Science post-Doctoral Research Fellowship awarded by the Katholieke Universiteit, Nijmegen, The Netherlands.

REFERENCES

1. A. C. and J. W. Baker, *Algebras of measures on a locally compact semigroup* II, J. London Math. Soc. 2 (1970), 651–659.
2. A. C. and J. W. Baker, *Algebras of measures on a locally compact semigroup* III, J. London Math. Soc. 4 (1972), 685–695.
3. J. W. Baker and R. J. Butcher, *The Stone–Čech compactification of a topological semigroup*, Proc. Cammb. Phil. Soc. 80 (1976), 103–117.
4. R. B. Burckel, *Weakly almost periodic functions on semigroups*, Gordon and Breach, New York, 1970.
5. P. Civin and B. Yood, *The second conjugate space of a Banach algebra as an algebra*, Pacific J. Math. 11 (1961), 847–870.
6. W. W. Comfort and K. A. Ross, *Pseudocompactness and uniform continuity in topological groups*, Pacific J. Math. 16 (1966), 483–496.
7. H. A. M. Dzinotyiweyi, *Algebras of measures on* C-*distinguished topological semigroups*, Proc. Camb. Phil. Soc. 84 (1978), 323–336.
8. H. A. M. Dzinotyiweyi, *On the analogue of the group algebra for locally compact semigroups*, J. London Math. Soc. 17 (1978), 489–506.

9. L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, New York, 1960.
10. E. E. Granirer, *Exposed points of convex sets and weak sequential convergence*, Mem. Amer. Math. Soc. 123 (1972).
11. A. Grothendieck, *Critères de compacité dans les espaces fonctionnels généraux*, Amer. J. Math. 74 (1952), 168–186.
12. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, vol. I, Springer-Verlag, Berlin, 1963.
13. R. Hirschfeld, *RIOTS*, Niew Arch. Wisk. 22 (1974), 1–43.
14. N. Macri, *The continuity of Arens product on the Stone–Čech compactification of semigroups*, Trans. Amer. Math. Soc. 191 (1974), 185–193.
15. P. Milnes, *Compactifications of semitopological semigroups*, J. Austral. Math. Soc. 15 (1973), 488–503.
16. A. L. T. Paterson, *Amenability and locally compact semigroups*, Mathematica Scandinavica 42 (1978), 271–288.
17. J. S. Pym, *Convolution algebras are not usually Arens-regular*, Quart, J. Math. Oxford (2) 25 (1974), 235–240.
18. G. L. G. Sleijpen, *Convolution measure algebras on semigroups*, Thesis Katholieke Universiteit, Nijmegen 1976.
19. G. L. G. Sleijpen, *Locally compact semigroups and continuous translations of measures*, Proc. Lond. Math. Soc. 37 (1978), 75–97.
20. J. L. Taylor, *The structure of convolution measure algebras*, Trans. Amer. Math. Soc. 119 (1965), 150–166.
21. N. J. Young, *The irregularity of multiplication in group algebras*, Quart. J. Math. Oxford (2) 24 (1973), 59–62.
22. N. J. Young, *Semigroup algebras having regular multiplication*, Studia Math. 47 (1973), 191–196.

MATHEMATISCH INSTITUUT
KATHOLIEKE UNIVERSITEIT
TOERNOOIVELD
NIJMEGEN
THE NETHERLANDS

AND

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NAIROBI
P.O. BOX 30197
NAIROBI
KENYA