MATH. SCAND. 46 (1980), 173–176

COMPACTNESS CRITERIA FOR SEMIGROUPS, A NOTE ON A PRECEDING PAPER

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This note is a sequel to matters studied in section 3 of our paper [2] and all terms (undefined here) are as defined in [2]. We say a semigroup S is a *topological* (or *semitopological*) *semigroup* if S is endowed with a Hausdorff topology with respect to which multiplication $(x,y) \to xy$ is a jointly (or separately, respectively) continuous mapping of $S \times S$ into S. For every semitopological semigroup S let $C(S)$ be the set of all bounded complex-valued continuous functions on S,

$$WAP(S) := \{f \in C(S) : \text{ the set } \{{}_xf : x \in S\} \text{ is weakly relatively compact}\}$$

and

$$WUC(S) := \{f \in C(S) : \text{ the maps } x \to {}_xf \text{ and } x \to f_x \text{ of } S \text{ into } C(S) \text{ are weakly continuous}\}$$

(Here the space $C(S)$ has the usual sup-norm and for all $f \in C(S)$ and $x \in S$ the functions ${}_xf, f_x \in C(S)$ are given by

$${}_xf(y) := f(xy) \text{ and } f_x(y) := f(yx) \qquad (y \in S)\,.)$$

Generalizing results of Burckel [1] and Granirer [3] we showed in [2] that if S is a topological semigroup supporting non-zero measures continuous under translation (i.e. $M_a(S)$ is non-zero) and such that

(*) for all compact $C,D \subseteqq S$ we have that $C^{-1}D$ and DC^{-1} are compact,

then S is locally compact and the following items are equivalent:

(a) S is compact;
(b) $WUC(S) = WAP(S)$;
(c) $C(S) = WAP(S)$.

In [2, Item 3.4 (iii)] we gave an example of a non-compact (pseudocompact) locally compact topological semigroup S with $M_a(S)$ zero, $C(S) = WAP(S)$ and

Received August 14, 1979.

satisfying property (*). Also $M_a(S)$ played a pivotal role in our proof of the equivalence of (a), (b) and (c). In this note we prove

THEOREM. *Let S be a σ-compact locally compact semitopological semigroup such that $x^{-1}K$ and Kx^{-1} are compact for all compact $K \subseteq S$ and $x \in S$. Then S is compact if and only if $C(S) = WAP(S)$.*

REMARK 1. Of course there are many σ-compact locally compact topological semigroups with $M_a(S)$ zero. For example the interval $[0, \infty)$ with maximum operation and the usual topology.

In view of the results of [2] and the preceding theorem, the following conjecture seems reasonable:

CONJECTURE 1. *If S is any σ-compact locally compact topological semigroup such that $x^{-1}K$ and Kx^{-1} are compact for all compact $K \subseteq S$ and $x \in S$, then $WUC(S) = WAP(S)$ implies that S is compact.*

If in the preceding conjecture S is a semitopological semigroup we strongly feel that one should be able to find a counterexample though so far our efforts in that direction have remained unfruitful.

The only examples we know of non-compact locally compact topological semigroups S satisfying property (*) and such that $C(S) = WAP(S)$, have $M_a(S)$ zero and are pseudocompact (see [2]). Numerous conversations with Professor Dr. A. C. M. van Rooij and later Dr. G. L. G. Sleijpen have led us to the following conjecture:

CONJECTURE 2. *If S is any locally compact topological semigroup satisfying property (*), then the following items are equivalent:*

(a) *S is pseudocompact*
(b) $WUC(S) = WAP(S)$
(c) $C(S) = WAP(S)$.

(That (c) $\Rightarrow$ (b) is the only (trivial) known part.)

REMARK 2. For a locally compact topological group G Granirer also studied the set $WAP(G)$ in [4]. (We thank the referee to our paper [2] for pointing out the reference [4] to us.) In particular we take this opportunity to announce that the question asked in [4, page 382] in connection with Theorem 14 of the same paper has been answered affirmatively by us. The latter will appear elsewhere.

Finally we prove our theorem. That part of our proof involving construction of sequences of compact sets is inspired by [2, lemma 3.1] and [6].

PROOF OF THEOREM. If S is compact it is easy (and well known) to show that $C(S) = WAP(S)$ (see for example [1]). Now suppose that S is not compact. Then our proof will be complete if we show that $C(S) \setminus WAP(S)$ is non-empty. To this end we first note that, since S is σ-compact, there is a sequence $\{U_n\}$ of compact neighbourhoods such that

$$S = \bigcup_{n=1}^{\infty} U_n \quad \text{and} \quad U_n \subset U_{n+1} \quad \text{for all } n \in \mathbf{N} .$$

We can assume, by the inductive hypothesis, that for some integer $p \in \mathbf{N}$ we have finite sequences $\{x_1, x_2, .., x_p\}$ and $\{y_1 y_2, ..., y_p\}$ in S, a finite subsequence $\{U_{n_1}, U_{n_2}, ..., U_{n_p}\}$ of $\{U_n\}$ and $\{O_{k,m}\}_{1 \leqq k < m \leqq p}$ such that if $i, j, k, m \in \{1, 2, ..., p\}$ then

(α) $k < m$ implies $U_{n_k} \subset U_{n_m}$, $O_{k,m} \subset U_{n_m}$ and $O_{k,m}$ is a compact neighbourhood of $x_k y_m$;
(β) $i > j$ and $k < m$ imply $x_i y_j \notin O_{k,m}$,
(γ) $k < m$ and $j < m$ imply $O_{k,m} \cap U_{n_j} = \emptyset$.

We now establish the preceding items with $p+1$ in place of p. Since $X_p := \{x_1, x_2, ..., x_p\}$ and $Y_p := \{y_1, y_2, ..., y_p\}$ are finite we have $U_{n_p} Y_p^{-1}$ and $X_p^{-1} U_{n_p}$ compact. So we can find x_{p+1}, $y_{p+1} \in S$ such that

$$(1) \qquad x_{p+1} Y_p \cap U_{n_p} = \emptyset ,$$

and $X_p y_{p+1} \cap U_{n_p} = \emptyset$. We can find a compact neighbourhood C of $X_p y_{p+1}$ and take $O_{k,p+1} = C$ so that

$$(2) \qquad O_{k,p+1} \cap U_{n_p} = \emptyset \quad \text{for all } k \in \{1, 2, ..., p\} .$$

Next we choose $U_{n_{p+1}}$ to be the first member of the sequence $\{U_n\}$ such that

$$(3) \qquad U_{n_{p+1}} \supset U_{n_p} \cup C .$$

It is now easy for the reader to note (by using (1), (2) and (3)) that (α), (β) and (γ) are still valid for $i, j, k, m \in \{1, 2, ..., p+1\}$. Hence repeating the argument countably many times we get infinite sequences such that (α), (β) and (γ) are valid for $i, j, k, m \in \mathbf{N}$.

Now for every $m \in \mathbf{N}$ if $m \geqq 2$ we can find a function f_m in $C(S)$ such that $0 \leqq f_m(x) \leqq 1$ $(x \in S)$, $f_m(x) = 0$ for $x \notin \bigcup_{k=1}^{m-1} O_{k,m}$ and $f_m(x_k y_m) = 1$ for $k \in \{1, 2, ..., m-1\}$.

We now consider the function $f := \sum_{m=1}^{\infty} f_m$. To see that f is continuous let V be any compact neighbourhood. Then for some $p \in \mathbf{N}$, with $p \geqq 3$, we must have $V \subset U_{n_p}$, by our construction of the sequence $\{U_{n_1}, U_{n_2}, \ldots\}$. So for all $x \in V$ we have (by (γ))

$$f(x) = \sum_{m=2}^{p-1} f_m(x)$$

and so f is continuous. It is trivial to note that f is bounded by 1 as a consequence of the fact that, $j \neq m$ implies that $O_{i,j} \cap O_{k,m} = \emptyset$ for $i \in \{1,2,\ldots,j-1\}$ and $k \in \{1,2,\ldots,m-1\}$ (by items (α) and (γ)), and our definition of the functions f_m.

Now from (α), (β) and (γ) we note that

$$f(x_k y_m) = \begin{cases} 1 & \text{if } k<m \\ 0 & \text{if } k>m\,. \end{cases}$$

The preceding equation together with Theorem 6 of Grothendieck [5] implies that $f \notin WAP(S)$. Thus $f \in C(S) \setminus WAP(S)$ and as remarked before this completes our proof.

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